Further results on local metric dimension of graphs

Kairui Nie[a,b], Xiuhao Fan[a], Sandi Klavžar[c,d,e,*]

[a]School of Mathematical Sciences,
Henan Institute of Science and Technology, Xinxiang 453003, P.R. China
[b]School of Mathematics and Statistics,
Henan Normal University, Xinxiang 453007, P.R. China
[c] Faculty of Mathematics and Physics, University of Ljubljana, Slovenia
[d] Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia
[e] Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia

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Abstract

In this paper, for every integer $k \geq 1$, we construct a bridgeless graph containing an edge whose deletion decreases the local metric dimension by $k-1$, and a connected graph containing a non-cut vertex of degree 2 whose deletion also decreases the local metric dimension by $k-1$. These two constructions disprove two conjectures of Okamoto, Phinezy, and Zhang from 2010. As a complementary result, we show that every connected graph whose local metric dimension equals its order minus its independence number is a split graph. We then give a direct characterization, in terms of laminar families, of the connected split graphs whose local metric dimension equals the size of the clique part.

Keywords: local metric dimension; edge deletion; vertex deletion; split graph; extremal graphs

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1 Introduction

The metric dimension [11, 21] serves as a core and fundamental concept within (metric) graph theory. Since its introduction, scholars have explored it extensively from structural, extremal, and algorithmic perspectives; see a selection of classical works [6, 7, 12, 13, 14], and a selection of recent papers [2, 5, 9, 19, 22, 24]. The state of the field through 2023 is summarized in an excellent overview article [23], while the review [16] considers metric dimension related parameters.

*Corresponding author
Email addresses: niekairui@163.com (K. Nie), fanxh2020@163.com (X. Fan), sandi.klavzar@fmf.uni-lj.si (S. Klavžar).
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Okamoto, Phinezy, and Zhang [17] followed with the local version of the dimension like this. The vertex $w \in V(G)$ *distinguishes* vertices x and y whenever $d_G(x,w) \neq d_G(y,w)$, where $d_G(\cdot,\cdot)$ denotes the shortest-path distance in G. A subset $W \subseteq V(G)$ forms a *local resolving set* (*LR-set* for short) for G when every pair of adjacent vertices in G can be distinguished by some vertex belonging to W. A local resolving set with minimum cardinality is referred to as a *local metric basis*, and its cardinality is the *local metric dimension* (*LMD* for short) of G, denoted by $\dim_\ell(G)$.

Unless otherwise specified, all graphs $G = (V(G), E(G))$ discussed in this work are connected. We will continue to use the standard symbols: $n(G)$ for the order of G; $N_G(v)$ and $N_G[v]$ for the open, resp. closed, neighborhood of v; and $\deg_G(v) = |N_G(v)|$ for the degree of v. Two vertices $u, v \in V(G)$ form a pair of *adjacent twins* provided $N_G[u] = N_G[v]$, while they are called *nonadjacent twins* whenever $N_G(u) = N_G(v)$. Note that in either case, the equality $d_G(u,z) = d_G(v,z)$ holds for every $z \in V(G) \setminus \{u, v\}$. In particular, every *LR-set* contains at least one vertex from each pair of adjacent twins. In the above notations, when no confusion can arise, we may omit the subscript G. As is also well established, $\alpha(G)$ and $\omega(G)$ are the independence number and the clique number of G, respectively. A graph G is *split* if its vertex set admits a *split partition* $V(G) = V_1 \cup V_2$, where V_1 induces a clique and V_2 an edgeless graph. We denote by $\mathrm{Sp}(m,n)$ a generic connected split graph with m vertices in its clique part and n vertices in its independent part.

A family $\mathcal{F}$ consisting of subsets of X is referred to as a *laminar family* whenever for arbitrary $A, B \in \mathcal{F}$, one of the three relations holds: $A \subseteq B$, $B \subseteq A$, or $A \cap B = \varnothing$.

Several fundamental bounds and extremal results for the LMD were established in [17]. Specifically, if $n(G) \geq 2$, then $1 \leq \dim_\ell(G) \leq n(G) - \alpha(G)$ and $\dim_\ell(G) \geq \lceil \log_2 \omega(G) \rceil$. Additionally, $\dim_\ell(G) = 1$ if and only if G is bipartite, whereas $\dim_\ell(G) = n(G) - 1$ if and only if G is complete. The bound $\dim_\ell(G) \leq n(G) - \alpha(G)$ is sharp for every admissible value of the independence number. Further investigations of the LMD have considered graph products, graphs with cut vertices, and several special graph classes; see [1, 4, 10, 15, 16, 20].

One of the problems raised in the study of the LMD is the effect of vertex or edge deletion. In [18] it was proved that if an edge e is a non-bridge in G, then $\dim_\ell(G - e) \leq \dim_\ell(G) + 2$. In addition, if z is a not a cut vertex of G, then $\dim_\ell(G - z) \leq \dim_\ell(G) + \deg_G(z)$. These two findings led the authors to the following conjectures.

Conjecture 1.1. [18, Conjecture 4.8] If e is a non-bridge edge of G, then

$$\dim_\ell(G) - \dim_\ell(G - e) \leq 2\,.$$

Conjecture 1.2. [18, Conjecture 4.7] If v is a non-cut vertex of G, then

$$\dim_\ell(G) - \dim_\ell(G - v) \leq \deg_G(v)\,.$$

In this article, we focus on graphs G satisfying $\dim_\ell(G) = n(G) - \alpha(G)$, for which the following problem was posed:

Problem 1.3. [16, Page 10] Characterize the graphs G for which $\dim_\ell(G) = n(G) - \alpha(G)$.

An earlier characterization of this equality case was obtained by Barragán-Ramírez [3] in terms of clustered nested covers. Here we give a direct structural formulation in terms of split graphs and laminar families. In this direction we add that Fitriani et al. [8] proved the sharp bounds $\lceil \log_2 m \rceil \leq \dim_\ell(\mathrm{Sp}(m,n)) \leq m$.

The layout of this paper is outlined below. In Section 2 we disprove Conjecture 1.1, while in Section 3 we disprove Conjecture 1.2. After that, we study graphs satisfying $\dim_\ell(G) = n(G) - \alpha(G)$ and characterize the split graphs $\mathrm{Sp}(m,n)$ for which $\dim_\ell(\mathrm{Sp}(m,n)) = m$.

2 LMD under edge deletion

In this section we disprove Conjecture 1.1. Consider the following family of bridgeless graphs. Suppose $k \geq 1$ and $m \geq 2$ are integers. For each index $i \in [k] = \{1, \ldots, k\}$, let D_i be the graph with

$$V(D_i) = \{p_{i,1}, p_{i,2}, \ldots, p_{i,7}\},$$
$$E(D_i) = \{p_{i,1}p_{i,2}, p_{i,1}p_{i,3}, p_{i,2}p_{i,3}, p_{i,2}p_{i,4}, p_{i,3}p_{i,5}, p_{i,4}p_{i,5}, p_{i,4}p_{i,6}, p_{i,6}p_{i,7}\},$$

and let Q_i be a complete graph with $V(Q_i) = \{q_{i,1}, q_{i,2}, \ldots, q_{i,m}\}$. Then the graph $G_{k,m}$ is defined by

$$V(G_{k,m}) = \{x, y\} \cup \bigcup_{i=1}^{k} (V(D_i) \cup V(Q_i)),$$

$$E(G_{k,m}) = \bigcup_{i=1}^{k} \big(E(D_i) \cup E(Q_i)\big) \cup \{xp_{i,1} : i \in [k]\} \cup \{p_{i,7}q_{i,j},\, yq_{i,j} : i \in [k],\, j \in [m]\} \cup \{xy\}.$$

Set also $e = xy$, so that the graph $G_{k,m} - e$ is constructed by removing the edge $e = xy$ from $G_{k,m}$. The graphs $G_{k,3}$ and $G_{k,3} - e$ are illustrated in Figure 1.

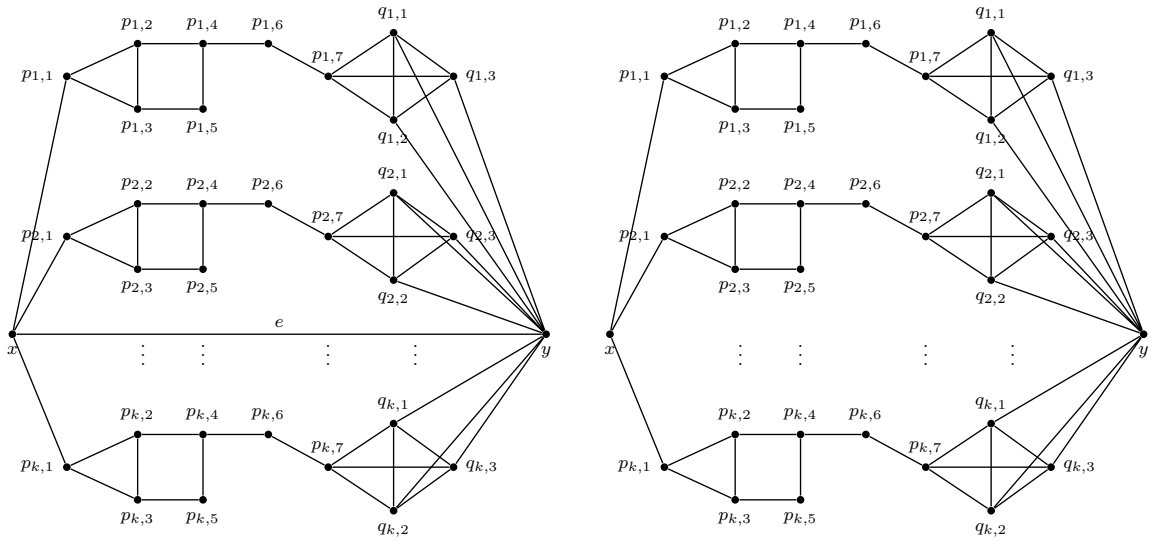


Figure 1: The graphs (i) $G_{k,3}$, and (ii) $G_{k,3} - e$.

It follows by the construction that every edge of $G_{k,m}$ lies on a cycle, hence $G_{k,m}$ is indeed bridgeless. We first determine LMD of $G_{k,m}$.

Lemma 2.1. *If $k \geq 1$ and $m \geq 2$, then $\dim_\ell(G_{k,m}) = mk$.*

Proof. Since all vertices of Q_i are adjacent twins, every *LR-set* S of $G_{k,m}$ must include no fewer than $m-1$ vertices from Q_i. Hence $|S \cap V(Q_i)| \geq m-1$ for every $i \in [k]$.

We first show that $S \cap V(D_i) \neq \varnothing$ for every $i \in [k]$. Indeed, $p_{i,2}$ and $p_{i,3}$ are adjacent, and every vertex in $V(G_{k,m}) \setminus V(D_i)$ has the same distance to $p_{i,2}$ and $p_{i,3}$. In particular,

$$d_{G_{k,m}}(p_{i,2}, x) = d_{G_{k,m}}(p_{i,3}, x) = 2,$$
$$d_{G_{k,m}}(p_{i,2}, y) = d_{G_{k,m}}(p_{i,3}, y) = 3,$$
$$d_{G_{k,m}}(p_{i,2}, q) = d_{G_{k,m}}(p_{i,3}, q) = 4$$

for every $q \in V(Q_i)$. For every $j \neq i$, vertices in $V(D_j) \cup V(Q_j)$ also have equal distances to $p_{i,2}$ and $p_{i,3}$, since these two vertices share equal distances to both x and y. Therefore, no vertex in $V(G_{k,m}) \setminus V(D_i)$ distinguishes the adjacent vertices $p_{i,2}$ and $p_{i,3}$. Consequently, every *LR-set* S must contain a vertex of $V(D_i)$, and hence $S \cap V(D_i) \neq \varnothing$ for every $i \in [k]$. Consequently,

$$\dim_\ell(G_{k,m}) \geq k(m-1) + k = mk.$$

To establish the reverse inequality, set $S = \bigcup_{i=1}^{k} \left(\{p_{i,2}\} \cup \left(V(Q_i) \setminus \{q_{i,m}\}\right)\right)$. Clearly, $|S| = mk$. We demonstrate that S is an *LR-set* of $G_{k,m}$. Any pair of adjacent vertices containing at least one element of S is immediately distinguished by the vertex in S. It suffices to examine adjacent vertex pairs for which neither vertex belongs to S. For every $i \in [k]$, the vertex $q_{i,1} \in V(Q_i) \cap S$ distinguishes the adjacent vertices $p_{i,1}$ and $p_{i,3}$, since

$$d_{G_{k,m}}(p_{i,1}, q_{i,1}) = 3 \neq 4 = d_{G_{k,m}}(p_{i,3}, q_{i,1}).$$

The vertex $p_{i,2} \in S$ distinguishes all the remaining adjacent pairs involving vertices of the ith copy. In particular,

$$d_{G_{k,m}}(p_{i,2}, p_{i,7}) = 3 \neq 4 = d_{G_{k,m}}(p_{i,2}, q_{i,m}),$$

so $p_{i,2}$ distinguishes $p_{i,7}$ and $q_{i,m}$. Moreover, because

$$d_{G_{k,m}}(p_{i,2}, y) = 3 \neq 4 = d_{G_{k,m}}(p_{i,2}, q_{i,m}),$$

$p_{i,2}$ distinguishes also y and $q_{i,m}$. Finally, $q_{i,1}$ distinguishes the adjacent vertices x and $p_{i,1}$, as well as x and y. Accordingly, every pair of adjacent vertices in $G_{k,m}$ can be distinguished by at least one vertex from S. This confirms that S is an *LR-set* of $G_{k,m}$. We therefore obtain $\dim_\ell(G_{k,m}) = mk$. □

We next determine LMD after deleting the edge $e = xy$.

Lemma 2.2. *If $k \geq 1$ and $m \geq 2$, then $\dim_\ell(G_{k,m} - e) = (m-1)k + 1$.*

Proof. The vertices of Q_i are pairwise adjacent twins in $G_{k,m} - e$. Accordingly, any *LR-set* of $G_{k,m} - e$ includes no fewer than $m-1$ vertices of $V(Q_i)$ for each $i \in [k]$. Therefore, $\dim_\ell(G_{k,m} - e) \geq (m-1)k$. Assume for contradiction that $\dim_\ell(G_{k,m} - e) = (m-1)k$. Then a minimum *LR-set* consists of exactly $m-1$ vertices from each $V(Q_i)$ and contains no other vertices. For every $q \in V(Q_i)$,

$$d_{G_{k,m}-e}(p_{i,1}, q) = d_{G_{k,m}-e}(p_{i,3}, q) = 5,$$

whereas for every $j \neq i$ and every $q \in V(Q_j)$,

$$d_{G_{k,m}-e}(p_{i,1}, q) = d_{G_{k,m}-e}(p_{i,3}, q) = 7.$$

Thus no vertex in such an *LR-set* distinguishes the adjacent vertices $p_{i,1}$ and $p_{i,3}$, a contradiction. Consequently,

$$\dim_\ell(G_{k,m} - e) \geq (m-1)k + 1.$$

For the reverse inequality, let $S' = \{x\} \cup \bigcup_{i=1}^{k} \left(V(Q_i) \setminus \{q_{i,m}\}\right)$. Then $|S'| = (m-1)k + 1$. We demonstrate that S' is a *LR-set* of $G_{k,m} - e$. For any pair of adjacent vertices with at least one member lying in S', the vertex contained in S' distinguishes them directly. Thus we only need to examine adjacent vertex pairs for which neither vertex lies in S'.

For each $i \in [k]$, the only adjacent pairs in $V(D_i)$ whose vertices have the same distance from x are $p_{i,2}, p_{i,3}$ and $p_{i,4}, p_{i,5}$. Such vertex pairs are distinguished by $q_{i,1} \in V(Q_i) \cap S'$, since

$$d_{G_{k,m}-e}(p_{i,2}, q_{i,1}) = 4 \neq 5 = d_{G_{k,m}-e}(p_{i,3}, q_{i,1}),$$

$$d_{G_{k,m}-e}(p_{i,4}, q_{i,1}) = 3 \neq 4 = d_{G_{k,m}-e}(p_{i,5}, q_{i,1}).$$

The remaining adjacent pairs with neither vertex in S' are $p_{i,7}, q_{i,m}$ and $y, q_{i,m}$. They are distinguished by x, since

$$d_{G_{k,m}-e}(x, p_{i,7}) = 5 \neq 6 = d_{G_{k,m}-e}(x, q_{i,m}),$$
$$d_{G_{k,m}-e}(x, q_{i,m}) = 6 \neq 7 = d_{G_{k,m}-e}(x, y).$$

Accordingly, every adjacent vertex pair of $G_{k,m} - e$ can be distinguished by at least one vertex belonging to S'. Hence S' is an *LR-set* of $G_{k,m} - e$, and thus $\dim_\ell(G_{k,m} - e) \leq (m-1)k+1$. Together with the lower bound, we obtain $\dim_\ell(G_{k,m} - e) = (m-1)k + 1$. □

Combining the preceding two lemmas leads us to the core theorem of this section which disproves Conjecture 1.1.

Theorem 2.3. *For every $k \geq 1$, there exists a connected bridgeless graph G containing an edge e such that*

$$\dim_\ell(G) - \dim_\ell(G - e) = k - 1.$$

In particular, the difference $\dim_\ell(G) - \dim_\ell(G - e)$ is unbounded for non-bridge edges.

Proof. Let $m \geq 2$ and consider the graph $G_{k,m}$ defined above. It is clear from the construction that every edge of $G_{k,m}$ lies on a cycle, and hence $G_{k,m}$ is bridgeless. By Lemmas 2.1 and 2.2, $\dim_\ell(G_{k,m}) - \dim_\ell(G_{k,m} - e) = mk - \big((m-1)k + 1\big) = k - 1$. □

3 LMD under vertex deletion

In this section we disprove Conjecture 1.2, for which we define a family of graphs similar to that introduced in the preceding section. Suppose $k \geq 1$ and $m \geq 2$ are integers. For each $i \in [k]$, let D_i be the graph with

$$V(D_i) = \{p_{i,1}, p_{i,2}, \ldots, p_{i,7}, r_i\},$$
$$E(D_i) = \{p_{i,1}p_{i,2}, p_{i,1}p_{i,3}, p_{i,2}p_{i,3}, p_{i,2}p_{i,4}, p_{i,3}p_{i,5}, p_{i,4}p_{i,5}, p_{i,4}p_{i,6}, p_{i,6}r_i, r_ip_{i,7}\}.$$

Let Q_i be a complete graph with $V(Q_i) = \{q_{i,1}, q_{i,2}, \ldots, q_{i,m}\}$. Let $D_1, \ldots, D_k, Q_1, \ldots, Q_k$ be pairwise vertex-disjoint, and let x, y, v be three additional vertices. Now, the graph $H_{k,m}$ is defined by

$$V(H_{k,m}) = \{x, y, v\} \cup \bigcup_{i=1}^{k} \big(V(D_i) \cup V(Q_i)\big),$$
$$E(H_{k,m}) = \bigcup_{i=1}^{k} \big(E(D_i) \cup E(Q_i)\big) \cup \{xp_{i,1} : i \in [k]\} \cup \{p_{i,7}q_{i,j},\ yq_{i,j} : i \in [k],\ j \in [m]\} \cup \{xv, vy\}.$$

The graphs $H_{k,3}$ and $H_{k,3} - v$ are illustrated in Figure 2.

By the construction, $H_{k,m}$ is connected and $\deg_{H_{k,m}}(v) = 2$. Moreover, v is not a cut vertex since, for every $i \in [k]$, $xp_{i,1}p_{i,2}p_{i,4}p_{i,6}r_ip_{i,7}q_{i,1}y$ is an x, y-path in $H_{k,m} - v$.

We first determine LMD of $H_{k,m}$.

Lemma 3.1. *If $k \geq 1$ and $m \geq 2$, then $\dim_\ell(H_{k,m}) = mk$.*

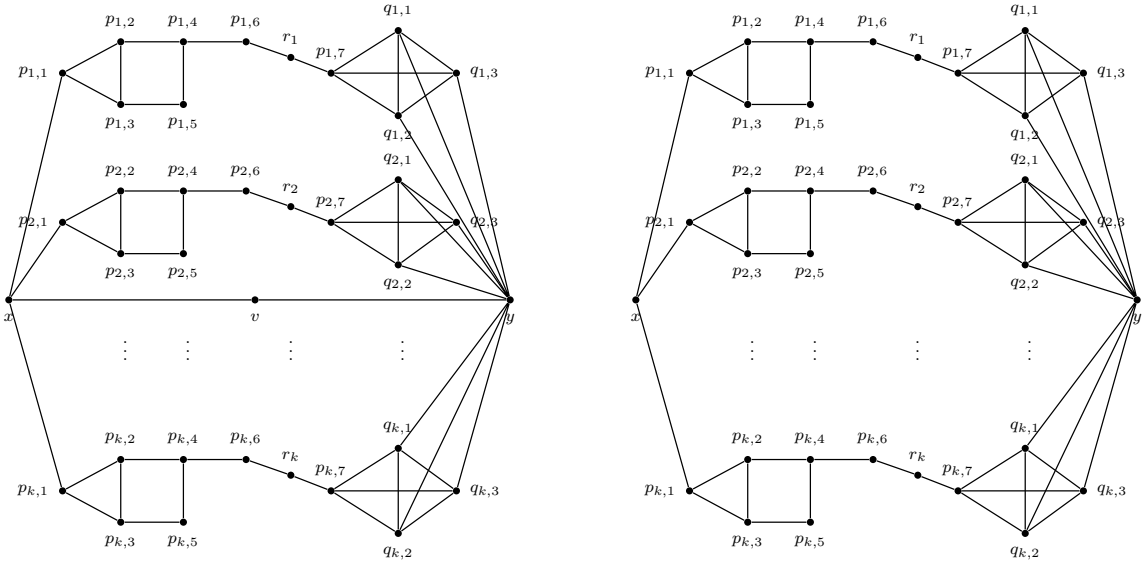


Figure 2: The graphs (i) $H_{k,3}$ and (ii) $H_{k,3} - v$.

Proof. Since the vertices of $V(Q_i)$ are adjacent twins, any *LR-set* S of $H_{k,m}$ must include no fewer than $m-1$ vertices from $V(Q_i)$. Hence $|S \cap V(Q_i)| \geq m-1$ for every $i \in [k]$.

We next show that $S \cap V(D_i) \neq \varnothing$ for every $i \in [k]$. The vertices $p_{i,2}$ and $p_{i,3}$ are adjacent, and every vertex in $V(H_{k,m}) \setminus V(D_i)$ has the same distance to $p_{i,2}$ and $p_{i,3}$. In particular,

$$\begin{aligned} d_{H_{k,m}}(p_{i,2}, x) &= d_{H_{k,m}}(p_{i,3}, x) = 2, \\ d_{H_{k,m}}(p_{i,2}, v) &= d_{H_{k,m}}(p_{i,3}, v) = 3, \\ d_{H_{k,m}}(p_{i,2}, y) &= d_{H_{k,m}}(p_{i,3}, y) = 4, \\ d_{H_{k,m}}(p_{i,2}, q) &= d_{H_{k,m}}(p_{i,3}, q) = 5 \end{aligned}$$

for every $q \in V(Q_i)$. For every $j \neq i$, vertices in $V(D_j) \cup V(Q_j)$ also have equal distances to $p_{i,2}$ and $p_{i,3}$, since $p_{i,2}$ and $p_{i,3}$ share equal distances to both x and y. Therefore no vertex in $V(H_{k,m}) \setminus V(D_i)$ distinguishes the adjacent vertices $p_{i,2}$ and $p_{i,3}$. Consequently, every *LR-set* S includes no fewer than one vertex of $V(D_i)$. It follows that

$$\dim_\ell(H_{k,m}) \geq k(m-1) + k = mk.$$

To verify the opposite inequality, let $S = \bigcup_{i=1}^{k} \left(\{p_{i,2}\} \cup \left(V(Q_i) \setminus \{q_{i,m}\}\right)\right)$. Then $|S| = mk$. We demonstrate that S is an *LR-set* of $H_{k,m}$. For any pair of adjacent vertices with at least one member lying in S, the vertex contained in S distinguishes them directly. Thus we only need to examine adjacent vertex pairs for which neither vertex lies in S.

For every $i \in [k]$, the vertex $q_{i,1} \in V(Q_i) \cap S$ distinguishes the adjacent vertices $p_{i,1}$ and $p_{i,3}$, since

$$d_{H_{k,m}}(p_{i,1}, q_{i,1}) = 4 \neq 5 = d_{H_{k,m}}(p_{i,3}, q_{i,1}).$$

The vertex $p_{i,2} \in S$ distinguishes all the remaining pairs of adjacent vertices in $V(D_i)$. Indeed,

$$\begin{aligned} d_{H_{k,m}}(p_{i,3}, p_{i,2}) &= 1 \neq 2 = d_{H_{k,m}}(p_{i,5}, p_{i,2}), \\ d_{H_{k,m}}(p_{i,4}, p_{i,2}) &= 1 \neq 2 = d_{H_{k,m}}(p_{i,5}, p_{i,2}), \\ d_{H_{k,m}}(p_{i,4}, p_{i,2}) &= 1 \neq 2 = d_{H_{k,m}}(p_{i,6}, p_{i,2}), \\ d_{H_{k,m}}(p_{i,6}, p_{i,2}) &= 2 \neq 3 = d_{H_{k,m}}(r_i, p_{i,2}), \\ d_{H_{k,m}}(r_i, p_{i,2}) &= 3 \neq 4 = d_{H_{k,m}}(p_{i,7}, p_{i,2}). \end{aligned}$$

Moreover,

$$\begin{aligned} d_{H_{k,m}}(p_{i,2}, p_{i,7}) &= 4 \neq 5 = d_{H_{k,m}}(p_{i,2}, q_{i,m}), \\ d_{H_{k,m}}(p_{i,2}, y) &= 4 \neq 5 = d_{H_{k,m}}(p_{i,2}, q_{i,m}), \end{aligned}$$

so $p_{i,2}$ distinguishes the adjacent vertices $p_{i,7}, q_{i,m}$ and $y, q_{i,m}$. Finally, $q_{i,1}$ distinguishes the remaining adjacent pairs $x, p_{i,1}$; x, v; and v, y. Indeed,

$$\begin{aligned} d_{H_{k,m}}(x, q_{i,1}) &= 3 \neq 4 = d_{H_{k,m}}(p_{i,1}, q_{i,1}), \\ d_{H_{k,m}}(x, q_{i,1}) &= 3 \neq 2 = d_{H_{k,m}}(v, q_{i,1}), \\ d_{H_{k,m}}(v, q_{i,1}) &= 2 \neq 1 = d_{H_{k,m}}(y, q_{i,1}). \end{aligned}$$

Accordingly, every adjacent vertex pair of $H_{k,m}$ can be distinguished by at least one vertex belonging to S. Hence S is an *LR-set* of $H_{k,m}$, and therefore $\dim_\ell(H_{k,m}) = mk$. □

We next determine LMD of $H_{k,m} - v$.

Lemma 3.2. *If $k \geq 1$ and $m \geq 2$, then $\dim_\ell(H_{k,m} - v) = (m-1)k + 1$.*

Proof. The vertices of $V(Q_i)$ remain adjacent twins in $H_{k,m} - v$. Accordingly, every *LR-set* of $H_{k,m} - v$ must include no fewer than $m-1$ vertices of $V(Q_i)$ for each $i \in [k]$, and hence $\dim_\ell(H_{k,m} - v) \geq (m-1)k$.

Assume for contradiction that $\dim_\ell(H_{k,m} - v) = (m-1)k$. Then a minimum *LR-set* consists of exactly $m-1$ vertices from each $V(Q_i)$ and contains no other vertices. For every $i \in [k]$ and every $q \in V(Q_i)$,

$$d_{H_{k,m}-v}(p_{i,1}, q) = d_{H_{k,m}-v}(p_{i,3}, q) = 6,$$

whereas for all $i, j \in [k]$ with $i \neq j$ and every $q \in V(Q_j)$,

$$d_{H_{k,m}-v}(p_{i,1}, q) = d_{H_{k,m}-v}(p_{i,3}, q) = 8.$$

Therefore no vertex in such an *LR-set* distinguishes the adjacent vertices $p_{i,1}$ and $p_{i,3}$. This is a contradiction. Consequently,

$$\dim_\ell(H_{k,m} - v) \geq (m-1)k + 1.$$

For the reverse inequality, let $S' = \{x\} \cup \bigcup_{i=1}^{k} (V(Q_i) \setminus \{q_{i,m}\})$. Then $|S'| = (m-1)k + 1$. We verify that S' forms an *LR-set* of $H_{k,m} - v$. For any pair of adjacent vertices with at least one member lying in S', the vertex in S' distinguishes them directly. Thus we only need to examine adjacent vertex pairs for which neither vertex belongs to S'.

For every $i \in [k]$, the only adjacent pairs in $V(D_i)$ whose vertices have the same distance from x are $p_{i,2}, p_{i,3}$ and $p_{i,4}, p_{i,5}$. They are distinguished by $q_{i,1} \in V(Q_i) \cap S'$, since

$$\begin{aligned} d_{H_{k,m}-v}(p_{i,2}, q_{i,1}) &= 5 \neq 6 = d_{H_{k,m}-v}(p_{i,3}, q_{i,1}), \\ d_{H_{k,m}-v}(p_{i,4}, q_{i,1}) &= 4 \neq 5 = d_{H_{k,m}-v}(p_{i,5}, q_{i,1}). \end{aligned}$$

The remaining adjacent pairs with neither vertex in S' are $p_{i,7}, q_{i,m}$ and $y, q_{i,m}$. They are distinguished by x, since

$$d_{H_{k,m}-v}(x, p_{i,7}) = 6 \neq 7 = d_{H_{k,m}-v}(x, q_{i,m}),$$
$$d_{H_{k,m}-v}(x, q_{i,m}) = 7 \neq 8 = d_{H_{k,m}-v}(x, y).$$

Accordingly, every adjacent vertex pair of $H_{k,m} - v$ can be distinguished by some vertex belonging to S'. Hence S' is an *LR-set* of $H_{k,m} - v$, and thus $\dim_\ell(H_{k,m} - v) = (m-1)k + 1$. □

Combining the preceding two lemmas, we arrive at the following central finding of this section, which disproves Conjecture 1.2.

Theorem 3.3. *Given an integer $k \geq 1$, there exists a connected graph H containing a non-cut vertex v of degree 2 such that*

$$\dim_\ell(H) - \dim_\ell(H - v) = k - 1.$$

In particular, for all $k \geq 4$, $\dim_\ell(H) - \dim_\ell(H - v) > \deg_H(v)$.

Proof. Let $m \geq 2$, and let $H = H_{k,m}$. By Lemmas 3.1 and 3.2, $\dim_\ell(H_{k,m}) - \dim_\ell(H_{k,m} - v) = mk - \big((m-1)k+1\big) = k-1$. In addition, since $\deg_{H_{k,m}}(v) = 2$, for every $k \geq 4$ we have $\dim_\ell(H_{k,m}) - \dim_\ell(H_{k,m} - v) = k - 1 > 2 = \deg_{H_{k,m}}(v)$. □

4 Graphs with $\dim_\ell(G) = n(G) - \alpha(G)$

Within this section, we provide a characterization for graphs that attain the equality $\dim_\ell(G) = n(G) - \alpha(G)$ using split graphs and laminar families. We first recall:

Lemma 4.1. [17, Observation 1.1] If G is a connected graph, then $\dim_\ell(G) \leq n(G) - \alpha(G)$.

We now determine a necessary structural condition for a graph satisfying equality in Lemma 4.1.

Lemma 4.2. *Let G be a connected graph with $n(G) \geq 2$. If $\dim_\ell(G) = n(G) - \alpha(G)$, then G is a split graph and $\omega(G) = n(G) - \alpha(G) + 1$.*

Proof. Suppose I is an independent set of G satisfying $|I| = \alpha(G)$. Set $S = V(G) \setminus I$. Clearly, if $uv \in E(G)$, then $S \cap \{u, v\} \neq \emptyset$. Hence S is an *LR-set* of G. As $|S| = n(G) - \alpha(G) = \dim_\ell(G)$, S is a local metric basis.

We first demonstrate that if $x, y \in S$ are nonadjacent vertices, then no vertex in I is adjacent simultaneously to x and y. Suppose for contradiction, there exists $u \in I$ adjacent to both x and y. Let $W = (S \setminus \{x, y\}) \cup \{u\}$. It follows that $|W| = |S| - 1 = n(G) - \alpha(G) - 1$. We verify that W is an *LR-set*.

Any pair of adjacent vertices containing a vertex of W is immediately distinguished by that vertex. Thus, it remains only to consider adjacent vertex pairs contained in $V(G) \setminus W = (I \setminus \{u\}) \cup \{x, y\}$. As I forms an independent set and $xy \notin E(G)$, every adjacent pair containing x or y is of the form xz or yz, where $z \in I \setminus \{u\}$. Since u is adjacent simultaneously to x and y, this yields

$$d_G(x, u) = 1 \neq d_G(z, u),$$

$$d_G(y, u) = 1 \neq d_G(z, u).$$

Thus u distinguishes all these adjacent pairs. Consequently, the set W is an *LR-set* of G, which contradicts the minimality of S. So, $N_G(x) \cap N_G(y) \cap I = \varnothing$, whenever $x, y \in S$ are nonadjacent.

We proceed to prove that each connected component of $G[S]$ is a complete graph. Let $x \in S$. As S constitutes a local metric basis, $S \setminus \{x\}$ fails to be an *LR-set*. Accordingly, there exists an adjacent vertex pair that is not distinguished by $S \setminus \{x\}$.

Neither vertex of this pair can belong to $S \setminus \{x\}$, since a vertex belonging to the set distinguishes itself from each of its neighbors. Therefore the pair must be x, u_x for some $u_x \in I$. That is, $d_G(x, z) = d_G(u_x, z)$ for every $z \in S \setminus \{x\}$. If $z \in N_{G[S]}(x)$, then $d_G(x, z) = 1 = d_G(u_x, z)$ and $u_x z \in E(G)$. Thus u_x is adjacent to every vertex of $N_{G[S]}(x)$.

Suppose that two vertices $y, z \in N_{G[S]}(x)$ are nonadjacent. Then $u_x \in I$ is a common neighbor of y and z, contradicting $N_G(y) \cap N_G(z) \cap I = \varnothing$. Thus, $N_{G[S]}(x)$ forms a clique for each $x \in S$. This implies that $G[S]$ contains no induced path of order three, and therefore every component of $G[S]$ is a complete graph.

We now prove that $G[S]$ has only one component. Suppose that it has at least two components. Note that no vertex of I can have neighbors in two distinct components of $G[S]$; otherwise, two nonadjacent vertices of S would have a common neighbor in I, which is impossible. Moreover, since I is independent, two distinct components of $G[S]$ cannot be connected through vertices of I. This contradicts the connectivity of G. Therefore $G[S]$ is connected, and hence S is a clique. Since I is independent, the decomposition $V(G) = S \cup I$ partitions $V(G)$ into a clique and an independent set. Thus G is a split graph.

Finally, let $x \in S$ and $u_x \in I$. Since S is a clique, $d_G(x, z) = 1$ for every $z \in S \setminus \{x\}$. By the choice of u_x, $d_G(x, z) = d_G(u_x, z)$, and hence u_x is adjacent to every vertex of $S \setminus \{x\}$. Since u_x is also adjacent to x, the set $S \cup \{u_x\}$ is a clique. Therefore, $\omega(G) \geq |S| + 1 = n(G) - \alpha(G) + 1$. Since I is independent, any clique contains at most one vertex of I. Thus, $\omega(G) \leq |S| + 1 = n(G) - \alpha(G) + 1$. Consequently, $\omega(G) = n(G) - \alpha(G) + 1$. □

By Lemma 4.2, it suffices to study the equality case of Lemma 4.1 within the class of split graphs.

Recall that a family $\mathcal{F}$ consisting of subsets of X is laminar when for any $A, B \in \mathcal{F}$, one of $A \subseteq B$, $B \subseteq A$, or $A \cap B = \varnothing$ holds. We further say that $\mathcal{F}$ *separates* two distinct elements $x, y \in X$ if there exists $F \in \mathcal{F}$ containing precisely one of these two elements. Now we have:

Lemma 4.3. *If $\mathcal{F}$ is a laminar family of subsets of X with t members that separates every pair of distinct elements of X, then these properties hold.*

(i) $|X| \leq t + 1$.

(ii) If every element of X belongs to some member of $\mathcal{F}$, then $|X| \leq t$.

Proof. Let $X' = \{x \in X : x \in F \text{ for some } F \in \mathcal{F}\}$. For every $x \in X'$, let F_x be a minimal member of $\mathcal{F}$ containing x. Since $\mathcal{F}$ is laminar, such a minimal member is unique.

We first show that distinct elements of X' have distinct minimal members. Suppose that $x, y \in X'$ are distinct and $F_x = F_y$. Since $\mathcal{F}$ separates x and y, there exists $F \in \mathcal{F}$ containing exactly one of them, say $x \in F$ and $y \notin F$. Because $x \in F \cap F_x$ and $\mathcal{F}$ is laminar, either $F \subseteq F_x$ or $F_x \subseteq F$. The latter is impossible, since $y \in F_x$ but $y \notin F$. Hence $F \subsetneq F_x$, which contradicts the minimality of F_x. Therefore, distinct elements of X' have distinct minimal members, and hence $|X'| \leq t$.

For (i), at most one element of X belongs to no member of $\mathcal{F}$; otherwise, two such elements could not be separated by $\mathcal{F}$. Thus, $|X| \leq |X'| + 1 \leq t + 1$.

For (ii), if every element of X belongs to some member of $\mathcal{F}$, then $X' = X$. Hence, $|X| = |X'| \leq t$. □

We now apply the preceding lemma to LMD of split graphs. Let $\mathrm{Sp}(m,n)$ be a generic connected split graph with $V(\mathrm{Sp}(m,n)) = V_1 \cup V_2$, where $V_1 = \{a_1, \ldots, a_m\}$ induces a complete graph and $V_2 = \{b_1, \ldots, b_n\}$ is an independent set. For every $b \in V_2$, let $S(b) = N(b) \cap V_1$, and put

$$U = \{b \in V_2 : S(b) = V_1\}.$$

The lemma below provides a sufficient condition ensuring $\dim_\ell(\mathrm{Sp}(m,n)) = m$.

Lemma 4.4. *Let $\mathrm{Sp}(m,n)$ be a connected split graph with $V(\mathrm{Sp}(m,n)) = V_1 \cup V_2$, where V_1 is the clique part and V_2 is the independent part. If $U \neq \varnothing$ and $\mathcal{S} = \{S(b) : b \in V_2 \setminus U\}$ is a laminar family, then*

$$\dim_\ell(\mathrm{Sp}(m,n)) = m.$$

Proof. Let W be an *LR-set* of $\mathrm{Sp}(m,n)$, and let $R = V_1 \setminus W$ and $B = W \cap (V_2 \setminus U)$. Then $W = (W \cap V_1) \cup B \cup (W \cap U)$.

Any two distinct vertices $a_i, a_j \in R$ are adjacent since V_1 induces a complete graph, and some vertex in W must distinguish them. No vertex of $W \cap V_1$ distinguishes a_i and a_j, since its distance from both vertices is 1. Also, no vertex of $W \cap U$ distinguishes them, as every vertex of U is adjacent to all vertices belonging to V_1. Thus, a vertex distinguishing a_i and a_j must belong to B.

For every $b \in V_2$, its distance from a vertex of V_1 is 1 or 2 according as they are adjacent or not. Therefore, there exists $b \in B$ adjacent to precisely one of a_i and a_j.

Let $\mathcal{L} = \{S(b) \cap R : b \in B\}$. We now show that $\mathcal{L}$ is laminar. Let $L, L' \in \mathcal{L}$, where $L = S(b) \cap R$ and $L' = S(b') \cap R$. Since $\mathcal{S}$ is laminar, one of the following three conditions holds: $S(b) \subseteq S(b')$, $S(b') \subseteq S(b)$, or $S(b) \cap S(b') = \varnothing$. This implies that at least one of the three possibilities holds:

$$(S(b) \cap R) \subseteq (S(b') \cap R),\ (S(b') \cap R) \subseteq (S(b) \cap R),\ \text{or } (S(b) \cap R) \cap (S(b') \cap R) = \varnothing.$$

Equivalently, $L \subseteq L'$, $L' \subseteq L$, or $L \cap L' = \varnothing$. Thus, $\mathcal{L}$ is also laminar. Moreover, for any two distinct vertices $a_i, a_j \in R$, some member from $\mathcal{L}$ contains exactly one of them. That is, $\mathcal{L}$ separates every pair of distinct vertices of R.

Suppose first that $W \cap U \neq \varnothing$. By Lemma 4.3(i), $|\mathcal{L}| \geq |R| - 1$. Since $|\mathcal{L}| \leq |B|$ and $W \cap U \neq \varnothing$, we obtain

$$|W| \geq |W \cap V_1| + |B| + 1 \geq m - |R| + (|R| - 1) + 1 = m.$$

Suppose now that $W \cap U = \varnothing$. Choose any $u \in U$. For every $a_i \in R$, the vertices a_i and u are adjacent, and they must be distinguished by some vertex of W. No vertex within $W \cap V_1$ can distinguish them, because its distance to both a_i and u equals 1. Thus, we can find a vertex $b \in B$ satisfying $d(b, a_i) \neq d(b, u)$. Since $b, u \in V_2$ and V_2 is independent, while u is adjacent to every vertex of V_1, we derive $d(b, u) = 2$. Hence, $d(b, a_i) = 1$, and so b is adjacent to a_i. Therefore, $a_i \in S(b)$. Since $a_i \in R$, we have $a_i \in S(b) \cap R$. Thus, every vertex of R belongs to some member of $\mathcal{L}$. By Lemma 4.3(ii), $|\mathcal{L}| \geq |R|$. Since $|\mathcal{L}| \leq |B|$, we have $|B| \geq |R|$. Hence,

$$|W| \geq |W \cap V_1| + |B| \geq m - |R| + |R| = m.$$

Therefore, in either case, $|W| \geq m$, and hence $\dim_\ell(\mathrm{Sp}(m,n)) \geq m$. As V_2 forms an independent set, $\alpha(\mathrm{Sp}(m,n)) \geq n$. By Lemma 4.1, $\dim_\ell(\mathrm{Sp}(m,n)) \leq m + n - \alpha(\mathrm{Sp}(m,n)) \leq m$. Combining these inequalities, we conclude that $\dim_\ell(\mathrm{Sp}(m,n)) = m$. □

We next show that the condition in Lemma 4.4 is also necessary.

Theorem 4.5. *Let* $\mathrm{Sp}(m,n)$ *be a connected split graph with* $V(\mathrm{Sp}(m,n)) = V_1 \cup V_2$*, where* V_1 *is the clique part and* V_2 *is the independent part. Then* $\dim_\ell(\mathrm{Sp}(m,n)) = m$ *if and only if* $U \neq \varnothing$ *and* $\mathcal{S} = \{S(b) : b \in V_2 \setminus U\}$ *is a laminar family.*

Proof. Assume first that $U \neq \varnothing$ and $\mathcal{S}$ is a laminar family. By Lemma 4.4, we have $\dim_\ell(\mathrm{Sp}(m,n)) = m$.

Conversely, assume that $\dim_\ell(\mathrm{Sp}(m,n)) = m$. We first show that $U \neq \varnothing$. Suppose that $U = \varnothing$. Choose any $a_i \in V_1$ and let $W = V_1 \setminus \{a_i\}$. We verify that W is an *LR-set*. Let x, y denote two adjacent vertices. If at least one of x and y lies in W, then that vertex distinguishes x and y. Thus, it remains to consider two vertices a_i and b, where $b \in V_2$. Since $U = \varnothing$, the vertex b is not adjacent to every vertex of V_1. Because $ba_i \in E(\mathrm{Sp}(m,n))$, there exists some vertex $a_j \in V_1 \setminus \{a_i\} = W$ which is not adjacent to b. Since V_1 induces a complete graph,

$$d(a_i, a_j) = 1 \neq 2 = d(b, a_j).$$

Thus, a_j distinguishes a_i and b. Hence W is an *LR-set*, and therefore

$$\dim_\ell(\mathrm{Sp}(m,n)) \leq |W| = m - 1,$$

a contradiction. Thus, $U \neq \varnothing$.

It remains to verify that $\mathcal{S}$ forms a laminar family. Assume, for contradiction, that this property fails for $\mathcal{S}$. Then there exist $b, b' \in V_2 \setminus U$ such that

$$S(b) \setminus S(b') \neq \varnothing,\ S(b') \setminus S(b) \neq \varnothing,\ S(b) \cap S(b') \neq \varnothing.$$

Hence, there exist three distinct vertices $a_i, a_j, a_k \in V_1$ satisfying

$$a_i \in S(b) \setminus S(b'),\ a_j \in S(b') \setminus S(b),\ a_k \in S(b) \cap S(b').$$

Let $W = (V_1 \setminus \{a_i, a_j, a_k\}) \cup \{b, b'\}$. Then $|W| = m - 1$. Our goal is to verify that W constitutes an *LR-set*.

Consider a pair of adjacent vertices x, y in $\mathrm{Sp}(m,n)$. Whenever either x or y belongs to W, that vertex distinguishes the pair. Hence, we assume $x, y \notin W$.

If $x, y \in \{a_i, a_j, a_k\}$, then b distinguishes a_i and a_j, b' distinguishes a_i and a_k, and b distinguishes a_j and a_k, since

$$d(a_i, b) = 1 \neq 2 = d(a_j, b),\ d(a_i, b') = 2 \neq 1 = d(a_k, b'),\ d(a_j, b) = 2 \neq 1 = d(a_k, b).$$

The last case is when one of x and y belongs to $\{a_i, a_j, a_k\}$ and the other belongs to $V_2 \setminus \{b, b'\}$. We may assume, without loss of generality, that $x = a_r \in \{a_i, a_j, a_k\}$ and $y = c \in V_2 \setminus \{b, b'\}$. By the choice of a_i, a_j, a_k, at least one of b and b' is adjacent to a_r. Let $z \in \{b, b'\}$ be such a vertex. Since a_r and c are adjacent and $z, c \in V_2$ are distinct, we have

$$d(a_r, z) = 1 \neq 2 = d(c, z).$$

Thus, z distinguishes a_r and c.

Therefore, all adjacent vertex pairs are distinguished by some vertex of W, and hence W is an *LR-set*. Consequently,

$$\dim_\ell(\mathrm{Sp}(m,n)) \leq |W| = m - 1,$$

which contradicts $\dim_\ell(\mathrm{Sp}(m,n)) = m$. Therefore, $\mathcal{S}$ forms a laminar family.

Hence, $\dim_\ell(\mathrm{Sp}(m,n)) = m$ if and only if $U \neq \varnothing$ and $\mathcal{S}$ is a laminar family. □

From Lemma 4.2 together with Theorem 4.5, we conclude:

Corollary 4.6. *If G has $n(G) \geq 2$, then $\dim_\ell(G) = n(G) - \alpha(G)$ if and only if G is a split graph with $V(G) = V_1 \cup V_2$, where V_1 induces a complete graph and V_2 is an independent set, such that $U \neq \varnothing$ and $\mathcal{S} = \{S(b) : b \in V_2 \setminus U\}$ is a laminar family.*

Proof. Assume the equality $\dim_\ell(G) = n(G) - \alpha(G)$ holds. By Lemma 4.2, G is a split graph. Choose a split partition $V(G) = V_1 \cup V_2$ with V_2 a maximum independent set, and let $|V_1| = m$. Then $m = n(G) - \alpha(G) = \dim_\ell(G)$. From Theorem 4.5, $U \neq \varnothing$ and $\mathcal{S}$ is a laminar family.

Conversely, assume that G is a split graph whose vertex set satisfies $V(G) = V_1 \cup V_2$, for which $U \neq \varnothing$ and $\mathcal{S}$ is a laminar family. Let $|V_1| = m$. By Theorem 4.5, $\dim_\ell(G) = m$. Since $U \neq \varnothing$, V_2 is a maximum independent set, and hence $\alpha(G) = |V_2|$. Therefore, $n(G) - \alpha(G) = |V_1| = m = \dim_\ell(G)$. □

Corollary 4.6 thus gives the announced direct characterization of the graphs attaining the bound $\dim_\ell(G) = n(G) - \alpha(G)$ in terms of split graphs and laminar families.

5 Conclusion

In this paper, we investigated the effects of edge deletion and vertex deletion on LMD. We constructed families of graphs showing that $\dim_\ell(G) - \dim_\ell(G - e)$ and $\dim_\ell(G) - \dim_\ell(G - v)$ are unbounded for non-bridge edges e and non-cut vertices v, respectively. These constructions disprove two conjectures of Okamoto, Phinezy, and Zhang. Using split graphs and laminar families, we also studied connected graphs satisfying $\dim_\ell(G) = n(G) - \alpha(G)$ and characterized the connected split graphs $\mathrm{Sp}(m, n)$ satisfying $\dim_\ell(\mathrm{Sp}(m, n)) = m$.

Conflict of Interest Statement

Sandi Klavžar is an Associate Editor of the Discrete Applied Mathematics journal and was not involved in the review and decision-making process of this article. The authors declare no other conflict of interest.

Data availability

Our manuscript has no associated data.

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