Strong isometric dimension, biclique coverings, and Sperner's Theorem

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Abstract

The strong isometric dimension of a graph G is the least number k such that G isometrically embeds into the strong product of k paths. Using Sperner's Theorem, the strong isometric dimension of the Hamming graphs $K_2 \square K_n$ is determined.

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1 Introduction

Motivated by an injective hull of a graph [9] and the game of "cops and robber" [12], Fitzpatrick and Nowakowski in [5] introduced the strong isometric dimension of a graph. Already in 1938 Schönberg [13] proved that this concept is well-defined, more precisely, he proved that every connected graph admits an isometric embedding into the strong product of paths, cf. [8, Proposition 5.2]. This result has also been independently obtained in [5]. Thus it is natural to define the strong isometric dimension, $\mathrm{idim}(G)$, of a graph G, as the least number k such that G embeds isometrically into the strong product of k paths. (For a general framework that for any class of graphs and for any graph product gives a different dimension concept see [11].)

Only few exact strong isometric dimensions of graphs are known: $\mathrm{idim}(K_{m,n}) = \lceil \log_2 m \rceil + \lceil \log_2 n \rceil$ [3], $\mathrm{idim}(C_n) = \lceil n/2 \rceil$, $\mathrm{idim}(Q_n) = 2^{n-1}$ [5], and $\mathrm{idim}(P) = 5$, where P is the Petersen graph [10]. In this note we prove the following result.

Theorem 1.1 *Let $k \geq 1$ and let $\binom{k}{\lfloor k/2 \rfloor} < n \leq \binom{k+1}{\lfloor (k+1)/2 \rfloor}$. Then $\mathrm{idim}(K_2 \,\square\, K_n) = k + 1$.*

In the next section we first recall concepts and notations needed in this note. Then we describe an explicit isometric embedding of $K_2 \,\square\, K_n$ into the strong product of paths that yields the upper bound of Theorem 1.1. In the subsequent section we then prove the corresponding lower bound. The proof consists of two key steps. The problem of determining $\mathrm{idim}(K_2 \,\square\, K_n)$ is first reduced to a covering problem of the complement of $K_2 \,\square\, K_n$ with bicliques, and then solved using Sperner's Theorem. We conclude with some remarks on related covering and decomposition problems.

2 Preliminaries and upper bound

By $d_G(u, v)$ we mean the standard graph distance, that is, the number of edges on a shortest u, v-path. The *diameter*, $\mathrm{diam}(G)$, of a connected graph G is the maximum distance between any two vertices of G. A subgraph H of G is an *isometric* subgraph of a graph G if $d_H(u, v) = d_G(u, v)$ for all $u, v \in H$. Let G and H be arbitrary graphs. A mapping $f : V(G) \to V(H)$ is an *isometric embedding* if $d_H(f(u), f(v)) = d_G(u, v)$ for any $u, v \in V(G)$.

The *strong product* $G = \boxtimes_{i=1}^{k} G_i$ of graphs $G_1, \ldots, G_k$ is the graph defined on the Cartesian product of the vertex sets of the factors, two distinct vertices $(u_1, \ldots, u_k)$ and $(v_1, \ldots, v_k)$ being adjacent if and only if u_i is equal or adjacent to v_i in G_i for $i = 1, \ldots, k$. The *Cartesian product* $G = \square_{i=1}^{k} G_i$ of graphs $G_1, \ldots, G_k$ is the graph that is also defined on the Cartesian product of the vertex sets of the factors. Two distinct vertices $(u_1, \ldots, u_k)$ and $(v_1, \ldots, v_k)$ are adjacent if and only if there exists an index j such that u_j is adjacent to v_j in G_j and $u_i = v_i$ for all $i \neq j$. Cartesian products of complete graphs are known as *Hamming graphs*. Note that the subgraph of $G = \Box_{i=1}^{k} G_i$ (as well as of $G = \boxtimes_{i=1}^{k} G_i$) induced by the vertices that differ from a given vertex u only in the ith coordinate is isomorphic to G_i.

The *strong isometric dimension*, $\mathrm{idim}(G)$, of a graph G is the least number k such that there is a set of k paths $\{P^{(1)}, \ldots, P^{(k)}\}$ for which G isometrically embeds into $\boxtimes_{i=1}^{k} P^{(i)}$.

Let $G = (V, E)$ be a graph and let $H_1 = (V_1, E_1), \ldots, H_k = (V_k, E_k)$ be subgraphs of G. If $E = E_1 \cup \ldots \cup E_k$, we say that G is *covered* by $H_1, \ldots, H_k$ or that the subgraphs $H_1, \ldots, H_k$ form a *covering* of G.

The *complement* $\overline{G}$ of a graph G is the graph on $V(G)$ with the edge set $\{xy \mid x, y \in V(G), x \neq y, xy \notin E(G)\}$, and by a *biclique* we mean a complete bipartite graph.

We now give an upper bound for the strong isometric dimension of $K_2 \,\Box\, K_n$.

Lemma 2.1 *Let* $n \leq \binom{k}{\lfloor k/2 \rfloor}$. *Then* $\mathrm{idim}(K_2 \,\Box\, K_n) \leq k$.

Proof. For the path on n vertices P_n let $V(P_n) = \{0, 1, \ldots, n-1\}$, where i is adjacent to $i+1$ for $i = 0, 1, \ldots, n-2$. Let $n = \binom{k}{\lfloor k/2 \rfloor}$ and set

$$X = \{(t_1, \ldots, t_k) \;\mid\; t_i \in \{0, 1\},\; \sum_{i=1}^{k} t_i = \lfloor k/2 \rfloor\}\,.$$

Note that $|X| = \binom{k}{\lfloor k/2 \rfloor} = n$. Let in addition

$$Y = \{(t_1 + 1, \ldots, t_k + 1) \;\mid\; (t_1, \ldots, t_k) \in X\}\,.$$

Now consider X and Y as vertex subsets of the graph $H = \boxtimes_{i=1}^{k} P_3^{(i)}$. We claim that $X \cup Y$ induces an isometric subgraph of H isomorphic to $K_2 \,\Box\, K_n$. Note first that both X and Y induce a complete subgraph on n vertices. Moreover, let $u = (t_1, \ldots, t_k) \in X$. Then u is in H adjacent to exactly one vertex of Y, namely to $(t_1 + 1, \ldots, t_k + 1)$.

For $n \leq \binom{k}{\lfloor k/2 \rfloor}$, $K_2 \,\Box\, K_n$ is an isometric subgraph of the graph $K_2 \,\Box\, K_{\binom{k}{\lfloor k/2 \rfloor}}$ and hence $\mathrm{idim}(K_2 \,\Box\, K_n) \leq \mathrm{idim}(K_2 \,\Box\, K_{\binom{k}{\lfloor k/2 \rfloor}}) \leq k$. $\Box$

3 Lower bound

In this section we prove that the upper bound of Lemma 2.1 is sharp. For this sake we first observe that for any $n \geq 2$, $K_2 \,\Box\, K_n$ is a graph of diameter 2 and recall the following result.

Theorem 3.1 *Let G be a graph with* $\operatorname{diam}(G) = 2$ *and let any edge of G be contained in an induced path on three vertices. Then* $\operatorname{idim}(G)$ *is equal to the smallest r such that the edges of $\overline{G}$ can be covered with r bicliques.*

Theorem 3.1 is due to Dewdney [3] who proved it in terms of the so-called adjacent isometric dimension of a graph. The result is in terms of the strong isometric dimension presented in [10].

Let $V = \{v_1, \ldots, v_n\}$ and $W = \{w_1, \ldots, w_n\}$ be the bipartition of the complete bipartite graph $K_{n,n}$. Let $K_{n,n}^-$, $n \geq 2$, be the graph obtained from $K_{n,n}$ by deleting a perfect matching M. We will always assume that $M = \{(v_i, w_i) \mid i = 1, \ldots, n\}$. Hence, by Theorem 3.1, $\operatorname{idim}(K_2 \,\square\, K_n)$ is the smallest r such that the edges of $K_{n,n}^-$ can be covered with r bicliques. Lemma 2.1 can therefore be rephrased as: Let $n \leq \binom{k}{\lfloor k/2 \rfloor}$, then $K_{n,n}^-$ can be covered with k bicliques. To complete the proof of Theorem 1.1 we thus need to show that if we have a covering of $K_{n,n}^-$ by k bicliques, then $n \leq \binom{k}{\lfloor k/2 \rfloor}$. We will prove this using Sperner's Theorem. For a recent related approach see [14], while for more information on Sperner theory we refer to [4].

Recall that an *antichain* $A_1, \ldots, A_n$ on a set A is a family of nonempty subsets of A such that $A_i \subseteq A_j$ implies that $i = j$.

Theorem 3.2 (Sperner) *Let $A_1, \ldots, A_n$ be an antichain on a k-set. Then $n \leq \binom{k}{\lfloor k/2 \rfloor}$. Moreover, for each $k \geq 1$, there exists an antichain that contains n sets for every $n \leq \binom{k}{\lfloor k/2 \rfloor}$.*

As noted above, the following lemma will complete the proof of our main result.

Lemma 3.3 *Let $n \geq 2$ and let $H_1, \ldots, H_k$ be a covering of $K_{n,n}^-$ by k bicliques. Then $n \leq \binom{k}{\lfloor k/2 \rfloor}$.*

Proof. For an arbitrary fixed covering $H_1, \ldots, H_k$ we construct a corresponding antichain in the following manner. To every vertex $v_i \in V$ we assign a set A_i that will consist of the subscripts of all covering bicliques containing v_i. Then we show that such a family of sets $A_1, \ldots, A_k$ is indeed an antichain on $A = \{1, \ldots, k\}$. The conclusion then follows from Theorem 3.2.

So suppose we have a fixed covering of $K_{n,n}^-$ by k bicliques $H_1, \ldots, H_k$. For $i = 1, \ldots, n$ we define $A_i = \{j \mid v_i \in H_j\}$. Obviously, $A_i \subseteq A$ as we have exactly k bicliques. To observe that each A_i is nonempty we notice that every vertex v_i belongs to at least one biclique H_j, otherwise the edges incident with v_i are not covered. This is impossible, as we assumed that $H_1, \ldots, H_k$ is a covering of $K_{n,n}^-$. Finally, we need to show that there is no pair of sets A_i and A_m such that $A_i \subseteq A_m$ while $i \neq m$.

To do that, we proceed by contradiction and suppose $A_i \subseteq A_m$ for some $i \neq m$. But if $j \in A_i$ and $j \in A_m$, then $v_i \in H_j$ and also $v_m \in H_j$. Therefore, every biclique that contains the vertex v_i must contain also the vertex v_m. However, $i \neq m$ and therefore the edge (v_i, w_m) has to belong to some biclique H_s and $s \in A_i$. Then $s \in A_m$. This means that $v_m \in H_s$. But now the biclique H_s contains both v_m and w_m and therefore also the edge (v_m, w_m). This is impossible, since (v_m, w_m) is not an edge of $K^-_{n,n}$. Therefore, no set A_i is contained in another set A_m and $A_1, \dots, A_n$ is an antichain on A (with k elements).

By Sperner's Theorem every antichain on A contains at most $\binom{k}{\lfloor k/2 \rfloor}$ sets. Thus $n \leq \binom{k}{\lfloor k/2 \rfloor}$ and the proof is complete. □

4 Concluding remarks

There are several angles from which one can look at a minimization problem related to coverings of a graph G by bicliques. Füredi and Kündgen [7] gave general bounds for the total number of edges used in the cover of any graph G by bicliques, as well as sharp bounds for certain classes of graphs such as 4-colorable graphs and random graphs.

Other graphs besides bicliques can also be used in the covering. Chung proved in [2] that the sum of the number of vertices of cliques used in an edge-disjoint cover of an n-vertex graph is at most $\lfloor \frac{n^2}{2} \rfloor$. Moreover, the biclique on $\lceil n/2 \rceil$ plus $\lfloor n/2 \rfloor$ vertices is the only extremal graph for this problem.

Chung [1] proved a conjecture by Bermond that $\lim_{n\to\infty} \rho(n)/n = 1$, where $\rho(n)$ denotes the smallest integer such that any graph with n vertices can be covered by $\rho(n)$ bicliques. Setting $\tau(n)$ to be the smallest number with the property that $K^-_{n,n}$ has a covering by $\tau(n)$ bicliques it follows from the results of this note that $\lim_{n\to\infty} \frac{\tau(n)}{n} = \lim_{k\to\infty} \frac{k}{2\binom{k}{\lfloor \frac{k}{2} \rfloor}} = 0$.

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