Transitive, locally finite median graphs with finite blocks

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Abstract

The subject of this paper are infinite, locally finite, vertex-transitive median graphs. It is shown that the finiteness of the Θ-classes of such graphs does not guarantee finite blocks. Blocks become finite if, in addition, no finite sequence of Θ-contractions produces new cut-vertices. It is proved that there are only finitely many vertex-transitive median graphs of given finite degree with finite blocks. An infinite family of vertex-transitive median graphs with finite intransitive blocks is also constructed and the list of vertex-transitive median graphs of degree four is presented.

Key words: Median graphs; infinite graphs; vertex-transitive graphs

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1 Introduction

Let u and v be vertices of a connected graph G. The set $I(u,v)$ of all vertices in G that lie on shortest u,v-paths is called the *interval* between u and v. G is a *median graph* if for every triple of vertices u,v,w of G there exists a unique vertex in $I(u,v) \cap I(u,w) \cap I(v,w)$. This vertex is called the *median* of u,v,w.

The structure of median graphs is well understood, see the survey [8], recent papers [4, 11], and references therein. Most of the numerous studies of these graphs focus on finite graphs, nonetheless, [1, 2, 12] are important references for the infinite case.

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There is at least one aspect of median graphs that is uninteresting on finite graphs but becomes intriguing for infinite graphs—regularity. The variety of k-regular finite median graphs is very modest: the k-cube Q_k is the only such graph [10]. The situation significantly changes in the case of infinite graphs. For instance, any finite or infinite median graph G of largest degree d gives rise to a d-regular median graph as follows. Let u be an arbitrary vertex of G of degree smaller than d. Then, at u, attach to G an infinite rooted tree in which the root is of degree $d - d(u)$ and any other vertex is of degree d. In fact, Bandelt and Mulder [2] observed that there are $2^{\aleph_0}$ cubic median graphs. Hence one has to impose additional structure to get more insight into infinite median graphs.

Natural candidates for a condition that is stronger than regularity are vertex-transitivity and distance-transitivity. By another result of [2], the only distance-transitive median graphs are hypercubes and regular trees. Thus we focus on vertex-transitivity.

Suppose that G is an infinite median graph. Following Tardif [12] let us call G *compact* if G does not contain any isometric ray, that is, a one-sided infinite path that is isometric. Tardif proved that a compact median graph contains a finite subcube that is invariant by any automorphism of G. Therefore a compact median graph is vertex-transitive if and only it is a cube. Since finite graphs are compact, we can formulate this remark as a lemma.

Lemma 1 *A vertex-transitive median graph is compact if and only if it is a hypercube.*

Consequently, median graphs of interest to us must be infinite, but not compact. This is assured, for example, if they are locally finite. We thus focus on *infinite, locally finite, vertex-transitive median graphs* in this paper.

The simplest such graphs are homogeneous trees, integer lattices, and Cartesian products of such graphs by hypercubes. Homogeneous trees are special cases of graphs with finite blocks. In our case all blocks have to be median graphs. We will first show that even if all the Θ-classes of a vertex-transitive median graphs are finite, the graph need not have finite blocks. On the other hand, if we assume in addition that no finite sequence of Θ-contractions produces new cut-vertices, then block are finite. As our main theorem we then prove that there only finitely many vertex-transitive median graphs of given finite degree with finite blocks. We also construct an infinite family of vertex-transitive median graphs with finite intransitive blocks. The paper concludes with the list of vertex-transitive median graphs of degree four. There are 13 such graphs.

2 Preliminaries

All graphs in this paper are finite or infinite simple graphs, that is, graphs without loops or multiple edges. The two way infinite path will be denoted P_∞ and the k-regular with T_k.

$G \,\Box\, H$ denotes the Cartesian product of the graphs G and H; Q_n is the n-th power of K_2 with respect to the Cartesian product. By a *ladder* of a graph G we mean an induced subgraph of G isomorphic to $P_n \,\Box\, K_2$ for some $n \geq 1$.

Median graphs have already been defined in the introduction. It follows easily from the definition that median graphs are bipartite and that $K_{2,3}$ is a forbidden subgraph for median graphs. We also note that the Cartesian product of median graphs is a median graph again, see e.g. [7].

We say two edges $e = xy$ and $f = uv$ of G are in the relation Θ, in symbols $e\Theta f$, if

$$d(x,u) + d(y,v) \;\neq\; d(x,v) + d(y,u)\,.$$

In median graphs Θ is an equivalence relation. If $ab \in E(G)$ we set

$$F_{ab} = \{uv \mid ab\,\Theta\, uv\}.$$

In other words, F_{ab} is the Θ-class of ab. We further define the sets

$$\begin{aligned}
W_{ab} &= \{w \mid w \in G,\, d(w,a) < d(w,b)\},\\
W_{ba} &= \{w \mid w \in G,\, d(w,b) < d(w,a)\},\\
U_{ab} &= \{u \mid u \in W_{ab},\, u \text{ is adjacent to a vertex in } W_{ba}\},\\
U_{ba} &= \{u \mid u \in W_{ba},\, u \text{ is adjacent to a vertex in } W_{ab}\}.
\end{aligned}$$

As such these sets are sets of vertices, but by abuse of language we shall use the same notation for the subgraphs they induce.

We further recall that a subgraph H of a graph G is *convex*, if all shortest uv-paths in G between any two vertices in $u, v \in H$ are also in H.

Now everything is ready for a characterization of median graphs by Mulder [9]. The formulation given here is that of [7]. It is a variant of Mulder's results that is also true for infinite graphs.

Theorem 2 *Let ab be an edge of a connected, bipartite graph G. Then G is a median graph if and only if the following three conditions are satisfied:*

(i) *F_{ab} is a matching defining an isomorphism between U_{ab} and U_{ba}.*
(ii) *U_{ab} is convex in W_{ab} and U_{ba} in W_{ba}.*
(iii) *W_{ab} and W_{ba} are median graphs.*

We wish to note that the removal of the edges in F_{ab} separates G into the two connected components W_{ab} and W_{ba}, that no two edges on a shortest path are in the relation Θ, and that any two edges e, f of a median graph that are in the relation Θ are connected by a ladder.

3 Regular median graphs with finite blocks

It is a folklore fact that G is a median graph if and only if every block of G is median. Hence it is natural to wonder how (median) blocks can be combined to obtain infinite, locally finite vertex-transitive median graphs. Consider first the median graph G of Figure 1.

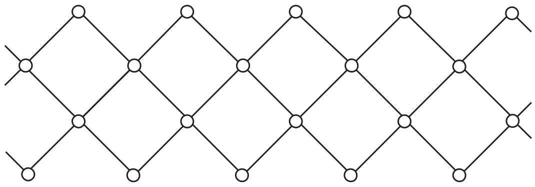

Figure 1: An infinite 2-connected median graph with finite Θ-classes

G has finite Θ-classes but contains an infinite block. (In fact, G is 2-connected.) Clearly G is not vertex-transitive. However, it has only two orbits, namely the set of vertices of degree two and those of degree four. We form a new graph G_1 by identifying each vertex of degree two in G with a vertex of degree four of a copy of G and every vertex of degree four with a vertex of degree two of a copy of G. The construction is illustrated in Figure 2 where an identification of the first type and another of the second type are performed in black vertices.

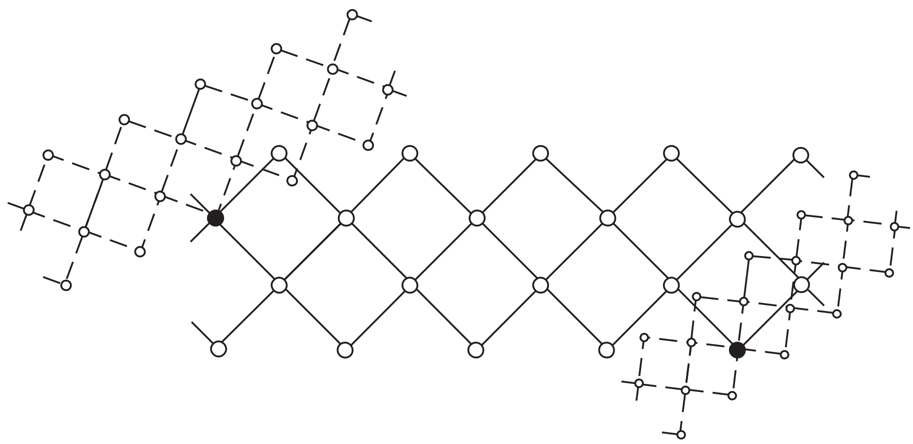

Figure 2: How to construct G_1

In addition to the vertices of degree six the new graph G_1 will again have vertices of degrees two and four. By the same construction as before we identify these vertices with vertices of copies of G to obtain a graph G_2. Continuing ad infinitum the limit graph G_∞ is clearly vertex-transitive (of degree 6). All of its blocks are isomorphic to G and every cut-vertex identifies vertices of two blocks, one vertex of degree four with one of degree two. Since the Θ-classes of a graph are those of its blocks it is clear that G_∞ has only finite Θ-classes.

This construction shows that vertex-transitivity alone does not ensure finite blocks, even if all Θ-classes are finite. A different condition though assures finiteness of blocks:

Proposition 3 *Let G be a median graph with finite Θ-classes. If no finite sequence of Θ-contractions produces new cut-vertices, then every block of G is finite.*

Proof. Suppose that B is an infinite block of G and let $e = ab$ be an edge of B. Then e is not a bridge, for otherwise e would contract to a cut-vertex. Hence there exists an edge $e_0 \in U_{ab}$. Let $e_1 \in U_{ba}$ be the edge opposite to e_0 and let F be the Θ-class containing e_0, e_1.

Let G_1 be the graph obtained from G by contracting F_{ab} and denote with f_1 the edge of G_1 obtained from e_0 and e_1. Let B_1 be the subgraph of G_1 induced by the contracted block B. By the theorem assumption, B_1 is a block of G_1. Since B_1 is median and 2-connected, f_1 is contained in a 4-cycle of G_1, say C. Let e_2 be the edge opposite to f_1 in C. Now contract the Θ-class containing the other two edges of C. Let f_2 be the contracted edge corresponding to f_1 and e_2. Continuing this procedure we obtain a sequence of edges $e_0, e_1, e_2, e_3, \ldots$ of G that are all in the same Θ-class, in contradiction to its finiteness. $\square$

We have observed above that vertex-transitivity does not ensure finite blocks. On the other hand, we are going to prove that there are only finitely many vertex-transitive median graphs of a given degree that have finite blocks.

We begin with the observation that k-regular median graphs have arbitrarily many orbits if they are sufficiently large. This result is a consequence of the tree-like structure of median graphs, and in particular of the fact that every finite median graph G has a subcube that is invariant under all automorphisms; see [7, Theorem 2.42].

Proposition 4 *Every finite median graph of largest degree k on at least $(2k)^k$ vertices has at least k orbits.*

Proof. Let G be a finite k-regular median graph and Q_c a subcube that is invariant under all automorphisms of G. Clearly $0 \le c \le k$. Therefore, no vertex of Q_c can be mapped by an automorphism into any vertex in $V(G) \setminus V(Q_c)$. Thus, $V(Q_c)$ consists of one or more orbits under the action of the automorphism group of G on $V(G)$. This implies that the set L_1 of vertices at distance 1 from Q_c consists of one or more orbits too. By induction this holds for the set L_r of vertices of any distance r from Q_c.

If G has fewer than k orbits, then we infer from the above that G has at most

$$|Q_c| + |L_1| + \cdots + |L_{k-2}|$$

vertices. Since $|Q_c| \leq 2^k$ and $|L_r| \leq |Q_c| \cdot (k-1)^r$ we obtain the estimate

$$|G| \;\leq\; 2^k\,[1 + (k-1) + (k-1)^2 + \cdots + (k-1)^{k-2}] \;<\; 2^k \cdot k^{k-1} \;<\; (2k)^k\,.$$

□

Note that this result is not true for general isometric subgraphs of hypercubes, as the cycle C_{2r} shows. It has degree 2, can be arbitrarily large, but has only one orbit. (For the definition of partial cubes, their relation to median graphs, and other properties see [3, 5, 6, 7] and references therein.)

Theorem 5 *For given k there are only finitely many k-regular vertex-transitive median graphs with finite blocks.*

Proof. We distinguish finite median graphs, infinite median graphs with vertex-transitive blocks and infinite median graphs with at least one intransitive block.

(a) As already mentioned, every regular finite median graph is a hypercube, thus Q_k is the only finite k-regular vertex-transitive median graph.

(b) Let G be a k-regular infinite vertex-transitive median graph with finite, vertex-transitive blocks. Then every block is a hypercube. By transitivity every vertex is a cut-vertex. Let v be an arbitrary vertex of G and $n_i(v)$ the number of blocks incident with v that are isomorphic to Q_i. Clearly $\sum_{i<k} i\, n_i = k$ and there are only finitely many possibilities for the vector $\mathbf{v} = (n_1(v), n_2(v), \ldots, n_{k-1}(v))$.

By transitivity $n_i(v) = n_i(w)$ for any $w \in G$. Hence $\mathbf{v}$ determines G and there are only finitely many graphs G for given k.

(c) Let G be a k-regular infinite vertex-transitive median graph with finite blocks and B a block with intransitive automorphism group. Since B is finite it has only finitely many orbits, say $O_i, 1 \leq i \leq k_B$. Let d_i denote the degree of the vertices in O_i. For every copy B' of B we denote the corresponding orbits by O'_i.

By transitivity every vertex v of B must be incident, for given $i, 1 \leq i \leq k_B$, with a vertex in the orbit O^i_i of a block B^i isomorphic to B; compare Figure 2. Also, set $d_B = \sum_{1 \leq i \leq k_B} d_i$. Clearly, $d_B \leq k$. If v is incident with a block C different from the $B^i, 1 \leq i \leq k_B$, then a similar construction yields blocks C_i isomorphic to C that are incident to v and the degree sum d_C with $d_B + d_C \leq k$.

Clearly v can be incident to no more than k blocks and each block can have no more than k orbits. To complete the proof we recall that every connected median graph of more than $(2k)^k$ vertices has more than k orbits by Proposition 4, because there are only finitely many graphs (and thus median graphs) on any given number of vertices. □

In the last case of the above proof regular infinite vertex-transitive median graph with finite intransitive blocks were treated. For further illustration construct an infinite family M_m, $m \geq 2$, of such graphs as follows: Let L_m be the ladder $P_m \Box K_2$, then every block B of M_m will be isomorphic to L_m. Note that L_m has $k = \lceil m/2 \rceil$ orbits. At every vertex of M_m we identify k blocks B such that every orbit is represented. Then M_m is a vertex-transitive median graph of degree $2 + 3(k-1)$. For instance, M_2 is shown in Figure 3.

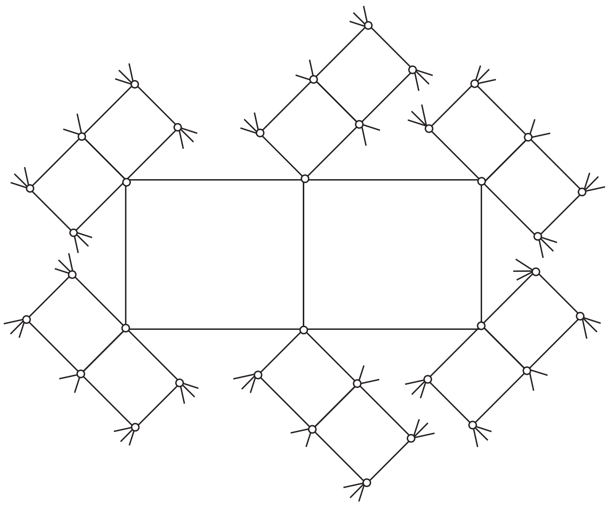

Figure 3: 5-regular vertex-transitive median graph with intransitive blocks

4 4-regular vertex-transitive median graphs

In this section we provide a complete list of 4-regular vertex-transitive median graphs. We first consider the case when the blocks are finite.

Proposition 6 *The 4-regular vertex-transitive median graphs with finite blocks are Q_4, T_4, and the graphs G_1, G_2, G_3 of Figure 4.*

Proof. Let G be a 4-regular vertex-transitive median graph with finite blocks. In G is finite then it is Q_4. Let G be infinite and u an arbitrary vertex of G. Then u cannot lie in a Q_4, for otherwise its degree would be greater than 4.

If u lies in no 4-cycle, then G is the 4-regular infinite tree T_4.

Suppose next that G contains 4-cycles but no Q_3. If u lies in exactly one 4-cycle, then G is the graph G_1 from Figure 4. Suppose u lies in at least two 4-cycles. If u

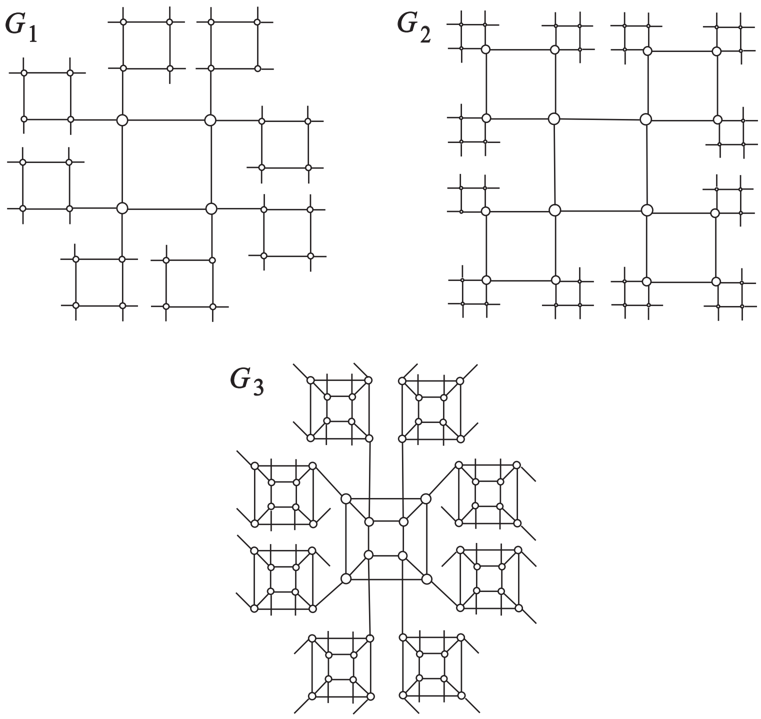

Figure 4: 4-regular vertex-transitive median graphs with finite blocks

is the intersection of two 4-cycles, G must be the graph G_2 from the same figure. Otherwise, two 4-cycles must intersect in an edge uv. Let $uvv'u'$ and uvv_1u_1 be the 4-cycles containing u. Note that if $u_1v' \in E(G)$ or $u'v_1 \in E(G)$ then we get a $K_{2,3}$. This means that the subgraph consisting of these two 4-cycles is induced. Therefore, considering the edge u_1v_1 we find two new vertices u_2, v_2 such that $u_1v_1v_2u_2$ is a 4-cycle. The edge u_2u' would lead to a Q_3. Continuing by induction we see that u_iu' cannot be an edge, for otherwise G would contain an odd cycle or an isometric even cycle. The convex closure of the latter cycle must be a hypercube (cf. [8]) which is not possible. Thus we obtain an infinite ladder, which is 2-connected.

Assume finally that G contains 3-cubes and let $Q = Q_3$ be a cube containing u. Let v be the neighbor of u not in Q. Then the edge uv is in no 4-cycle containing an edge of Q, for otherwise we end up with an infinite block. So G must be the graph G_3 from Figure 4. $\square$

Now the list of all 4-regular vertex-transitive median graphs.

Theorem 7 *Let G be a vertex-transitive, 4-regular median graph. Then G is one of the following graphs: the Q_4, the 4-regular tree, $P_\infty \Box P_\infty$, or one of the graphs $G_1, \ldots, G_{10}$ from Figures 4 and 5.*

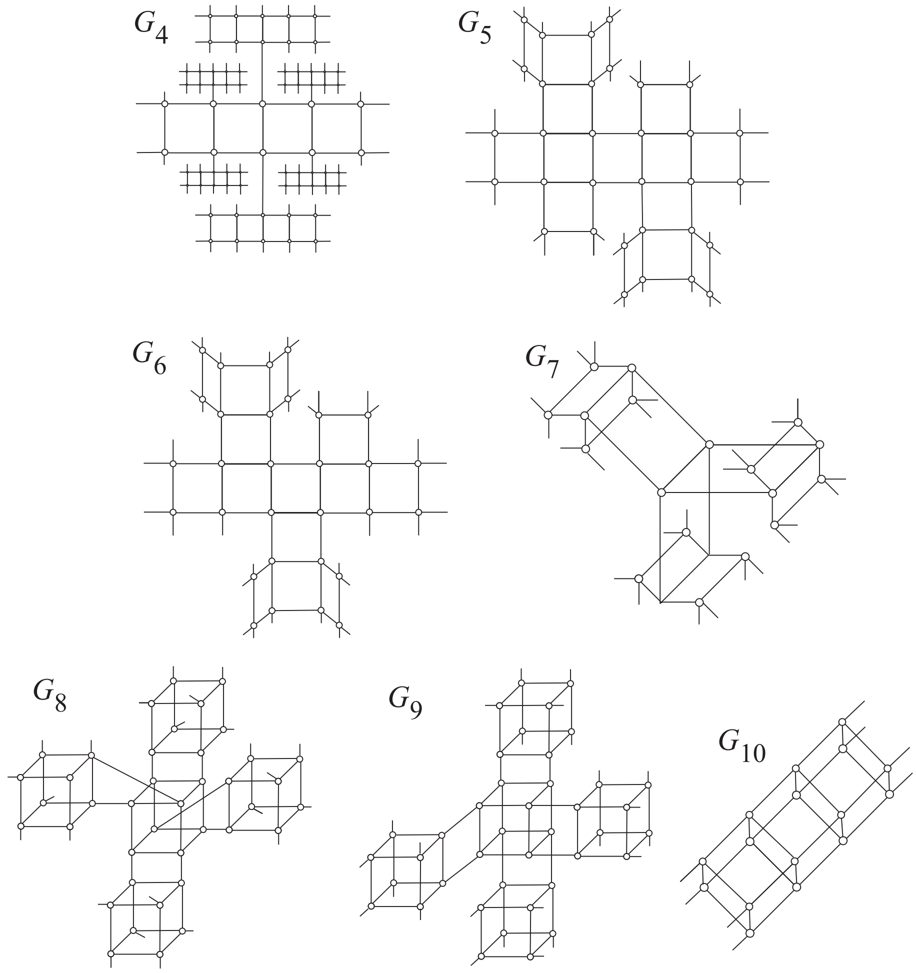

Figure 5: Additional 4-regular vertex-transitive median graphs

Proof. From Proposition 6 we already know all vertex-transitive, 4-regular median graph with finite blocks. So suppose that u is an arbitrary vertex of G and that G contains infinite blocks.

Suppose that G contains 4-cycles but no Q_3. Then from the proof of Proposition 6 we know that u lies in an infinite ladder L. Let v be the fourth neighbor of u, that is, the neighbor of u not in L. If u lies in exactly two 4-cycles we get the graph G_4 from Figure 5. Suppose u lies in exactly three 4-cycles and that no three 4-cycles share a common edge, see Figure 6. Then there are two possibilities how the situation can be extended, see Figure 6 again. This gives us the graphs G_4 and G_5 from Figure 5.

It the three 4-cycles that contain u share an edge, the graph G_7 appears. Note that $G_7 = T_3 \,\Box\, K_2$.

The last subcase is when u lies in four 4-cycles. Then the two possibilities as shown in Figure 7 lead to a $K_{2,3}$ and the 3-cube minus a vertex, respectively. The latter case is not possible because the convex closure of it would be a Q_3. Hence in this subcase we obtain $P_\infty \Box P_\infty$.

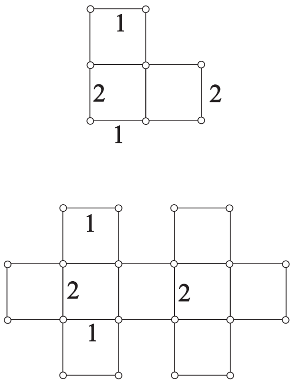

Figure 6: The situation when a vertex is in three 4-cycles

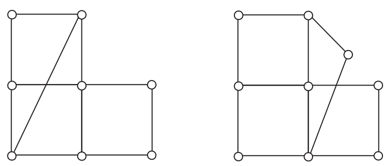

Figure 7: Two forbidden configurations

Assume now that u lies in a $Q = Q_3$. Let v be the neighbor of u that is not in Q. The case when uv is a bridge was treated in Proposition 6. Hence assume there is a path between v and Q different from the edge vu. A shortest such path must be of length 2, for otherwise the degree of G would be larger than 4. (Having in mind that the convex closure of an isometric cycle in a median graph is a hypercube.) Hence we have a 4-cycle C attached to the 3-cube Q. By the transitivity, every vertex of Q must have such an attachment. Suppose that C is not in a 3-cube. Then there are two ways how this can be done, either the attachments are done on the edges of the same Θ-class of Q or in two edges from one class and in two edges from another. This gives the graphs G_8 and G_9. Finally, if C lies in a 3-cube Q', then $|Q' \cap Q| = 4$, for otherwise the degree of G would be more than four. But then G_{10} is the graph. (Note that G_{10} is the Cartesian product of C_4 by P_∞.) $\Box$

5 Concluding remarks

The (unique) extendibility in a tree-like fashion used in this paper is a consequence of the fact that a median graph has no convex cycles of length more than four. In fact, as we already mentioned, the convex closure of such a cycle (having in mind that convex subgraphs are isometric) is a hypercube.

We have no example of a strip that is a median graph and not of the form $Q_n \Box P_\infty$. However, we cannot show that no other such strips exist, except for strips that are Cayley graphs or strips that have an infinite Θ-class. This will be treated in a forthcoming paper.

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