Generalized Fibonacci cubes

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Abstract

Generalized Fibonacci cube $Q_d(f)$ is introduced as the graph obtained from the d-cube Q_d by removing all vertices that contain a given binary string f as a substring. In this notation the Fibonacci cube Γ_d is $Q_d(11)$. The question whether $Q_d(f)$ is an isometric subgraph of Q_d is studied. Embeddable and non-embeddable infinite series are given. The question is completely solved for strings f of length at most five and for strings consisting of at most three blocks. Several properties of the generalized Fibonacci cubes are deduced. Fibonacci cubes are, besides the trivial cases $Q_d(10)$ and $Q_d(01)$, the only generalized Fibonacci cubes that are median closed subgraphs of the corresponding hypercubes. For admissible strings f, the f-dimension of a graph is introduced. Several problems and conjectures are also listed.

Key words. hypercube, Fibonacci cube, generalized Fibonacci cube, isometric embedding, isometric dimension.

AMS subject classifications. 05C12, 11C75

1 Introduction

Fibonacci cubes form a class of graphs with many appealing properties. They admit a recursive decomposition into smaller Fibonacci cubes which in turn implies that the order of a Fibonacci cube is the corresponding Fibonacci number. This class of graphs was introduced as a model for interconnection network [10]. It was studied from several points of view, see [1, 4, 5, 9, 12, 13, 14, 17, 18] for their structural properties. Fibonacci cubes can be recognized in $O(m \log n)$ time (where n is the order and m the size of a given graph) [19], earlier an $O(mn)$ was presented in [20].

The Fibonacci cube Γ_d, $d \geq 1$, is defined as follows. The vertex set of Γ_d is the set of all binary strings $b_1 b_2 \ldots b_d$ containing no two consecutive 1's. Two vertices are adjacent in Γ_d if they differ in precisely one bit. Fibonacci cubes can also be described as the simplex graphs of the complement of paths, cf. [2], and as the graphs of certain distributive lattices [8]. Looking from the other side, Γ_d is a graph obtained from the d-cube Q_d by removing all strings that contain 11 as a substring. This point of view rises to the following general approach.

Suppose f is an arbitrary binary string and $d \geq 1$. Then we introduce the *generalized Fibonacci cube*, $Q_d(f)$, as the graph obtained from Q_d by removing all vertices that contain f as a substring. We point out that the term "generalized Fibonacci cubes" has been used in [11] for the graphs $Q_d(1^s)$ that were further studied in [15, 22]. Since our definition is more general, we have decided to use the same name for all the graphs $Q_d(f)$.

In Sections 3 and 4 we study the question for which strings f, $Q_d(f)$ is an isometric subgraph of Q_d. We give several embeddable and non-embeddable infinite series, where f is of arbitrary length. With some additional efforts, we apply these results in Section 5 to classify the embeddability for strings f of length at most five. Then, in Section 6, we give several properties of these graphs. As an example we compute the number of vertices, edges and squares in $Q_d(110)$. We also prove that Fibonacci cubes and the paths $Q_d(10)$ and $Q_d(01)$ can be characterized among the generalized Fibonacci cubes with the property that they are median closed subgraphs of the corresponding hypercubes.

Graphs isometrically embeddable into hypercubes naturally yield the isometric dimension of a graph. Two closely related dimensions are the lattice dimension [6] and the Fibonacci dimension [2], where the latter is defined as the smallest d (if such a d exists) for which G isometrically embeds into Γ_d. Now, suppose that for a given string f and for any d, $Q_d(f)$ lies isometrically in Q_d. Then we can define a new graph dimension $\dim_f(G)$ as the smallest integer d' such that G embeds isometrically into $Q_{d'}(f)$. This aspect of generalized Fibonacci cubes is treated in Section 7.

We conclude the paper with several conjectures and problems for further investigation. In particular, we pose a conjecture that would significantly increase the number of embeddable generalized Fibonacci cubes and ask about the computational complexity of determining the newly introduced dimensions.

2 Preliminaries

In this section we introduce the concepts needed in this paper and prove some preliminary results that narrow the strings f that need to be considered.

For a binary string b we denote its *(binary) complement* with $\overline{b}$. With e_i we denote the binary string with 1 in the i-th position and 0 elsewhere. For binary strings b and c of equal length let $b+c$ denote their sum computed bitwise modulo 2. In particular, $b+e_i$ is the string obtained from b by reversing its i-th bit. For a binary string $b = b_1b_2 \dots b_d$ let $b^R = b_d b_{d-1} \dots b_1$ be the *reverse* of b. A non-extendable sequence of contiguous equal digits in a string b is called a *block* of b.

For a (connected) graph G, the *distance* $d_G(u, v)$ between vertices u and v is the usual shortest path distance. The set of vertices lying on shortest u, v-paths is called the *interval* between u and v and denoted $I_G(u, v)$.

The d-cube Q_d is the graph whose vertices are all binary strings of length d, two vertices being adjacent if they differ in exactly one position. More formally, $b = b_1b_2 \dots b_d$ is adjacent to $c = c_1c_2 \dots c_d$ if there exists an index i such that $b_i \neq c_i$ and $b_j = c_j$ for $j \neq i$. Recall that $d_{Q_d}(b, c)$ is the number of bits in which the strings b and c differ. Let $b_j = 1$ and $c_j = 0$ for $j = i_1, \dots, i_k$ and $b_j = 0$ and $c_j = 1$ for $j = i_{k+1}, \dots, i_p$. Then

$$P : b \to (b + e_{i_1}) \to (b + e_{i_1} + e_{i_2}) \to \cdots \to (b + e_{i_1} + e_{i_2} + \cdots + e_{i_p})$$

is a b, c-path in Q_d of length $p = d_{Q_d}(b, c)$. Such a path (that is, a path where we first change each bit of b from 1 to 0 for which $b_i = 1$ and $c_i = 0$, and then change from 0 to 1 the other bits in which b and c differ) is called a *canonical b, c-path*.

A subgraph H of G is called *isometric* if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$. We will write

$$H \hookrightarrow G$$

to denote that H is an isometric subgraph of G and $H \not\hookrightarrow G$ that this is not the case. For instance, $\Gamma_d \hookrightarrow Q_d$. To see this, let b and c be arbitrary vertices of Γ_d and let P be a canonical b, c-path in Q_d. Then it is straightforward that P lies in Γ_d, hence Γ_d is isometric in Q_d.

Let $b = uvw$ be a binary string obtained by a concatenation of u, v, and w, where u and w are allowed to be the empty string. Then we say that v is a *factor* of b. Let f be a binary string and define the *generalized Fibonacci cube* $Q_d(f)$ to be the induced subgraph of Q_d defined with the vertex set

$$V(Q_d(f)) = \{b \mid b \in V(Q_d), f \text{ is not a factor of } b\}.$$

Note that the notation f is selected since it denotes a **f**orbidden factor of the binary strings. Observe also that $\Gamma_d = Q_d(11)$. The graph $Q_4(101)$ is depicted in Figure 1.

Lemma 2.1 *Let f be a binary string and let $1 \leq d \leq |f|$, where $|f|$ denotes the length of f. Then $Q_d(f) \hookrightarrow Q_d$.*

Proof. If $d < |f|$ then $Q_d(f) = Q_d$. Let $d = |f|$, then $Q_d(f)$ is the d-cube with the vertex f removed. Since Q_d is vertex transitive, we may without loss of generality assume that $f = 00 \dots 0$, hence $V(Q_d(f)) = V(Q_d) \setminus \{00 \dots 0\}$. Let $b, c \in V(Q_d(f))$ and let P be a shortest b, c-path in Q_d. If P does not contain $00 \dots 0$ we are done, so suppose this is not the case. Then we may further assume that P contains the subpath

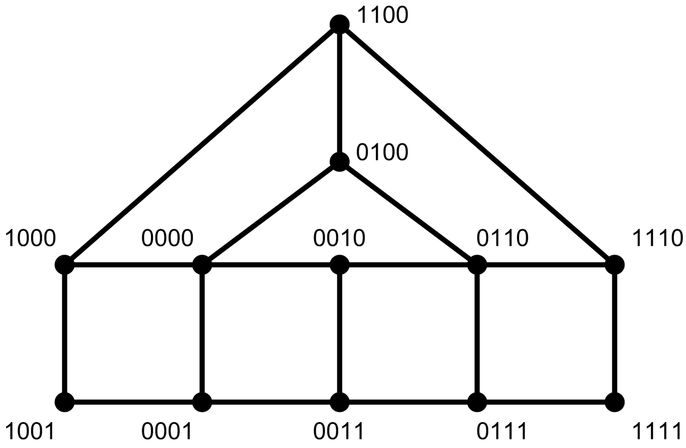

Figure 1: Generalized Fibonacci cube $Q_4(101)$

$10\ldots0 \to 00\ldots0 \to 01\ldots0$. It can be replaced by $10\ldots0 \to 11\ldots0 \to 01\ldots0$. This new b,c-path is of the same length as P and lies in $Q_d(f)$. ■

Lemma 2.2 *Let f be a binary string and $d \geq 1$. Then $Q_d(f)$ is isomorphic to $Q_d(\overline{f})$.*

Proof. Note that f is a factor of b if and only if $\overline{f}$ is a factor of $\overline{b}$. Hence it follows easily that the assignment $b \mapsto \overline{b}$ is an isomorphism between $Q_d(f)$ and $Q_d(\overline{f})$. ■

For instance, $\Gamma_d \cong Q_d(00) \cong Q_d(11)$.

Lemma 2.3 *Let f be a nonempty binary string and $d \geq 1$. Then $Q_d(f)$ is isomorphic to $Q_d(f^R)$.*

Proof. Again f is a factor of b if and only if f^R is a factor of b^R. Hence the assignment $b \mapsto b^R$ is an isomorphism between $Q_d(f)$ and $Q_d(f^R)$. ■

Let $b, c \in Q_d(f)$ and $p \geq 2$. Then u and v are called *p-critical words for $Q_d(f)$* if $d_{Q_d}(u,v) = p$, but none of the neighbors of b in $I_{Q_d}(b,c)$ belongs to $Q_d(f)$ or none of the neighbors of c in $I_{Q_d}(b,c)$ belongs to $Q_d(f)$. The next lemma gives a tool to be used throughout the paper to prove that $Q_d(f) \not\hookrightarrow Q_d$.

Lemma 2.4 *Let f be a nonempty binary string. If there exist p-critical words for $Q_d(f)$ ($p \geq 2$), then $Q_d(f) \not\hookrightarrow Q_d$.*

Proof. Let b and c be p-critical words for $Q_d(f)$. Then none of the neighbors of b or none of the neighbors of c in $I_{Q_d}(b,c)$ belongs to $Q_d(f)$, which means that $d_{Q_d}(b,c) = p < d_{Q_d}(f)(b,c)$ and therefore $Q_d(f) \not\hookrightarrow Q_d$. ■

3 Forbidden factors with at most three blocks

In this section we characterize generalized Fibonacci cubes $Q_d(f)$ such that $Q_d(f) \hookrightarrow Q_d$, where f contains at most three blocks. The cases with one and three blocks are rather straightforward, two blocks need some more arguments. We begin with the simpler cases.

Proposition 3.1 *Let $s \geq 1$. Then $Q_d(1^s) \hookrightarrow Q_d$.*

Proof. For $s = 1$ we have $Q_d(1) \cong K_1$ and there is nothing to be proved. Let $s \geq 2$ and consider arbitrary vertices b and c of $Q_d(1^s)$ with $d_{Q_d}(b, c) = p$. We need to show that $d_{Q_d(1^s)}(b, c) = p$ as well. Note that this will in particular imply that $Q_d(1^s)$ is connected. Let P be a canonical b, c-path in Q_d. By the construction, if some vertex of P would contain 1^s as a factor, 1^s would also be a factor of c. Since this is not the case we conclude that P lies entirely in $Q_d(1^s)$. ■

Proposition 3.2 *Let $r, s, t \geq 1$ and let $d \geq r + s + t + 1$. Then $Q_d(1^r0^s1^t) \not\hookrightarrow Q_d$.*

Proof. Suppose first that $d = r + s + t + 1$. Select vertices $b = 1^r10^{s-1}11^t$ and $c = 1^r00^{s-1}01^t$. Note that $b, c \in Q_d(1^r0^s1^t)$ and that they differ in two bits. The only vertices on the two shortest b, c-paths are $1^r00^{s-1}11^t = 1^r0^s1^t1$ and $1^r10^{s-1}01^t = 11^r0^s1^t$, but none of them is a vertex of $Q_d(1^r0^s1^t)$. Thus b and c are 2-critical words for $Q_d(1^r0^s1^t)$ and hence by Lemma 2.4, $Q_d(1^r0^s1^t) \not\hookrightarrow Q_d$.

Attaching an appropriate number of 1's to the front of b and c, we get 2-critical words for $Q_d(1^r0^s1^t)$ for any $d > r + s + t + 1$. ■

We now move to forbidden factors consisting of two blocks.

Theorem 3.3 *Let $d \geq 2$. Then*

(i) For $r \geq 1$, $Q_d(1^r0) \hookrightarrow Q_d$.
(ii) For $s \geq 2$, $Q_d(1^20^s) \hookrightarrow Q_d$ if and only if $d \leq s + 4$.
(iii) If $r, s \geq 3$, then $Q_d(1^r0^s) \hookrightarrow Q_d$ if and only if $d \leq 2r + 2s - 3$.

Proof. We first prove (i). For $r = 1$ the vertices of $Q_d(10)$ are $11\ldots1$, $01\ldots1$, $\ldots$, $00\ldots0$, hence $Q_d(10) \cong P_{d+1} \hookrightarrow Q_d$. Let $r \geq 2$ and let b and c be vertices of $Q_d(1^r0)$. We proceed by induction on $p = d_{Q_d}(b, c)$ and need to prove that $d_{Q_d(1^r0)}(b, c) = p$ as well. Suppose $p = 1$, that is, $d_{Q_d}(b, c) = 1$. Then by definition, b is adjacent to c in $Q_d(1^r0)$.

Let $p \geq 2$ and let i be the index of the leftmost bit in which b and c differ. We may without loss of generality assume that $b_i = 1$ and $c_i = 0$. (If $b_i = 0$ and $c_i = 1$ we proceed analogously by considering the neighbor $c_i + e_i$ of c_i.) Let $b' = b + e_i$. The only possibility that b' would not belong to $Q_d(1^r0)$ is that b'_i is preceded by s 1's. By the way the index i is selected, $c_i = 0$ is then also preceded by r 1's. But this would mean that $c \notin Q_d(1^r0)$. We conclude that $b' \in Q_d(1^r0)$. Since b' differs from c in $p - 1$

bits, induction implies that there exists a b,c-path in $Q_d(1^r0)$ of length p which proves (i). In the rest of the proof we thus need to consider the cases when $r, s \geq 2$.

Claim. Let $r, s \geq 2$ and $d \leq 2r+2s-3$. Then $Q_d(1^r0^s) \hookrightarrow Q_d$ if and only if it is not the case that $r=2$, $s \geq 4$, and $d > s+4$.
Let b and c be different vertices of $Q_d(1^r0^s)$ and let i be the index of the leftmost bit in which b and c differ. We may without loss of generality assume that $b_i = 1$ and $c_i = 0$. Let $b' = b + e_i$. Then $d_{Q_d}(b', c) < d_{Q_d}(b, c)$. Therefore, if $b' \in Q_d(1^r0^s)$ induction implies that $d_{Q_d(1^r0^s)}(b', c) = d_{Q_d}(b', c)$ and consequently $d_{Q_d(1^r0^s)}(b, c) = d_{Q_d}(b, c)$. So, suppose that $b' \notin Q_d(1^r0^s)$. Then b' contains a substring of the form $x_{b'} = 1^r0^k b'_i 0^{s-1-k}$, where $b'_i = 0$, and the corresponding substring x_b of b is $1^r0^k10^{s-1-k}$. In the corresponding substring x_c of c, at least one of the last $s-1-k$ bits must be 1, for otherwise $c \notin Q_d(1^r0^s)$ would hold. We distinguish two cases.

Case 1: x_c contains a bit 1 that is not the last bit of x_c.
Then $s-1-k \geq 2$ and therefore $k \leq s-3$. Then we can change this bit in b to obtain a vertex from $Q_d(1^r0^s)$ at distance $d_{Q_d}(b,c)-1$ from c unless $r=2$, $s \geq 4$ and $d > s+4$. Assume $r=2$, $s \geq 4$ and $d > s+4$. Let $k = d-s-4$. Then $1 \leq k \leq s-3$. Select vertices $b = 1^20^k100^s$ and $c = 1^20^k010^s$. Note that $b, c \in Q_d(1^20^s)$ and that they differ in two bits. The only neighbors of b in $I_{Q_d}(b,c)$ are $1^20^k000^s$ and $1^20^k110^s$. But none of them belongs to $Q_d(1^20^s)$. Thus b, c are 2-critical for $Q_d(1^r0^s)$ and hence by Lemma 2.4, $Q_d(1^r0^s) \not\hookrightarrow Q_d$.

Case 2: The last bit of x_c is 1 and it is preceded with a block of $s-1$ zeros.
Now change the last bit of x_b in b. If the new vertex would not be in $Q_d(1^r0^s)$, then the length of b would be at least $r+(s-2)+r+s = 2r+2s-2$ if $s-1-k=1$ and would be at least $r+(s-1)+r+s = 2r+2s-1$ if $s-1-k \geq 2$. We conclude that $Q_d(1^r0^s) \hookrightarrow Q_d$ for any $d \leq 2r+2s-3$ and the claim is proved.

The claim proves (ii) for $s=3$ if $d \leq s+4 = 2r+2s-3$, and (iii) if $d \leq 2r+2s-3$. For the other cases of (ii), firstly assume $s=2$. Then we have $s+4 = 6 > 2r+2s-3 = 5$. Hence we need to prove that $Q_d(1^20^2) \hookrightarrow Q_d$ if $d=6$ and $Q_d(1^20^2) \not\hookrightarrow Q_d$ if $d>6$. For $d>6$, it is proved in Case 1 below and for $d=6$, it is checked by computer. When $s \geq 4$, we have $s+4 < 2r+2s-3$ and the claim proves the lemma for $d \leq 2r+2s-3$. Hence it remains to prove that for $r \neq 2$ or $s \neq 2$ there holds $Q_d(1^r0^s) \not\hookrightarrow Q_d$ if $d > 2r+2s-3$. This will be done in Case 2 below.

Case 1: $r = s = 2$.
Suppose first that $d=7$. Select vertices $b = 1^21010^2$ and $c = 1^20100^2$. Note that $b, c \in Q_d(1^20^2)$ and that they differ in three bits. The only neighbors of b in $I_{Q_d}(b,c)$ are 1^20010^2, 1^21110^2 and 1^21000^2. But none of them belongs to $Q_d(1^20^2)$ and therefore b, c are 3-critical words for $Q_d(1^20^2)$. Attaching an appropriate number of 1's to the front of b and c, we get 3-critical words for $Q_d(1^20^2)$ for $d>7$.

Case 2: $r>2$ or $s>2$.
Suppose first that $d = 2r+2s-2$. Then vertices $b = 1^r0^{s-2}101^{r-2}0^s$ and $c = 1^r0^{s-2}011^{r-2}0^s$ are 2-critical for $Q_d(1^r0^s)$ and hence by Lemma 2.4, $Q_d(1^r0^s) \not\hookrightarrow Q_d$.

Attaching an appropriate number of 1's to the front of b and c, we get 2-critical words for $Q_d(1^r0^s)$ when $d > 2r + 2s - 2$. ■

Note that Theorem 3.3 covers all the cases in view of Lemma 2.2 and Lemma 2.3. For instance, $Q_d(1^20^s) \cong Q_d(0^21^s) \cong Q_d(1^s0^2) \cong Q_d(0^s1^2)$ and hence the same embedding conclusion holds in each of these cases.

4 Forbidden factors with more than three blocks

We now move to forbidden factors consisting of more than three blocks. We do not have a complete solution but prove embeddability of several infinite series and give infinite families that are not embeddable. Let us start with the latter.

Proposition 4.1 *Let $s \geq 1$. Then $Q_d((10)^s1) \not\hookrightarrow Q_d$ for $d \geq 4s$.*

Proof. The case $s = 1$ has already been treated in Proposition 3.2. Assume in the rest that $s \geq 2$. Let $d = 4s$ and set

$$\begin{aligned} b &= (10)^{s-1}100(10)^{s-1}1, \\ c &= (10)^{s-1}111(10)^{s-1}1\,. \end{aligned}$$

Considering that the only neighbors of b in $I_{Q_d}(b,c)$ are $(10)^{s-1}110(10)^{s-1}1 = (10)^{s-1}1(10)^s1$ and $(10)^{s-1}101(10)^{s-1}1 = (10)^s1(10)^{s-1}1$, b, c are 2-critical words for $Q_d((10)^s1)$ and hence by Lemma 2.4, $Q_d(f) \not\hookrightarrow Q_d$. If $d > 4s$ attach an appropriate number of 1's to the front of b and c to get 2-critical words for $Q_d((10)^s1)$. ■

Proposition 4.2 *Let $r, s \geq 1$. Then $Q_d((10)^r1(10)^s) \not\hookrightarrow Q_d$ for $d \geq 2r + 2s + 3$.*

Proof. Select 2-critical words for $Q_d((10)^r1(10)^s)$, $b = (10)^r100(10)^s$ and $c = (10)^r111(10)^s$ for $Q_d((10)^r1(10)^s)$ and then it is proved by Lemma 2.4. ■

We now give two infinite families of embeddable graphs.

Theorem 4.3 *Let $s \geq 2$. Then $Q_d(1^s01^s0) \hookrightarrow Q_d$.*

Proof. Let b and c be vertices of $Q_d(1^s01^s0)$. We again proceed by induction on $p = d_{Q_d}(b,c)$, the case $p = 1$ being trivial. Hence let $p \geq 2$ and let i be the index of the leftmost bit in which b and c differ.

Suppose that $b_i = 1$ and $c_i = 0$. Let $b' = b + e_i$. The only possibility that b' would not belong to $Q_d(1^s01^s0)$ is that b'_i is preceded by 1^s and followed by 1^s0. The vertices b and c must differ also on an index j such that $i < j \leq i + s + 1$, because otherwise c would contain 1^s01^s0 as a factor. Without loss of generality, we can assume that $i = s + 1$, that is, b starts with 1^s11^s0 and c starts with 1^s0. We distinguish two cases and proceed by induction on the distance.

Case 1: $s+2 \le j \le 2s+1$.
The vertex $b' = b + e_j$ belongs to $Q_d(1^s01^s0)$, since b' starts with $1^s11^x01^y0$ with $x+y+1 = s$, and cannot contain 1^s01^s0 as a factor. (Note that it is possible that $x = 0$ or $y = 0$.) Since b' differs from c in $p-1$ bits, the induction assumption on the distance implies that there exists a b, c-path in $Q_d(1^s01^s0)$ of length p.

Case 2: $j = 2s+2$.
In this case, the vertex b starts with 1^s11^s0, while the vertex c starts with 1^s01^s1. Consider two vertices $\widetilde{b}$ and $\widetilde{c}$ obtained by removing the first $s+1$ bits from b and c. These two vertices are at distance $p-1$. By the induction assumption on the distance, one can find a $\widetilde{b}, \widetilde{c}$-path of length $p-1$ in $Q_{d-s-1}(1^s01^s0)$. As $\widetilde{b}$ and $\widetilde{c}$ start with 1^s, following the same bit changes we can construct a shortest path from $b = 1^s1\widetilde{b}$ to $1^s1\widetilde{c}$ in $Q_d(1^s01^s0)$. Finally, we change the $(s+1)$-th bit of $1^s1\widetilde{c}$ and get a b, c-path in $Q_d(1^s01^s0)$ of length p. ■

Theorem 4.4 *Let $s \ge 1$. Then $Q_d((10)^s) \hookrightarrow Q_d$.*

Proof. The case $s = 1$ follows from Theorem 3.3(i), so we can assume that $s \geqslant 2$.

Let b and c be vertices of $Q_d((10)^s)$ and suppose b and c differ in $p \ge 1$ bits. If $p = 1$ then b is adjacent to c in $Q_d((10)^s)$. Assume that $p \ge 2$ and let i be the index of the leftmost bit in which b and c differ.

Suppose that $b_i = 1$ and $c_i = 0$. Let $b' = b + e_i$. The only possibility that b' would not belong to $Q_d((10)^s)$ is that b'_i is preceded by $(10)^x1$ and followed by $(10)^y$, where $x + y + 1 = s$ and y is strictly greater than zero. The vertices b and c must differ also on an index j such that $i < j \le i + 2y$, because otherwise c would contain $(10)^s$ as a factor. We may assume that j is the first such index. Without loss of generality, we can assume that $i = 2x + 2$, that is, b starts with $(10)^x11(10)^y$ and c starts with $(10)^x10$. We distinguish two cases.

Case 1: $2x+3 \le j \le 2s-1$.
We claim that the vertex $b' = b + e_j$ belongs to $Q_d((10)^s)$. For this sake note that b' starts with $(10)^x11(10)^z00(10)^{y-z-1}$ or $(10)^x11(10)^z11(10)^{y-z-1}$ with $z < y$, and cannot contain $(10)^s$ as a factor. (Observe that it is possible that $z = 0$ or $z = y-1$.)

Case 2: $j = 2s$.
In this case b starts with $(10)^x11(10)^{y-1}10$, c starts with $(10)^x10(10)^{y-1}11$ and b' starts with $(10)^x11(10)^{y-1}11$. The only case when b' would not belong to $Q_d((10)^s)$ is that b'_{2s} is followed by $0(10)^{s-1}$. Now, we can again distinguish two cases based on the position of the next bit in which b and c differ. If this position is between $2s+1$ and $4s-2$, we can proceed as in Case 1 and find an appropriate bit such that after changing it the obtained vertex belongs to $Q_d((10)^s)$. Therefore, assume that b and c again differ on the last bit position $4s-1$.

Consider two vertices $\widetilde{b}$ and $\widetilde{c}$ obtained by removing the first $2s-2$ bits from b and c. These two vertices are at distance $p-1$. Note that $\widetilde{b}$ starts with $100(10)^{s-2}10$ and $\widetilde{c}$ starts with $110(10)^{s-2}11$. Using induction we can find a $\widetilde{b}, \widetilde{c}$-path of the length

$p-1$ in $Q_{d-2s+2}((10)^s)$. Since the first bits and the third bits of both $\widetilde{b}$ and $\widetilde{c}$ are 1 and 0, respectively, following the same bit changes, we can construct a shortest path from $b = (10)^x 11(10)^{y-1}\widetilde{b}$ to $(10)^x 11(10)^{y-1}\widetilde{c}$ in $Q_d((10)^s)$, without introducing any appearances of $(10)^s$. Finally, we change the $(2x+2)$-th bit and get a b,c-path in $Q_d((10)^s)$ of length p. ■

5 Classification of strings of length at most five

The results from previous sections can be applied to fill the following table which classifies isometry of $Q_d(f)$ in Q_d for strings f of length at most five. In the table, strings that yield isometric embeddings are written in bold. Note that the table covers all the strings up to the complement and reversal, cf. Lemmata 2.2 and 2.3.

length	forbidden factor
1	**1** (Proposition 3.1)
2	**11** (Proposition 3.1) **10** (Theorem 3.3(i))
3	**111** (Proposition 3.1) **110** (Theorem 3.3(i)) 101 (Proposition 3.2)
4	**1111** (Proposition 3.1) **1110** (Theorem 3.3(i)) **1100** ($d \leq 6$, Theorem 3.3(ii)), 1100 ($d \geq 7$, Theorem 3.3(ii)) **1010** (Theorem 4.4) 1101, 1001 (Proposition 3.2)
5	**11111** (Proposition 3.1) **11110** (Theorem 3.3(i)) **11100** ($d \leq 7$, Theorem 3.3(ii)), 11100 ($d \geq 8$, Theorem 3.3(ii)) 11001, 11101, 11011, 10001 (Proposition 3.2) **10110** ($d \leq 6$, Lemma 2.1 and computer check for $d = 6$) 10110 ($d \geq 7$, Proposition 4.2) **10101** ($d \leq 7$, Lemma 2.1 and computer check for $d = 6, 7$) 10101 ($d \geq 8$, Proposition 4.1) **11010** (Proposition 5.1)

Table 1: Classification of embeddability of generalized Fibonacci cubes with forbidden factors of length at most 5

As we can see from Table 1, the only case not covered by the results from previous sections is 11010. We cover it with the next result.

Proposition 5.1 *For every $d \geq 1$, it holds $Q_d(11010) \hookrightarrow Q_d$.*

Proof. Let b and c be vertices of $Q_d(11010)$ and suppose b and c differ in $p \geq 1$ bits. If $p = 1$ then b is adjacent to c in $Q_d(11010)$. To imply induction later in the proof, we also need to consider the case $p = 2$. Assume that $p \geq 2$ and let i be the index of the leftmost bit in which b and c differ.

Suppose that $b_i = 1$ and $c_i = 0$. Let $b' = b + e_i$. The only possibility that b' would not belong to $Q_d(11010)$ is that b'_i is preceded by 11 and followed by 10. Without loss of generality, we can assume that b and c start with 11110 and 110, respectively. If c starts with 1100, we can simply change the fourth bit, since b' would start with 11100 and therefore $b' \in Q_d(11010)$. Therefore, the vertex c starts with 11011.

Now, let $b' = b + e_5$. The only possibility that b' would not belong to $Q_d(11010)$ is that b starts with 11110010. In this case c must differ with b in 6-th, 7-th or 8-th position and $p \geq 3$. We distinguish three cases.

Case 1: b and c differ in the 7-th bit.
The vertex $b' = b + e_7$ starts with 11110000 and belongs to $Q_d(11010)$.

Assume in the rest that b and c agree on the 7-th bit.

Case 2: b and c differ in the 6-th bit.
The vertex $b' = b + e_6$ starts with 11110110, and the only possible case when $b' \notin Q_d(11010)$ is when b starts with 1111001010 and c starts with 1101111. Consider two new vertices $\widetilde{b}$ and $\widetilde{c}$ obtained by cutting off the first five bits from b and c, respectively. Using induction, one can find a $\widetilde{b}, \widetilde{c}$-path of length $p - 2$ in $Q_{d-5}(11010)$. Following the same bit changes, we can construct a shortest path from $b = 11110\widetilde{b}$ to $11110\widetilde{c}$ in $Q_d(11010)$. Finally, we change the third and the fifth bit of $11110\widetilde{c}$ and get a b, c-path in $Q_d(11010)$ of length p.

In the past case we may thus assume that b and c agree also on the 6-th bit.

Case 3: b and c differ in the 8-th bit.
The vertex $b' = b + e_8$ belongs to $Q_d(11010)$, except when b starts with 11110010010 or with 111100101010. Here, we can again apply the induction argument, by considering two new vertices $\widetilde{b}$ and $\widetilde{c}$ that are obtained by cutting off the first six bits from b and c, respectively. Using induction, one can find a $\widetilde{b}, \widetilde{c}$-path of length $p - 2$ in $Q_{d-6}(11010)$. Following the same bit changes we can construct a shortest path from $b = 111100\widetilde{b}$ to $111100\widetilde{c}$ in $Q_d(11010)$. Finally, we change the third and the fifth bit of $111100\widetilde{c}$ and get a b, c-path in $Q_d(11010)$ of length p. ■

6 Some properties of generalized Fibonacci cubes

In this section we have a closer look to the basic properties of generalized Fibonacci cubes in particular to their orders and sizes. The results support the name we selected for these graphs. We also prove that the classical Fibonacci cubes stand out by the property that they are the only graphs (besides the trivial case of paths) among the generalized Fibonacci cubes that are median closed in the corresponding hypercubes. We begin with:

Proposition 6.1 *Let $f \neq 01, 10$ be a binary string of length greater than one and let $Q_d(f) \hookrightarrow Q_d$, $d \geq 1$. Then the maximum degree and diameter of $Q_d(f)$ are equal to d.*

Proof. Without loss of generality, we can assume that f contains at least two 1's. The vertex 0^d belongs to $Q_d(f)$, and all of its neighbors contain exactly one 1 in binary representation – proving that the maximum degree equals d.

If f contains two adjacent 1's, one can consider a path from $v = 10101\ldots$ to $\overline{v} = 01010\ldots$ by passing through the vertex 0^d. If f contains two adjacent 0's, one can consider similar path of length d, passing through vertex 1^d. Finally, if f does not have two equal consecutive digits, and one can consider a path from 0^d to 1^d by complementing digits from left to right. Therefore, in each of the cases the diameter of $Q_d(f)$ is at least d. Since by the assumption $Q_d(f)$ is isometric in Q_d, the diameter is at most d. ∎

When the length of a forbidden string is three, we have two non-isomorphic cases that yield isometric embeddings: $f = 111$ and $f = 110$.

Let $G_d = Q_d(111)$ and let $S(G_d)$ be the set of 4-cycles of G_d. Then the following recurrent formulas hold

$$\begin{aligned} |V(G_d)| &= |V(G_{d-1})| + |V(G_{d-2})| + |V(G_{d-3})|, & (1)\\ |E(G_d)| &= |E(G_{d-1})| + |E(G_{d-2})| + |E(G_{d-3})| + |V(G_{d-2})| + 2|V(G_{d-3})|, & (2)\\ |S(G_d)| &= |S(G_{d-1})| + |S(G_{d-2})| + |S(G_{d-3})| + |E(G_{d-2})| + \\ &\quad +2|E(G_{d-3})| + |V(G_{d-3})|. & (3) \end{aligned}$$

The starting values are $|V(G_0)| = 1$, $|V(G_1)| = 2$, $|V(G_2)| = 4$ for the number of vertices, $|E(G_0)| = 0$, $|E(G_1)| = 1$, $|E(G_2)| = 4$ for the number of edges, and $|S(G_0)| = 0$, $|S(G_1)| = 0$, $|S(G_2)| = 1$ for the number of squares.

Let us partition the set of vertices of G_d into three classes $V(G_d) = A_d \cup B_d \cup C_d$, where A_d, B_d, and C_d are the subsets of vertices that start with 0, with 10, and with 110, respectively. Since the vertices in G_d do not contain 111 as a factor, every vertex belongs to exactly one of the classes A_d, B_d, and C_d.

The formula (1) follows since $|A_d| = |V(G_{d-1})|$, $|B_d| = |V(G_{d-2})|$ and $|C_d| = |V(G_{d-3})|$. For the number of edges of G_d, we need to count edges connecting the induced subgraphs A_d and B_d, B_d and C_d, and A_d and C_d. Since every two vertices from different classes are at distance at least one, the number of such edges are $|V(G_{d-2})|$, $|V(G_{d-3})|$ and $|V(G_{d-3})|$, respectively. This proves the relation (2). Similarly, for the number of squares we need to include the squares from A_d, B_d, C_d and the squares between pairs (A_d, B_d), (B_d, C_d), (A_d, C_d). Finally, we need to count the squares that contain the vertices from all three classes: for every vertex $110v$ from C_d, we have that vertices $000v, 010v, 110v, 100v$ form a square. Therefore, the relation (3) follows. For more information on the graphs $Q_d(111)$ and, more generally, $Q_d(1^r)$, see [11, 15, 22].

Let $H_d = Q_d(110)$. Then the following recurrent formulas hold

$$
\begin{aligned}
|V(H_d)| &= |V(H_{d-1})| + |V(H_{d-2})| + 1, & (4)\\
|E(H_d)| &= |E(H_{d-1})| + |E(H_{d-2})| + |V(H_{d-2})| + 2, & (5)\\
|S(H_d)| &= |S(H_{d-1})| + |S(H_{d-2})| + |E(H_{d-2})| + 1. & (6)
\end{aligned}
$$

The starting values are $|V(H_0)| = 1$, $|V(H_1)| = 2$ for the number of vertices, $|E(H_0)| = 0$, $|E(H_1)| = 1$ for the number of edges, and $|S(H_0)| = 0$, $|S(H_1)| = 0$ for the number of squares.

In order to prove the above relations, we can apply the same arguments as for G_d using the partition $H_d = A_d \cup B_d \cup C_d$, where A_d is the subset of vertices that start with bit 0, B_d is the subset of vertices that start with 10, and C_d is the one-element set of all vertices from H_d that start with 11.

It can easily be proved by induction that $|V(H_d)| = F_{d+3} - 1$, where F_d are the Fibonacci numbers. We next count the number of edges of H_d.

Proposition 6.2 *For any $d \geq 0$,*

$$|E(H_d)| = -1 + \sum_{i=1}^{d+1} F_i F_{d+2-i}.$$

Proof. For $d = 0$ and $d = 1$, we have $|E(H_0)| = -1 + 1 \cdot 1 = 0$ and $|E(H_1)| = -1 + 1 \cdot 1 + 1 \cdot 1 = 1$. Hence, the equality holds for $d = 0, 1$. Let $d \geq 2$ and assume that it holds for all indices smaller than d. Using (5) and the inductions hypothesis, we get

$$
\begin{aligned}
|E(H_d)| &= \left(-1 + \sum_{i=1}^{d} F_i F_{d+1-i}\right) + \left(-1 + \sum_{i=1}^{d-1} F_i F_{d-i}\right) + F_{d+1} + 1\\
&= -1 + F_{d+1} + F_d F_{d+1-d} + \sum_{i=1}^{d-1} F_i (F_{d+1-i} + F_{d-i})\\
&= -1 + F_{d+1} \cdot F_1 + F_d \cdot F_2 + \sum_{i=1}^{d-1} F_i F_{d+2-i}\\
&= -1 + \sum_{i=1}^{d+1} F_i F_{d+2-i}.
\end{aligned}
$$

This concludes the inductive proof. ■

Using [12, Corollary 4], it follows that

$$|E(H_d)| = -1 + \frac{(d+1)F_{d+2} + 2(d+2)F_{d+1}}{5}.$$

Similarly one can prove the following closed formula for the number of squares. The proof goes along the same lines as the proof of [12, Proposition 5] and is thus omitted.

Proposition 6.3 *For any $d \geq 0$,*

$$|S(H_d)| = -\frac{3(d+1)}{25}F_{d+2} + \left(\frac{(d+1)^2}{10} + \frac{3(d+1)}{50} - \frac{1}{25}\right)F_{d+1}.$$

Notice that $|S(Q_d(110))| = |S(Q_{d+1}(11))| = |S(\Gamma_{d+1})|$, while $|V(Q_d(110))| = |V(\Gamma_{d+1})| - 1$ and $|E(Q_d(110))| = |E(\Gamma_{d+1})| - 1$.

Recall that a connected graph G is a *median graph* if for every triple u, v, w of its vertices $|I(u,v) \cap I(u,w) \cap I(v,w)| = 1$. A subgraph H of a graph G is *median closed* if, with any triple of vertices of H, their median is also in H. A connected graph is a median graph if and only if it is a median closed, induced subgraph of some hypercube [16]. From this point of view the Fibonacci cubes stand out among the classes of graphs considered in this paper by the following result.

Proposition 6.4 *Let f be a nonempty binary string of length $|f| \geq 2$ and let $d \geq |f|$. Then $Q_d(f)$ is a median closed subgraph of Q_d if and only if $|f| = 2$. In other words, the only median closed generalized Fibonacci cubes are paths and Fibonacci cubes.*

Proof. Let $|f| = 2$. Then we have already observed that for any d, $Q_d(10)$ and $Q_d(01)$ are paths of length d and hence median closed subgraph of Q_d. In addition, $Q_d(11) \cong Q_d(00) \cong \Gamma_d$ which are also such subgraphs of Q_d [12].

Let $|f| \geq 3$ and let $f = f_1 f_2 f_3 \dots f_{|f|}$. Define $g = \overline{f_{|f|}}$ and set

$$\begin{aligned}
x &= \overline{f_1} f_2 f_3 \dots f_{|f|} g \dots g, \\
y &= f_1 \overline{f_2} f_3 \dots f_{|f|} g \dots g, \\
z &= f_1 f_2 \overline{f_3} \dots f_{|f|} g \dots g,
\end{aligned}$$

where in each of x, y, and z the bit g appears $d - |f|$ times. (Note that if $|f| = d$ then the length of x, y, z is $|f|$.) Then each of x, y, z is a vertex of $Q_d(f)$. Indeed, the only possibility that f would be a factor of these vertices is that f would start before the $|f|$-th bit and end after it. But this is not possible since by construction the last bits of f and such a string are different.

Since x, y, z are pairwise at distance 2, the unique candidate for their median is $f_1 f_2 f_3 \dots f_{|f|} g \dots g$. This vertex does not belong to $Q_d(f)$ and therefore $Q_d(f)$ is not a median graph. ■

7 f-dimension

For a connected graph G, the *isometric dimension*, $\mathrm{idim}(G)$, is the smallest integer d such that G admits an isometric embedding into Q_d. If there is no such d we set $\mathrm{idim}(G) = \infty$. It is well-known that $\mathrm{idim}(G)$ is the number of the so-called Θ-classes of G and that it can be determined in polynomial time, the fastest algorithm being of quadratic complexity [7].

Let f be a nonempty binary string and suppose that $Q_d(f) \hookrightarrow Q_d$ for any $d \geq 1$. Then we define the *f-dimension* of a graph G, $\dim_f(G)$, as the smallest integer d such that G admits an isometric embedding into $Q_d(f)$, and set $\dim_f(G) = \infty$ if there is no such d.

Proposition 7.1 *Let f be a nonempty binary string, $f \neq 1, 0, 10, 01$. If $Q_d(f) \hookrightarrow Q_d$ for any $d \geq 1$, then for any connected graph G, $\dim_f(G) < \infty$ if and only if $\mathrm{idim}(G) < \infty$.*

Proof. Suppose $d = \dim_f(G) < \infty$. Then G isometrically embeds into $Q_d(f)$. By the assumption $Q_d(f)$ isometrically embeds into Q_d, hence G isometrically embeds into Q_d.

Conversely, let $d = \mathrm{idim}(G) < \infty$ and consider G isometrically embedded into Q_d. We distinguish two cases.

Case 1: 11 or 00 is a factor of f.
Suppose 11 is a factor of f and let $d = \mathrm{idim}(G)$. To each vertex $b = b_1 b_2 \ldots b_d$ of G (embedded into Q_d) assign the vertex $\widetilde{b} = b_1 0 b_2 0 \ldots 0 b_d$. Let $\widetilde{G}$ be the subgraph of Q_{2d-1} induced by the vertex set

$$V(\widetilde{G}) = \{\widetilde{b} \mid b \in V(G)\}\,.$$

Note first that $\widetilde{G}$ is isomorphic to G. Moreover, for any $b \in V(G)$, $\widetilde{b}$ does not contain 11 as a factor and hence also do not contain f, therefore $\widetilde{b}$ can be considered as a vertex of $Q_{2d-1}(f)$. Hence we may consider $\widetilde{G}$ as a subgraph of $Q_{2d-1}(f)$. Then

$$d_{\widetilde{G}}(\widetilde{b}, \widetilde{c}) = d_G(b, c) = d_{Q_d}(b, c) = d_{Q_{2d-1}}(\widetilde{b}, \widetilde{c}) = d_{Q_{2d-1}(f)}(\widetilde{b}, \widetilde{c})\,,$$

where the last equality holds since by the assumption of the proposition, $Q_{2d-1}(f) \hookrightarrow Q_{2d-1}$. Hence $\widetilde{G}$ is isometric in $Q_{2d-1}(f)$ and therefore $\dim_f(G) \leq 2d - 1 < \infty$.

If 00 is a factor of f we proceed analogously by inserting 1 between consecutive bits of b.

Case 2: Neither 11 nor 00 is a factor of f.
In this case, $f = 01010\cdots$ or $f = 10101\cdots$ and f contains at least three bits by the assumption. Moreover, $f \neq 010$ by Proposition 3.2. Hence f contains at least two 1's. Now to each vertex $b = b_1 b_2 \ldots b_d$ of G assign the vertex $\widetilde{b} = b_1 00 b_2 00 \ldots 00 b_d$. Then if $\widetilde{b}_i = \widetilde{b}_j = 1$ we have $|i - j| > 2$, hence $\widetilde{b}$ can be considered as a vertex of $Q_{3d-2}(f)$. By the same arguments as in the first case we conclude that $\dim_f(G) \leq 3d - 2 < \infty$. ■

Note that the proof of Proposition 7.1 yields that

$$\mathrm{idim}(G) \leq \dim_f(G) \leq 3\,\mathrm{idim}(G) - 2\,.$$

It is also clear that the upper bound is a general estimate that can be improved in specific cases.

As already mentioned, the special case $\dim_{11}(G)$ was introduced in [2] as the *Fibonacci dimension* of a graph. This dimension has many interesting properties. Among others—rather surprisingly for the area of isometric embeddings—it is NP-complete to decide whether $\dim_{11}(G)$ is equal to $\mathrm{idim}(G)$.

A different but related version of a dimension is the following. Let G be a graph and let f be a binary string. Then define $dim_f^{-1}(G)$ as the largest d such that $Q_d(f)$ isometrically embeds into G. This inverse dimension has been studied in [3] for the case $f = 11$, that is, for the Fibonacci cubes, where it was proved that deciding whether $dim_{11}^{-1}(G) = d$ is NP-complete even if G is given as an induced subgraph of Q_d.

8 Concluding remarks

We first pose the conjecture announced in the introduction.

Conjecture 8.1 *If $Q_d(f) \hookrightarrow Q_d$ then $Q_d(ff) \hookrightarrow Q_d$.*

It is NP-hard to determine $\dim_{11}(G)$ for an arbitrary graph. Hence we pose:

Problem 8.2 *Suppose that f is a binary string for which $\dim_f$ is well-defined. What is the complexity of determining $\dim_f(G)$ for an arbitrary graph?*

We feel that the answer to Problem 8.2 is NP-hard in all the case except perhaps for $f = (10)^s$.

Suppose that for some f and for some d, $Q_d(f) \not\hookrightarrow Q_d$. It would still be possible that $Q_d(f)$ embeds into some $Q_{d'}$ where $d' > d$. Hence we pose:

Problem 8.3 *Suppose $Q_d(f) \not\hookrightarrow Q_d$. Is there a dimension d' such that $Q_d(f)$ is an isometric subgraph of $Q_{d'}$?*

With respect to Problem 8.3 we are inclined to believe that the answer is negative in most (if not all) cases.

For instance, let $d \geq 4$ and consider the vertices $u = 1^{d-3}000$, $v = 1^{d-3}001$, $x = 1^{d-3}110$, $y = 1^{d-3}111$, and edges $e = uv$, $f = xy$ of $Q_d(101)$. Then $d_{Q_d(101)}(v, y) \neq 2$ and hence

$$v = 1^{d-3}001 \;\rightarrow\; 1^{d-3}000 \;\rightarrow\; 1^{d-3}100 \;\rightarrow\; 1^{d-3}110 \;\rightarrow\; 1^{d-3}111 = y$$

is a shortest path in $Q_d(101)$. Thus e is not in relation Θ with f. On the other hand, we can find a ladder in $Q_d(101)$ from e to f which implies that $e\Theta^* f$,

$$\begin{array}{cccccccc} 1^d \rightarrow & 01^{d-1} \rightarrow & 001^{d-2} \rightarrow & \ldots \rightarrow & 0^{d-1}1 \rightarrow & 10^{d-2}1 \rightarrow & \ldots \rightarrow & 1^{d-3}001 \\ 1^{d-1}0 \rightarrow & 01^{d-2}0 \rightarrow & 001^{d-3}0 \rightarrow & \ldots \rightarrow & 0^d \rightarrow & 10^{d-1} \rightarrow & \ldots \rightarrow & 1^{d-3}000. \end{array}$$

Hence by Winkler's theorem [21] we conclude that $Q_d(101)$, $d \geq 4$, is not an isometric subgraph of any hypercube.

For the final remark, recall that $|V(Q_d(110))| = |V(Q_{d+1}(11))| - 1$, $|E(Q_d(110))| = |E(Q_{d+1}(11))| - 1$, and $|S(Q_d(110))| = |S(Q_{d+1}(11))|$. According to Proposition 6.1, the diameter and the maximum degree of $Q_d(110)$ are d, while the diameter and the maximum degree of $Q_{d+1}(11)$ are $d+1$, see Figure 2 where the Fibonacci cube $Q_5(11)$ is confronted with the 110-Fibonacci cube $Q_4(110)$. Hence $Q_d(110)$ and $Q_{d+1}(11)$ appear quite similar and it might be interesting to give a further insight into this fact.

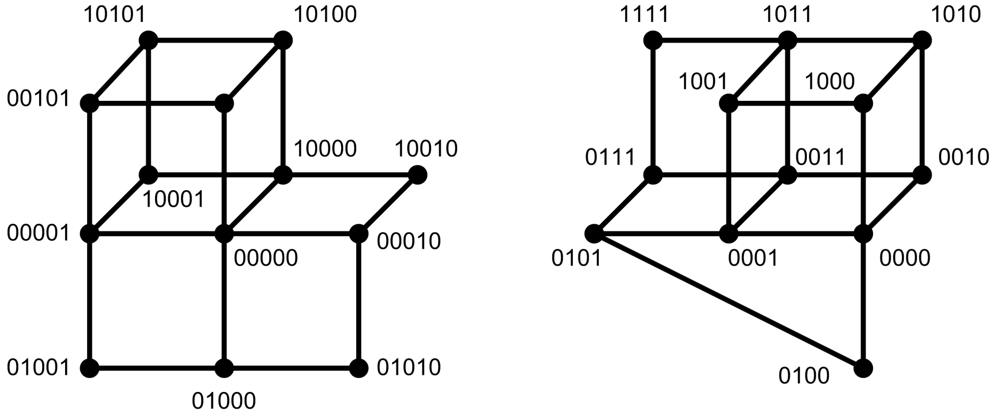

Figure 2: Fibonacci cube $Q_5(11)$ and 110-Fibonacci cube $Q_4(110)$

Acknowledgements

This work was supported by Research Grant 144007 of Serbian Ministry of Science, and by the Research Grant P1-0297 of Ministry of Higher Education, Science and Technology Slovenia. The work was partially done during a visit of A.I. at the University of Ljubljana supported by the grant P1-0297 of Ministry of Higher Education, Science and Technology Slovenia and during a sabbatical visit of Y.R. at the Faculty of Mathematics and Physics, University of Ljubljana, supported by a research grant for 2009/10 from the University of Incheon, Korea.

We are grateful to Petr Gregor for several useful remarks and comments on the earlier draft of the paper.

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