Asymptotic number of isometric generalized Fibonacci cubes

Sandi Klavžar
Faculty of Mathematics and Physics
University of Ljubljana, Slovenia
and
Faculty of Natural Sciences and Mathematics
University of Maribor, Slovenia
e-mail: sandi.klavzar@fmf.uni-lj.si

Sergey Shpectorov
School of Mathematics
University of Birmingham, United Kingdom
e-mail: s.shpectorov@bham.ac.uk

Abstract

For a binary word f, let $Q_d(f)$ be the subgraph of the d-dimensional cube Q_d induced on the set of all words that do not contain f as a factor. Let $\mathcal{G}_n$ be the set of words f of length n that are *good* in the sense that $Q_d(f)$ is isometric in Q_d for all d. It is proved that $\lim_{n\to\infty} |\mathcal{G}_n|/2^n$ exists. Estimates show that the limit is close to 0.08, that is, about eight percent of all words are good.

Key words: hypercube; generalized Fibonacci cube; isometric embedding; combinatorics on words

AMS subject classifications: 05A16, 05C12, 68R15

1 Introduction

The n-cube Q_n, also known as the n-dimensional hypercube, encompasses the binary strings of length n. Hence it is not surprising that the n-cubes form one of the most studied classes of graphs; that they are one of the fundamental models used in computer science, in particular for designing interconnection networks; and that they are applicable elsewhere.

The edges of Q_n intrinsically capture the (binary) Hamming distance between the binary strings. Consequently, hypercubes form one of the central classes in metric graph theory. However, a serious limitation of hypercubes for practical (and theoretical) purposes is that they are very rare—only one such graph exists for each power of 2. That is, if we wish to model a situation with n nodes, where n

is not close to a power of 2, we usually cannot make use of hypercubes. Hence different modifications of hypercubes and different subgraphs of hypercubes were proposed such that the important properties (for whatever purpose) of hypercubes are preserved. The most important such class of graphs introduced by Graham and Pollack [5] back in 1971 is the class of partial cubes, the graphs defined as isometric subgraphs of hypercubes. These graphs have metric properties analogous to those of hypercubes and can thus be equally well used for routing and similar tasks.

Fibonacci cubes, introduced by Hsu [8] in 1993, are defined as graphs obtained from n-cubes by removing all vertices that contain two consecutive ones. In this way significantly smaller graphs than hypercubes are obtained but they still have nice "interconnection" properties, cf. [8]. Moreover, Fibonacci cubes are partial cubes, see [10], and have been studied from numerous points of view [6, 8, 11, 14, 17, 18].

Instead of removing all vertices that contain the factor 11, one may remove all vertices that contain an arbitrary fixed factor f. We say that the word f is good if the subgraph $Q_d(f)$ obtained from Q_d by removing the vertices with factor f is isometric in Q_d for all d. Note that if $Q_d(f)$ is not isometric then the same holds for all $d' > d$, hence for a bad word f, the subgraph $Q_d(f)$ can be isometric in Q_d for only a finite number of dimensions d. Denoting with $\mathcal{G}_n$ the set of good words of length n and with $\mathcal{B}_n$ the set of bad words of length n, we are interested in the asymptotic behavior of $|\mathcal{G}_n|/2^n$ and $|\mathcal{B}_n|/2^n$. In other words, if the words are long, is the number of good/bad words negligible comparing to the number of all words? We prove that neither of these cases occur by demonstrating that there are considerable number of both good and bad words. In fact, we show that, as the length n goes to infinity, the proportion of good words has a limit strictly between 0 and 1 (and also provide more precise estimates for the limit). Thus, the generalized Fibonacci cubes $Q_d(f)$ for good words f constitute a large new explicit family of partial cubes, see [1, 2, 3, 4, 15] for a sample of other classes of partial cubes.

In order to estimate the limit proportion of good/bad words we will study r-error overlaps of words and demonstrate that, in particular, 2-error overlaps play a crucial role. Note that a similar concept has been studied before. The words that admit 0-error overlap are known in the literature as the *non bifix-free words* aka *bordered words*, see [7, 13]. The numbers of such words of length n are gives as sequence `A094536` in [16] while `A003000` of [16] gives the numbers of bifix-free words.

The rest of paper is organized as follows. In the next section concepts needed in this paper and some preliminary observations are given. In Section 3 we introduce words with r-error overlap. We focus on the words called stutters which are defined as words that have an r-error overlap, $r \leq 2$, of length at least half the length of the word. We show that the proportion of the stutters among all words becomes negligible for large n. Then, in Section 4, we prove that the density of the set of all words of length n having a 2-error overlap converges to a limit value α. In the following section we prove that every bad word has a 2-error overlap and from this fact we deduce that the density of bad words converges to α as well. In Section 6 we then prove that α lies between 0.919975 and 0.924156.

2 Preliminaries

Let $B = \{0, 1\}$ and call elements of B *bits*. An element of B^d is called a *binary word* (or simply a *word*) of length d. We will use the product notation for words meaning concatenation, for example, 1^s is the word $11 \cdots 1$ of length s. A word f is a *factor* of a word w if $w = w_1 f w_2$ for some words w_1 and w_2.

Let $d \geq 1$ be a fixed integer. Then all words of length d are vertices of the d-dimensional cube Q_d, where two words are adjacent whenever they differ in exactly one position. For a word f of length n, let $Q_d(f)$ be the subgraph of Q_d induced on the set of all words of length d that do not contain a factor f, see [9]. The graph $Q_d(f)$ is called the *generalized Fibonacci cube* defined by the forbidden word f. In this notation, $Q_d(11)$ denotes the Fibonacci cubes. (We note that the name "generalized Fibonacci cubes" was earlier used for the restricted family $Q_d(1^s)$, $s \geq 1$, see [12].)

We will consider the usual shortest path distance and write $d_G(u, v)$ for the distance in a graph G between u and v. Recall that a subgraph of a graph is called *isometric* if the distance between any two vertices of the subgraph is independent of whether it is computed in the subgraph or in the entire graph G. We will write $H \hookrightarrow G$ to denote that H is an isometric subgraph of G. Call a word f *bad* if there exists a dimension d such that $Q_d(f)$ is not isometric in Q_d (notationally, $Q_d(f) \not\hookrightarrow Q_d$). For instance, it is proved in [9] that $(10)^s 1$ ($s \geq 1$), and $1^r 0^s$ ($r, s \geq 2$), are bad words.

Lemma 2.1 *Suppose that $Q_d(f) \not\hookrightarrow Q_d$ for some dimension d. Then $Q_{d'}(f) \not\hookrightarrow Q_{d'}$ for all $d' \geq d$.*

Proof. Let $d' = d + r$, $r \geq 1$. Since $Q_d(f) \not\hookrightarrow Q_d$, there exist vertices u, v of $Q_d(f)$ such that $d_{Q_d(f)}(u, v) > d_{Q_d}(u, v)$. If the first bit of f is 1, set $\widehat{u} = 0^r u$ and $\widehat{v} = 0^r v$, otherwise set $\widehat{u} = 1^r u$ and $\widehat{v} = 1^r v$. Then $\widehat{u}, \widehat{v} \in Q_{d'}(f)$ and

$$d_{Q_{d'}(f)}(\widehat{u}, \widehat{v}) = d_{Q_d(f)}(u, v) > d_{Q_d}(u, v) = d_{Q_{d'}}(\widehat{u}, \widehat{v}),$$

hence $Q_{d'}(f) \not\hookrightarrow Q_{d'}$. ■

So for every bad f there exists the smallest dimension $d = d(f)$ for which non-isometricity holds. If we are not interested in the exact value of $d(f)$, we can always talk about a *sufficiently large* dimension d.

The word f is called *good* if it is not bad. That is, f is good if $Q_d(f) \hookrightarrow Q_d$ for all dimensions d. For instance, 1^s, 10^s, and $(10)^s$, are good words for any $s \geq 1$, see [9], hence in particular the Fibonacci cube $Q_d(11)$ isometrically embeds into Q_d. Set finally

$$\mathcal{G}_n = \{f \in B^n \mid f \text{ is good}\} \quad \text{and} \quad \mathcal{B}_n = \{f \in B^n \mid f \text{ is bad}\}.$$

3 r-Error overlaps and stutters

For a word f of some length n, let $f_{s,k}$ be the factor of f of length k starting from position $s+1$. Here, clearly, $k \le n$ and $s \in \{0, 1, \ldots, n-k\}$. For example, $b_k(f) := f_{0,k}$ is the beginning of f of length k. Similarly, $e_k(f) := f_{n-k,k}$ is the end part of f of the same length k. Suppose that $b_k(f)$ and $e_k(f)$ agree in all but r positions. Then we say that f has an *r-error overlap of length k*. If f has an r-error overlap for some length k then we simply say that f has an r-error overlap. We say that f is a *stutter* if f has an r-error overlap of length k, where $r \le 2$ and $\frac{n}{2} \le k \le n-1$. For instance, the word 110100011 is not a stutter, while the word 001010111 of length $n = 9$ is a stutter because it has a 2-error overlap of length $5 \ge 9/2$. Among short words it is easy to find stutters, but for long words we have:

Lemma 3.1 *The proportion of stutters among all words of length n tends to zero as n goes to infinity. That is, denoting with $\mathcal{S}_n$ the set of all stutters of length n,*

$$\lim_{n\to+\infty} \frac{|\mathcal{S}_n|}{2^n} = 0.$$

Proof. Suppose $k \ge \frac{n}{2}$ and let $s(k)$ be the number of stutters f in $\mathcal{S}_n$ that have an r-error overlap of length exactly k for some $r \le 2$. It is easy to see that such a word f is fully specified by the r locations of the "errors" within $b_k(f)$ and by the bits in the last $n-k$ positions of f. The number of choices of $r \le 2$ positions within the k positions in $b_k(f)$ is $1 + k + \frac{k(k-1)}{2} = \frac{k^2+k+2}{2}$. Hence $s(k) = (k^2+k+2)2^{n-k-1}$.

We can now estimate the number of stutters from above as

$$|\mathcal{S}_n| = \sum_{k=\lceil \frac{n}{2} \rceil}^{n-1} s(k) = \sum_{k=\lceil \frac{n}{2} \rceil}^{n-1} (k^2+k+2)2^{n-k-1} \le n(2n^2)2^{n/2-1} = n^3 2^{n/2}.$$

Therefore, the density of stutters tends to zero, since $\frac{|\mathcal{S}_n|}{2^n} \le \frac{n^3 2^{n/2}}{2^n} = n^3 2^{-n/2}$. ■

4 2-Error overlaps

In this section we study the density of the set $\mathcal{T}_n$ of all words of length n having a 2-error overlap. We say that a word $f \in \mathcal{T}_n$ is *split* if f has a 2-error overlap of length $k \le \frac{n}{2}$. In this case, the beginning and the end part of f, that realize the 2-error overlap, are disjoint. Let $\mathcal{T}_n^s$ be the set of all split words from $\mathcal{T}_n$.

Lemma 4.1 *If $f \in \mathcal{T}_n - \mathcal{T}_n^s$ then f is a stutter.*

Proof. Follows directly from the definitions. ■

The next result allows us to conclude that the set of split words with a 2-error overlap has a limit density as the length n goes to infinity.

Proposition 4.2 *We have* $|\mathcal{T}^s_{n+1}| \geq 2|\mathcal{T}^s_n|$.

Proof. Consider the mapping $\phi : B^{n+1} \to B^n$ defined by erasing the bit in position $\lfloor \frac{n}{2} \rfloor + 1 = \lceil \frac{n+1}{2} \rceil$. Clearly, if $f \in B^{n+1}$ and $\phi(f)$ has a 2-error overlap of some length $k \leq \frac{n}{2}$ then f also has a 2-error overlap of the same length k. In particular, $\phi^{-1}(\mathcal{T}^s_n) \subseteq \mathcal{T}^s_{n+1}$. Since every fiber of this mapping has size two, the claim follows. ■

Corollary 4.3 *The sequence* $\alpha_n = \frac{|\mathcal{T}^s_n|}{2^n}$ *is monotonically increasing and bounded from above by* 1. *In particular, it has a limit* $\alpha \leq 1$.

In fact, this number α is also the limit density of the words with a 2-error overlap.

Corollary 4.4 *The sequence* $\beta_n = \frac{|\mathcal{T}_n|}{2^n}$ *converges to the same limit value* α.

Proof. Let $\gamma_n = \frac{|\mathcal{S}_n|}{2^n}$. According to Lemma 3.1, the sequence γ_n tends to zero. Hence both α_n and $\alpha_n + \gamma_n$ converge to the same limit, α. On the other hand, clearly, $\alpha_n \leq \beta_n$, since $\mathcal{T}^s_n \subseteq \mathcal{T}_n$, and also $\beta_n \leq \alpha_n + \gamma_n$ by Lemma 4.1. So the claim follows. ■

5 Density of bad words

In this section we establish that the density of the set of bad words also tends to the same limit value α. For this sake the following result is crucial.

Theorem 5.1 *If* f *is bad then* f *has a* 2*-error overlap.*

Proof. Suppose f is bad and choose d sufficiently large, so that $Q_d(f) \not\hookrightarrow Q_d$. Let w and w' be vertices of $Q_d(f)$ such that $d_{Q_d(f)}(w, w') > d_{Q_d}(w, w')$. We will assume that $m = d_{Q_d}(w, w')$ is as small as possible. Clearly, $m \geq 2$. Let $i_1 < i_2 < \cdots < i_m$ be the positions in which w and w' differ. For a subset S of $V = \{i_1, i_2, \ldots, i_m\}$, let $w(S)$ be the word obtained from w by switching the bits in all positions contained in S. In particular, $w(\emptyset) = w$ and $w(V) = w'$. To simplify the notation, we will write $w(i)$ instead of $w(\{i\})$, $w(i, j)$ instead of $w(\{i, j\})$, and so on.

Note that all words $w(S)$ lie on the shortest paths in Q_d from w to w'. In particular, in view of the minimality of m, none of the words $w(S)$, where $\emptyset \neq S \neq V$, is contained in $Q_d(f)$. So all these words contain occurrences of the word f. For each $i \in \{i_1, i_2, \ldots, i_m\}$, let f_i be one occurrence of f in $w(i)$. Let also $s_i + 1$ be the first position that f_i occupies within $w(i)$. Using the notation from Section 3, we can write $w(i)_{s_i,n} = f$, where n is the length of f.

Clearly every f_i contains the corresponding position i. Define a digraph D on V drawing an arc from i to j if and only if f_i covers the position j (i.e., $s_i + 1 \leq j \leq s_i + n$).

Claim: There exist i and j in V with arcs both from i to j and from j to i.

Note first that every $i \in V$ is a source of an arc, for otherwise w' would contain f. In addition, all arcs from i_1 are directed to the right and all the arcs from i_m to the left. Let $j = i_s$ be the first vertex from which there is an arc pointing left. It is clear that $i = i_{s-1}$ and j fulfill the assertion of the claim.

The assertion of the theorem is now essentially a direct corollary of the claim. Indeed, let i and j be as in the claim. Then both factors f_i (of the word $w(i)$) and f_j (of $w(j)$) are equal to f and they both contain positions i and j. So they have an overlap. Furthermore, within this overlap, they disagree only in the positions i and j. Hence f has a 2-error overlap. ■

This theorem is in particular useful for a short argument that certain words are good. For instance, it implies immediately that the already mentioned words 1^s, 10^s, and $(10)^s$, are good. Additional instances of good words are 11010 and 1^s01^s0, $s \geq 1$. To prove these facts much more work was needed in [9].

According to Theorem 5.1, $\mathcal{B}_n \subseteq \mathcal{T}_n$. We next show that in fact $\mathcal{B}_n$ nearly coincides with $\mathcal{T}_n$.

Proposition 5.2 *If f has a 2-error overlap and it is good then f is a stutter.*

Proof. Let n be the length of f and suppose it has a 2-error overlap of length k. Then $b_k(f)$ and $e_k(f)$ disagree in some two positions i and j (out of k). Set $d = 2n - k$ and let $w, w' \in B^d$ be defined as follows. Let w be $b_{n-k}(f)$ followed by $f(i)$ (recall from the proof of Theorem 5.1 that $f(i)$ is the word obtained from f by switching its ith bit) and, similarly, let w' be the same $b_{n-k}(f)$ followed by $f(j)$. It is easy to see that w and w' disagree in positions $n-k+i$ and $n-k+j$. So they are at distance two in Q_d. Notice, furthermore, that the two common neighbours of w and w' in Q_d are the words $w(n-k+i) = w'(n-k+j)$ and $w(n-k+j) = w'(n-k+i)$. Both these words contain f; indeed, $w(n-k+i)$ contains f as its end part (suffix of length n), while $w(n-k+j)$ contains f as its beginning (prefix of length n). In particular, the common neighbours of w and w' are not contained in $Q_d(f)$. Since f is good, $Q_d(f) \hookrightarrow Q_d$, which means that either $w \notin Q_d(f)$, or $w' \notin Q_d(f)$.

We can now show that f is a stutter. In view of the symmetry between i and j (and hence between w and w') we may assume that $w \notin Q_d(f)$, that is, w contains a factor f. This factor overlaps with either $b_n(w)$ or with $e_n(w)$ (or with both) in $s > \frac{n}{2}$ positions. As both these words differ from f in one position, we conclude that f has an r-error overlap of length s with $r \leq 1$ (in fact, one can see that $r = 1$). As $s > \frac{n}{2}$, we have that f is a stutter, as claimed. ■

As a corollary of Proposition 5.2, we obtain that the limit density of bad words exists and it is equal to the same α, as in the preceding section. Let $\delta_n = \frac{|\mathcal{B}_n|}{2^n}$.

Corollary 5.3 *The density δ_n converges to α, that is,*

$$\lim_{n\to\infty} \delta_n = \alpha.$$

Proof. By Theorem 5.1, $\delta_n \le \beta_n$. On the other hand, by Proposition 5.2, $\delta_n \ge \beta_n - \gamma_n$. Since γ_n tends to zero, it follows that the sequence δ_n has the same limit as β_n, namely, α. ■

Before we leave this section, we remark that the proof of Proposition 5.2 has a further consequence. Suppose f is a bad word, but it is not a stutter. By Theorem 5.1, f has a 2-error overlap of some length k. Set $d = 2n - k$ and select vertices w and w' as in the proof of Proposition 5.2. Since f is not a stutter, the final argument in the proof means that both w and w' are contained in $Q_d(f)$, while, again as in the proof, the two common neighbours of w and w' are not. That is, we have $2 = d_{Q_d}(w, w') < d_{Q_d(f)}(w, w')$. This means that badness of nearly every bad word f can be established in a cube Q_d, with $d < 2n$, with a pair of vertices at distance two from each other. This confirms observations from [9] where good and bad words of small length were considered. Namely, in most of the cases for which non-isometricity was established, the distance condition was shown to fail for distance 2. However, there exist bad words f for which distance 2 is preserved in $Q_d(f)$ but still $Q_d(f) \not\hookrightarrow Q_d(f)$.

6 Estimates for the limit value α

We are now finding some estimates for the limit density α. Estimates from below are easy: we can use that the sequence α_n (densities of split words with a 2-error overlap) is monotonically increasing. Furthermore it is easy to see that $\alpha_{2k+1} = \alpha_{2k}$ so we only need to consider even n. Manifestly, $\alpha_4 = 0.25$. With the use of a computer a few further values were found, see Table 1.

In particular, $\alpha_{30} = 0.919975\ldots$ and hence we can say that $\alpha > 0.919975$. So over 90% of all long words are bad. In fact, the data in Table 1 seems to suggest that the value of α is close to 0.92.

The starting values of the sequence β_n (involving words of length n having a 2-error overlap) are shown in Table 2. By Corollary 4.4 this sequence also converges to α. Note, however, that it is not monotone.

To get an upper bound for α we will use similar ideas as above. Again we can only focus on the words of even length $n = 2k$. Let T_n be the number of nonsplit words of length n. If w is such a word then inserting two new bits in the middle produces a word of length $n + 2$ which is either again nonsplit or it has a 2-error overlap of length exactly $k+1$. The number of words of the latter sort is $2^{k+1}\binom{k+1}{2}$ because we can choose $k + 1$ bits arbitrarily and then the second half must be the same as the first half but with two positions changed. Therefore we can write

$$4T_n \le T_{n+2} + 2^{k+1}\binom{k+1}{2}.$$

Switching to the densities $\mu_n = T_n/2^n$ (notice that $\mu_n = 1 - \alpha_n$) we get

$$\mu_n \le \mu_{n+2} + \frac{k(k+1)}{2^{k+2}}$$

n	$\lvert\mathcal{T}_n^s\rvert$	$\alpha_n = \lvert\mathcal{T}_n^s\rvert/2^n$
4	4	0.250000
6	34	0.531250
8	182	0.710938
10	830	0.810547
12	3518	0.858887
14	14538	0.887329
16	59074	0.901398
18	238534	0.909935
20	958714	0.914301
22	3845886	0.916931
24	15408114	0.918395
26	61689006	0.919238
28	246881258	0.919704
30	987815218	0.919975

Table 1: Some values of α_n

which implies that

$$\mu_{n+2} \geq \mu_n - \frac{k(k+1)}{2^{k+2}} .$$

Since $\alpha_n = 1 - \mu_n$ we get

$$\alpha_{n+2} \leq \alpha_n + \frac{k(k+1)}{2^{k+2}} .$$

Combining these relations from n to $n + 2m$ we get

$$\alpha_{n+2m} \leq \alpha_n + \sum_{i=k}^{k+m-1} \frac{i(i+1)}{2^{i+2}} .$$

Now we take m to infinity and obtain the following estimate for the limit α:

$$\alpha \leq \alpha_n + \sum_{i=k}^{\infty} \frac{i(i+1)}{2^{i+2}} .$$

Using computer with $k = 15$ we get that the sum here is at most 0.004181. Together with the value $\alpha_{30} = 0.919975$ from Table 6 this yields an upper limit of 0.924156. Hence we have the following result:

Theorem 6.1 *The value of the limit α is between* 0.919975 *and* 0.924156*.*

As a consequence we see that for large n the number of good words is approximately 8% of all words of that length.

n	$\lvert\mathcal{T}_n\rvert$	$\beta_n = \lvert\mathcal{T}_n\rvert/2^n$
4	8	0.500000
5	22	0.687500
6	46	0.718750
7	98	0.765625
8	210	0.820313
9	430	0.839844
10	886	0.865234
11	1790	0.874023
12	3638	0.888184
13	7350	0.897217
14	14830	0.905151
15	29758	0.908142
16	59802	0.912506
17	119802	0.914017
18	240362	0.916908
19	480966	0.917370
20	963302	0.918676
21	1927382	0.919047
22	3857746	0.919758
23	7715446	0.919753
24	15437078	0.920122
25	30873042	0.920088
26	61759618	0.920290
27	123512490	0.920240
28	247051278	0.920338
29	494077866	0.920292
30	988213906	0.920346

Table 2: Some values of β_n

Acknowledgements

We are grateful to Ciril Petr and Marko Petkovšek for the computation of the data in Tables 1 and 2 and also to Marko for pointing out the connection with bifix-free words. The work of the first author was supported by the Research Grant P1-0297 of Ministry of Higher Education, Science and Technology Slovenia. He is also grateful to the University of Birmingham for his stay there.

References

[1] L Beaudou, S. Gravier, K. Meslem, Isometric embeddings of subdivided complete graphs in the hypercube, SIAM J. Discrete Math. 22 (2008) 1226–1238.

[2] B Brešar, Characterizing almost-median graphs, European J. Combin. 28 (2007) 916–920.

[3] B. Brešar, W. Imrich, S. Klavžar, H.M. Mulder, R. Škrekovski, Tiled partial cubes, J. Graph Theory 40 (2002) 91–103.

[4] D. Eppstein, Cubic partial cubes from simplicial arrangements, Electron. J. Combin. 13 (2006) #R79, 14 pp.

[5] R.L. Graham, H. Pollak, On the addressing problem for loop switching, Bell System Tech. J. 50 (1971) 2495–2519.

[6] P. Gregor, Recursive fault-tolerance of Fibonacci cube in hypercubes, Discrete Math. 306 (2006) 1327–1341.

[7] L.J. Guibas, M.A. Odlyzko, Periods in strings, J. Combin. Theory Ser. A 30 (1981) 19–42.

[8] W.-J. Hsu, Fibonacci cubes—a new interconnection technology, IEEE Trans. Parallel Distrib. Syst. 4 (1993) 3–12.

[9] A. Ilić, S. Klavžar, Y. Rho, Generalized Fibonacci cubes, Discrete Math., in press. DOI:10.1016/j.disc.2011.02.015.

[10] S. Klavžar, On median nature and enumerative properties of Fibonacci-like cubes, Discrete Math. 299 (2005) 145–153.

[11] S. Klavžar, M. Mollard, M. Petkovšek, The degree sequence of Fibonacci and Lucas cubes, Discrete Math. 311 (2011) 1310–1322.

[12] J. Liu, W.-J. Hsu, M.J. Chung, Generalized Fibonacci cubes are mostly Hamiltonian, J. Graph Theory 18 (1994) 817–829.

[13] M. Lothaire, Algebraic Combinatorics on Words, Encyclopedia of Mathematics and its Applications, 90. Cambridge University Press, Cambridge, 2002.

[14] E. Munarini, N. Salvi Zagaglia, Structural and enumerative properties of the Fibonacci cubes, Discrete Math. 255 (2002) 317–324.

[15] N. Polat, Netlike partial cubes. I. General properties, Discrete Math. 307 (2007) 2704–2722.

[16] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, http://www.research.att.com/~njas/sequences/

[17] A. Taranenko, A. Vesel, Fast recognition of Fibonacci cubes, Algorithmica 49 (2007) 81–93.

[18] H. Zhang, L. Ou, H. Yao, Fibonacci-like cubes as Z-transformation graphs, Discrete Math. 309 (2009) 1284–1293.