

# The (non-)existence of perfect codes in Fibonacci cubes

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## Abstract

The Fibonacci cube  $\Gamma_n$  is obtained from the  $n$ -cube  $Q_n$  by removing all the vertices that contain two consecutive 1s. It is proved that  $\Gamma_n$  admits a perfect code if and only if  $n \leq 3$ .

**Key words:** Error-correcting code; perfect code; efficient dominating set; Fibonacci cube.

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## 1 Introduction

A *1-perfect code* of a graph  $G$  is a subset  $C \subseteq V(G)$  such that every vertex of  $G$  is either in  $C$  or adjacent to precisely one member of  $C$ . This concept generalizes to  $r$ -perfect codes,  $r \geq 1$ , but since we will exclusively deal with 1-perfect codes, we will call them simply *perfect codes*. Another name frequently used for a perfect code is an *efficient dominating set*.

The study of codes in graphs which was initiated by Biggs [3] presents a generalization of the problem of the existence of (classical) error-correcting codes. For instance, Hamming codes and Lee codes correspond to codes in the Cartesian product of complete graphs and cycles, respectively. For further results on perfect codes in Cartesian products, lexicographic products, and strong products see [22, 24, 1], respectively. In addition, for the characterization of perfect codes in the direct product of cycles see [18, 27].

Classes of graphs similar to products for which perfect codes were investigated include direct graph bundles [9] and twisted tori [12]. Perfect codes were also investigated in other classes of graphs, notably on Sierpiński graphs [5, 14], Cayley graphs [6], cubic vertex-transitive graphs [19], circulant graphs [7, 21], and AT-free and dually chordal graphs [4].

Kratochvíl [20] proved a remarkable result that there are no nontrivial perfect codes over complete bipartite graphs with at least three vertices. In this note we establish a similar non-existence result for Fibonacci cubes. These cubes are interesting here from (at least) two reasons. First, as (isometric) subgraphs of hypercubes (recall that Hamming codes are perfect codes in hypercubes) they are close to Cartesian product graphs. Second, they form an appealing model for interconnection networks [10].

Fibonacci cubes have been extensively studied and found several applications: see the survey [13]. The interest for Fibonacci cubes continues: recent research of them includes asymptotic properties [15], connectivity issues [2], and the structure of their disjoint induced hypercubes [8]. From the algorithmic point of view, Ramras [23] investigated congestion-free routing of linear permutations on Fibonacci cubes, while Vesel [25] designed a linear time recognition algorithm for this class of graphs.

The result of this note reads as follows.

**Theorem 1.1** *The Fibonacci cube  $\Gamma_n$ ,  $n \geq 0$ , admits a perfect code if and only if  $n \leq 3$ .*

In the rest of this section we formally introduce the concepts needed, while in the next section Theorem 1.1 is proved. We conclude this note with some ideas for further research.

A *Fibonacci string* of length  $n$  is a binary string  $b_1 \dots b_n$  with  $b_i \cdot b_{i+1} = 0$  for  $1 \leq i < n$ . Fibonacci strings are thus binary strings that contain no consecutive 1s. A *Fibonacci string of weight  $k$*  is a Fibonacci string with precisely  $k$  ones. The *Fibonacci cube  $\Gamma_n$* ,  $n \geq 1$ , is the graph whose vertices are all the Fibonacci strings of length  $n$ , two vertices being adjacent if they differ in precisely one position. (In other words,  $\Gamma_n$  is the subgraph of the  $n$ -cube  $Q_n$  induced by all vertices that contain no two consecutive 1s.) For convenience we also set  $\Gamma_0 = K_1$ .

By  $\Gamma_{n,k}$  we denote the vertices of  $\Gamma_n$  of weight  $k$ . It is easy to observe that

$$|\Gamma_{n,k}| = \binom{n-k+1}{k}.$$

For a more general result of this type see [16]. For  $i \in \{0, 1\}$  we denote by  $\Gamma_{n,k}^i$  the vertices of  $\Gamma_{n,k}$  that start with  $i$ . Observe that the vertices of  $\Gamma_{n,k}^0$  are of the form  $0\alpha$ , where  $\alpha$  is a Fibonacci string of weight  $k$  and length  $n-1$ . Consequently,

$$|\Gamma_{n,k}^0| = |\Gamma_{n-1,k}| = \binom{n-k}{k}.$$

By a similar argument we infer that

$$|\Gamma_{n,k}^1| = |\Gamma_{n-2,k-1}| = \binom{n-k}{k-1}.$$

## 2 Proof of Theorem 1.1

It can be easily checked by hand that each of  $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3$  contains a perfect code and that none of  $\Gamma_4$  and  $\Gamma_5$  does. It remains to prove that  $\Gamma_n$  does not admit a perfect code for any  $n \geq 6$ . For the sake of this we first show:

**Lemma 2.1** *If  $n \geq 6$  and  $\mathcal{C}$  is a perfect code of  $\Gamma_n$ , then  $0^n \notin \mathcal{C}$ .*

**Proof.** Suppose on the contrary that  $0^n \in \mathcal{C}$ . Then all the vertices in  $\Gamma_{n,1}$  are dominated by  $0^n$ . Hence  $\Gamma_{n,2} \cap \mathcal{C} = \emptyset$ . Consequently, each vertex of  $\Gamma_{n,2}$  must be dominated by a vertex of  $\Gamma_{n,3}$ . The only vertices in  $\Gamma_{n,3}$  that dominate the vertices in  $\Gamma_{n,2}^1$  are in  $\Gamma_{n,3}^1$ . Since each vertex  $v \in \Gamma_{n,3}^1$  has precisely two neighbors in  $\Gamma_{n,2}^1$  we have

$$|\mathcal{C} \cap \Gamma_{n,3}^1| = \frac{|\Gamma_{n,2}^1|}{2} = \frac{n-2}{2}.$$

Therefore, the number of undominated vertices in  $\Gamma_{n,3}^1$  so far is

$$|\Gamma_{n,3}^1| - \frac{(n-2)}{2} = \binom{n-3}{2} - \frac{(n-2)}{2} = \frac{n^2 - 8n + 14}{2}.$$

These vertices can only be dominated by the vertices of  $\Gamma_{n,4}^1$ . Moreover, each such vertex dominates precisely three of the undominated vertices of  $\Gamma_{n,3}^1$ . Hence we have that

$$|\mathcal{C} \cap \Gamma_{n,4}^1| = \frac{n^2 - 8n + 14}{6}.$$

But the last expression is not an integer. This contradiction proves the lemma.  $\square$

Suppose now that  $\mathcal{C}$  is a perfect code of  $\Gamma_n$ . Then by Lemma 2.1 we have that  $0^n \notin \mathcal{C}$ . This implies that  $|\mathcal{C} \cap \Gamma_{n,1}| = 1$ . Denote with  $a$  this unique vertex. The remaining  $n-1$  vertices of  $\Gamma_{n,1}$  must be dominated by the vertices of  $\Gamma_{n,2}$ . Since a vertex of  $\Gamma_{n,2}$  dominates precisely two vertices of  $\Gamma_{n,1}$ , it follows that  $n$  is odd and that there are

$$\begin{aligned} |\Gamma_{n,2}| - (n-1)/2 - (d-1) &= \binom{n-1}{2} - (n-1)/2 - (d-1) \\ &= (n^2 - 4n - 2d + 5)/2 \end{aligned} \tag{1}$$

undominated vertices in  $\Gamma_{n,2}$  where  $d \in \{n-2, n-1\}$  is the degree of  $a$ . These vertices must be dominated by the vertices of  $\Gamma_{n,3}$  and hence the expression (1) must be divisible

by 3. Setting  $d = n - 1$  and using the fact that  $n = 2k + 1$  for some integer  $k$ , (1) is reduced to  $2k^2 - 4k + 1$ . Since 3 does not divide  $2k^2 - 4k + 1$  for any  $k > 0$ , it follows that  $d = n - 2$ . Consequently,  $a \notin \{10^{n-1}, 0^{n-1}1\}$ . The fact that  $d = n - 2$  implies that there are  $(n - 3)^2/2$  undominated vertices in  $\Gamma_{n,2}$  that need to be dominated by the vertices of  $\Gamma_{n,3}$ . This implies that  $6|(n - 3)^2$ , that is,  $n = 6k + 3$  for some  $k > 0$ .

In what follows we split the proof into two parts depending on whether  $a$  starts with 00 or 01. Suppose first that  $a$  starts with 00. Then it dominates precisely  $n - 4$  vertices of  $\Gamma_{n,2}^0$  and a single vertex of  $\Gamma_{n,2}^1$  (see Fig. 1.) In order to dominate  $\Gamma_{n,1}^1 = \{10^{n-1}\}$ , the perfect code  $\mathcal{C}$  must contain at least one vertex of  $\Gamma_{n,2}^1$ . Moreover, since every vertex of  $\Gamma_{n,2}^1$  is adjacent to  $10^{n-1}$ , we must have  $|\mathcal{C} \cap \Gamma_{n,2}^1| = 1$ . Since the vertex  $a$  dominates precisely one vertex in  $\Gamma_{n,2}^1$ , there are  $|\Gamma_{n,2}^1| - 2 = n - 4$  vertices in  $\Gamma_{n,2}^1$  that must be dominated by the vertices in  $\Gamma_{n,3}^1$  and since each such vertex dominates precisely two elements of  $\Gamma_{n,2}^1$  it follows that  $\mathcal{C}$  must contain  $(n - 4)/2$  vertices of  $\Gamma_{n,3}^1$ . The fact that  $n$  is odd implies that the last expression is not an integer thus deriving a contradiction.

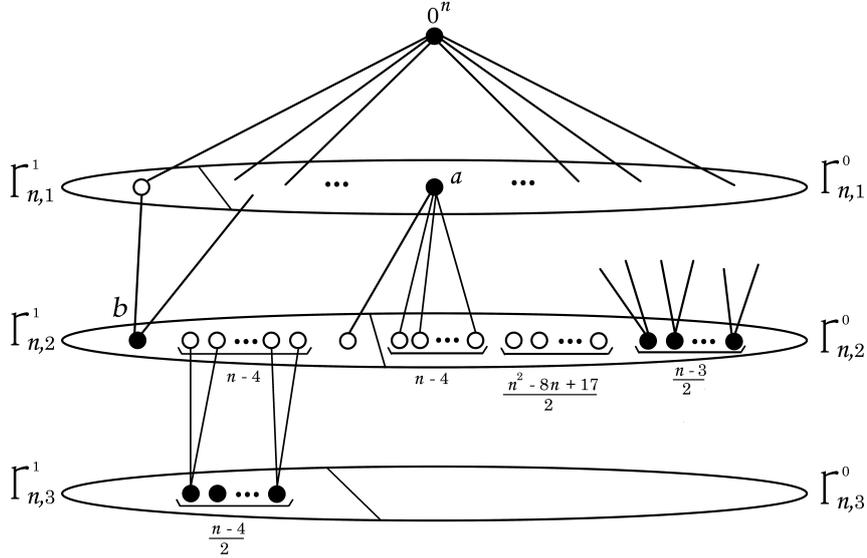


Figure 1: The case when  $a$  starts with 00

We are left with the case for which  $a$  starts with 01. In this case  $a$  actually starts with 010 and dominates precisely  $n - 3$  vertices of  $\Gamma_{n,2}^0$ . As before, in order to dominate  $\{10^{n-1}\} = \Gamma_{n,1}^1$ , the set  $\mathcal{C}$  must contain precisely one vertex of  $\Gamma_{n,2}^1$ , call it  $b$ . Notice that this vertex dominates  $10^{n-1}$  as well as precisely one vertex in  $\Gamma_{n,1}^0$ . Observe furthermore that the number of undominated vertices in  $\Gamma_{n,2}^1$  is  $(n - 2) - 1$  which in turn imply that

$$|\mathcal{C} \cap \Gamma_{n,3}^1| = \frac{n - 3}{2}.$$

Finally, we compute the number of undominated vertices of  $\Gamma_{n,3}^1$ . Observe that  $b$  dominates  $n - 4 - t$  vertices in  $\Gamma_{n,3}^1$ , where  $t \in \{0, 1\}$  depending on whether  $b$  starts with 100 or not. Since  $a \in \mathcal{C}$ , every vertex of  $\mathcal{C} \cap \Gamma_{n,2}^0$  starts with 00 and hence has precisely one neighbor in  $\Gamma_{n,3}^1$ . The set  $\Gamma_{n,1}$  contains precisely one vertex of  $\mathcal{C}$  namely  $a$ . In addition, the vertex  $b$  dominates precisely two vertices of  $\Gamma_{n,1}$ . Therefore there are  $n - 3$  undominated vertices in  $\Gamma_{n,1}$  implying that  $|\mathcal{C} \cap \Gamma_{n,2}^0| = (n - 3)/2$ . It thus follows that  $(n - 3)/2$  vertices of  $\Gamma_{n,3}^1$  are dominated by  $\Gamma_{n,2}^0$  and that consequently

$$\begin{aligned} |\Gamma_{n,3}^1| - (n - 3) - (n - 4 - t) &= \binom{n - 3}{2} - \frac{n - 3}{2} - \frac{n - 3}{2} - (n - 4 - t) \\ &= (n^2 - 11n + 2t + 26)/2 \end{aligned} \tag{2}$$

vertices of  $\Gamma_{n,3}^1$  remain undominated by  $\mathcal{C}$ . These vertices must be dominated by the vertices in  $\Gamma_{n,4}^1$ , each vertex dominating precisely 3 vertices. Since  $n = 6k + 3$  for some integer  $k$  the expression (2) reduces to  $18k^2 - 15k + t + 1$  which is equivalent to  $t + 1 \pmod{3} \neq 0$  for  $t \in \{0, 1\}$ . Hence the expression (2) is not divisible by 3 and we have derived a final contradiction which proves Theorem 1.1.

### 3 Concluding remarks

Fibonacci cubes  $\Gamma_n$  generalize to *generalized Fibonacci cubes*  $\Gamma_n(f)$ , where  $f$  is a forbidden binary string [11], see also [17, 26]. More precisely, the graph  $\Gamma_n(f)$  is obtained from the  $n$ -cube  $Q_n$  by removing all strings that contain  $f$  as a substring. Note that in this notation,  $\Gamma_n = \Gamma_n(11)$ . It would be interesting to see whether some generalized Fibonacci cubes admit perfect codes.

Note that the proof of Theorem 1.1 showed that for any  $n \geq 6$ , no perfect code of  $\Gamma_n$  can dominate the vertices in  $\Gamma_{n,3}$ . Since vertices in  $\Gamma_{n,3}$  can only be dominated by vertices in  $\Gamma_{n,2} \cup \Gamma_{n,4}$  this shows that a very small local structure of  $\Gamma_n$  already forbids the existence of a perfect code. In the case of the non-existence of perfect codes in generalized Fibonacci cubes maybe such local obstructions can also be constructed.

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