

# Characterization of general position sets and its applications to cographs and bipartite graphs

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## Abstract

A vertex subset  $S$  of a graph  $G$  is a general position set of  $G$  if no vertex of  $S$  lies on a geodesic between two other vertices of  $S$ . The cardinality of a largest general position set of  $G$  is the general position number  $\text{gp}(G)$  of  $G$ . It is proved that  $S \subseteq V(G)$  is in general position if and only if the components of  $G[S]$  are complete subgraphs, the vertices of which form an in-transitive, distance-constant partition of  $S$ . If  $\text{diam}(G) = 2$ , then  $\text{gp}(G)$  is the maximum of  $\omega(G)$  and the maximum order of an induced complete multipartite subgraph of the complement of  $G$ . As a consequence,  $\text{gp}(G)$  of a cograph  $G$  can be determined in polynomial time. If  $G$  is bipartite, then  $\text{gp}(G) \leq \alpha(G)$  with equality if  $\text{diam}(G) \in \{2, 3\}$ . A formula for the general position number of the complement of an arbitrary bipartite graph is deduced and simplified for the complements of trees, of grids, and of hypercubes.

**Key words:** general position set; graph of diameter 2; cograph; bipartite graph; bipartite complement

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## 1 Introduction

Motivated by the century old Dudeney's no-three-in-line problem [6] (see [11, 14, 17] for recent developments on it) and by the general position subset selection problem [7, 16] from discrete geometry, the natural related problem was introduced to graph theory in [12] as follows. Let  $G = (V(G), E(G))$  be a graph. Then we wish to find a largest set of vertices  $S \subseteq V(G)$ , called a *gp-set* of  $G$ , such that no vertex of  $S$  lies on a geodesic (in  $G$ ) between two other vertices of  $S$ . The *general position number* (*gp-number* for short),  $\text{gp}(G)$ , of  $G$  is the number of vertices in a gp-set of  $G$ .

As it happens, the same concept has already been studied two years earlier in [20] under the name *geodetic irredundant sets*. The concept was formally defined in a different, more technical language, see the preliminaries below. In [20] graphs  $G$  with  $\text{gp}(G) \in \{2, n(G) - 1, n(G)\}$  were characterized and several additional results about the general position number were deduced. The term general position problem was coined in [12], where different general upper and lower bounds on the gp-number are proved. In the same paper it is demonstrated that in a block graph the set of simplicial vertices forms a gp-set and that the problem is NP-complete in the class of all graphs. In the subsequent paper [13] the gp-number is determined for a large class of subgraphs of the infinite grid graph, for the infinite diagonal grid, and for Beneš networks.

In this paper we continue the investigation of general position sets in graphs. In the following section definitions and preliminary observations are listed. In Section 3 we prove a characterization of general position sets and demonstrate that some earlier results follow directly from the characterization. In the subsequent section we consider graphs of diameter 2. We prove that if  $G$  is such a graph, then  $\text{gp}(G)$  is the maximum of the clique number of  $G$  and the maximum order of an induced complete multipartite subgraph of the complement of  $G$ . In the case of cographs the latter invariant can be replaced by the independence number. As a consequence,  $\text{gp}(G)$  of a cograph  $G$  is polynomial. Moreover, we determine a formula for  $\text{gp}(G)$  for graphs with at least one universal vertex. In Section 5 we consider bipartite graphs and their complements. If  $G$  is bipartite, then  $\text{gp}(G) \leq \alpha(G)$  with equality if  $\text{diam}(G) \in \{2, 3\}$ . We prove a formula for the general position number of the complement of a bipartite graph and simplify it for the complements of trees, of grids, and of hypercubes. In particular,  $\text{gp}(\overline{T}) = \max\{\alpha(T), \Delta(T) + 1\}$  for a tree  $T$ .

## 2 Preliminaries

Let  $G$  be a connected graph and  $u, v \in V(G)$ . The *distance*  $d_G(u, v)$  between  $u$  and  $v$  is the minimum number of edges on a  $u, v$ -path. The maximum distance between all pairs of vertices of  $G$  is the *diameter*  $\text{diam}(G)$  of  $G$ . A  $u, v$ -path of length  $d_G(u, v)$  is called an  $u, v$ -*geodesic*. The *interval*  $I_G[u, v]$  between vertices  $u$  and  $v$  of a graph  $G$  is the set of vertices  $x$  such that there exists a  $u, v$ -geodesic which contains  $x$ . For  $S \subseteq V(G)$  we set  $I_G[S] = \bigcup_{u, v \in S} I_G[u, v]$ . To simplify the writing, we may omit the index  $G$  in the above notation provided that  $G$  is clear from the context.

A set of vertices  $S \subseteq V(G)$  is a *general position set* of  $G$  if no three vertices of  $S$  lie on a common shortest path. A gp-set is thus a largest general position set. Call a vertex  $v \in T \subseteq V(G)$  to be an *interior vertex* of  $T$ , if  $v \in I[T - \{v\}]$ . Now,  $T$  is a general position set if and only if  $T$  contains no interior vertices. In this way general position sets were introduced in [20] under the name geodetic irredundant sets.

The set  $S$  is *convex* in  $G$  if  $I[S] = S$ . The *convex hull*  $H(S)$  of  $S$  is the smallest convex set that contains  $S$ , and  $S$  is a *hull set* of  $G$  if  $H(S)$  is the whole vertex set of  $G$ . A smallest hull set is a *minimum hull set* of  $G$ , its cardinality is the *hull number*  $h(G)$  of  $G$ . A hull set  $S$  in a graph  $G$  is a *minimal hull set* if no proper subset of  $S$  is a hull set of  $G$ . The number of vertices in a largest minimal hull set of  $G$  is its *upper hull number*  $h^+(G)$ . It is clear that in any graph  $G$ , every minimum hull set is also minimal, therefore  $h(G) \leq h^+(G)$ . The following fact is obvious.

**Observation 2.1** *Let  $G$  be a connected graph,  $S$  a minimal hull set,  $x, y \in S$ , and  $P$  a  $x, y$ -geodesic. If  $z \in P$ , where  $z \neq x, y$ , then  $z \notin S$ .*

It follows from this observation that every minimal hull set is a general position set. Consequently,

$$2 \leq h(G) \leq h^+(G) \leq \text{gp}(G) \leq n(G),$$

where  $n(G) = |V(G)|$ . The related difference can be arbitrary large though. For instance, if  $n \geq 2$ , then  $h^+(K_{n,n}) = 2$  and  $\text{gp}(K_{n,n}) = n$ .

With respect to convexities we mention the following parallel concept to the general position number, where in the definition of the interior vertex we replace ‘‘I’’ with ‘‘H’’. More precisely, the *rank* of a graph  $G$  is the cardinality of a largest set  $S$  such that  $v \notin H(S - \{v\})$  for every  $v \in S$ , see [9]. Actually, the graph rank can be studied for any convexity, cf. [21], the one defined here is the rank w.r.t. the geodesic convexity.

If the open neighborhood  $N(x)$  of a vertex  $x \in V(G)$  induces a complete graph, then  $x$  is called *simplicial*. In other words,  $x$  is simplicial if and only if  $N(x)$  is convex in  $G$ . The subgraph of  $G$  induced by  $S \subseteq V(G)$  is denoted by  $G[S]$ . The order of a largest complete subgraph of a graph  $G$  is  $\omega(G)$  and the order of

its largest independent set is  $\alpha(G)$ . A vertex of degree  $n(G) - 1$  is *universal*. The complement of  $G$  will be denoted with  $\bar{G}$ . The *join*  $G + H$  of graphs  $G$  and  $H$  is the graph obtained by first taking the disjoint union of  $G$  and  $H$ , and then adding all possible edges between vertices of  $G$  and  $H$ . Set finally  $[n] = \{1, \dots, n\}$ , where  $n \in \mathbb{N}$ .

### 3 The characterization

In this section we characterize general position sets in graphs. For this sake the following concepts are needed.

Let  $G$  be a connected graph,  $S \subseteq V(G)$ , and  $\mathcal{P} = \{S_1, \dots, S_p\}$  a partition of  $S$ . Then  $\mathcal{P}$  is *distance-constant* if for any  $i, j \in [p]$ ,  $i \neq j$ , the value  $d(u, v)$ , where  $u \in S_i$  and  $v \in S_j$  is independent of the selection of  $u$  and  $v$ . (We note that in [9, p. 331] the distance-constant partition is called “distance-regular”, but we decided to rather avoid this naming because distance-regular graphs form a well-established term, cf. [4].) If  $\mathcal{P}$  is a distance-constant partition, and  $i, j \in [p]$ ,  $i \neq j$ , then let  $d(S_i, S_j)$  be the distance between the sets  $S_i$  and  $S_j$ , that is, the distance between two arbitrary vertices pairwise from them. Finally, we say that a distance-constant partition  $\mathcal{P}$  is *in-transitive* if  $d(S_i, S_k) \neq d(S_i, S_j) + d(S_j, S_k)$  holds for arbitrary pairwise different  $i, j, k \in [p]$ . With these concepts in hand we can characterize general position sets as follows.

**Theorem 3.1** *Let  $G$  be a connected graph. Then  $S \subseteq V(G)$  is a general position set if and only if the components of  $G[S]$  are complete subgraphs, the vertices of which form an in-transitive, distance-constant partition of  $S$ .*

**Proof.** Let  $S \subseteq V(G)$ . Let  $G_1, \dots, G_p$  be the components of  $G[S]$  and let  $\mathcal{P}$  be the partition of  $S$  induced by the vertex sets of the components, that is,  $\mathcal{P} = \{V(G_1), \dots, V(G_p)\}$ . To simplify the notation let  $V_i = V(G_i)$  for  $i \in [p]$ , so that  $\mathcal{P} = \{V_1, \dots, V_p\}$ .

Suppose first that  $G_1, \dots, G_p$  are complete subgraphs of  $G$  and that  $\mathcal{P}$  forms an in-transitive, distance-constant partition of  $S$ . We claim that  $S$  is a general position set. Assume by the way of contradiction that  $S$  contains three vertices  $u, v, w$ , such that  $v$  lies in  $I(u, w)$ . Since  $G_1, \dots, G_p$  are complete subgraphs,  $u$  and  $w$  lie in different parts of  $\mathcal{P}$ , say  $u \in V_i$  and  $w \in V_j$ , where  $i, j \in [p]$ ,  $i \neq j$ . Since  $\mathcal{P}$  is distance-constant, we infer that  $v \notin V_i$  as well as  $v \notin V_j$ . Therefore,  $v \in V_k$  for some  $k \in [p]$ ,  $k \neq i, j$ . But then  $d(V_i, V_j) = d(V_i, V_k) + d(V_k, V_j)$ , a contradiction with the assumption that  $\mathcal{P}$  is an in-transitive partition.

Conversely, let  $S$  be a general position set. If  $G_i$  is not complete for some  $i \in [p]$ , then  $G_i$  contains an induced  $P_3$ , say  $uvw$ . But this means that  $S$  is not a general

position set. Hence  $G_i$  is complete for every  $i \in [p]$ . Next, let  $u, v \in V_i$  and  $w \in V_j$  for  $i, j \in [p]$ ,  $i \neq j$ . Since  $G_i$  and  $G_j$  are complete,  $uv \in E(G)$  and hence  $|d(u, w) - d(v, w)| \leq 1$ . Moreover, neither  $v$  can be on a  $u, w$ -geodesic, nor  $u$  lies on a  $v, w$ -geodesic and consequently  $d(u, w) = d(v, w)$ . Since  $u, v$  are arbitrary vertices of  $G_i$  and  $w$  an arbitrary vertex of  $G_j$ , this means that  $d(V_i, V_j) = d(u, w) = d(v, w)$  is well defined. Consequently,  $\mathcal{P}$  is a distance-constant partition. Finally,  $\mathcal{P}$  must also be an in-transitive partition. If this would not be the case, then there would exist components (complete subgraphs)  $G_i, G_j$ , and  $G_k$  of  $\mathcal{P}$  such that  $d(V_i, V_k) = d(V_i, V_j) + d(V_j, V_k)$ . But this means that if  $u \in V_i, v \in V_j$ , and  $w \in V_k$ , then  $v$  would lie on a  $u, w$ -geodesic. This contradiction implies that  $\mathcal{P}$  is indeed an in-transitive partition.  $\square$

Theorem 3.1 in particular implies some earlier results. First, it immediately implies [12, Lemma 3.5] asserting that the set of simplicial vertices of a given graph lies in a general position. Also, setting  $d(e, f) = \min\{d(u, x), d(u, y), d(v, x), d(v, y)\}$  for edges  $e = uv$  and  $f = xy$  of a graph  $G$ , we obtain:

**Corollary 3.2** [12, Proposition 4.4] *Let  $G$  be a graph with  $\text{diam}(G) \geq 2$ . If  $F \subseteq E(G)$  is such that  $d(e, e') = \text{diam}(G)$ ,  $e, e' \in F$ ,  $e \neq e'$ , then  $\text{gp}(G) \geq 2|F|$ .*

**Proof.** For  $e \in F$  let  $x_e$  and  $y_e$  be the end-vertices of  $e$ . Then, having in mind that  $\text{diam}(G) \geq 2$ , it is straightforward to see that  $\{\{x_e, y_e\} : e \in F\}$  forms an in-transitive, distance-constant partition.  $\square$

## 4 Graphs of diameter 2

Graphs of diameter 2 form one of the most interesting classes of graph theory, after all, as it is well-known, almost all graphs have diameter 2. They are still extensively investigated, the papers [2, 3, 22] are examples of recent developments on these graphs. In this section we are going to use Theorem 3.1 in the case of graphs of diameter 2. For this sake we denote with  $\eta(G)$  the maximum order of an induced complete multipartite subgraph of  $\overline{G}$ . Note that  $K_n$  is complete multipartite, and that  $\eta(K_n) = 1$  and  $\omega(K_n) = n$ .

**Theorem 4.1** *If  $\text{diam}(G) = 2$ , then  $\text{gp}(G) = \max\{\omega(G), \eta(G)\}$ .*

**Proof.** Since the vertices of an arbitrary complete subgraph of a graph  $G$  form a general position set of  $G$ , we have  $\text{gp}(G) \geq \omega(G)$ . Suppose  $H$  is a complete multipartite subgraph of  $\overline{G}$ . Then in  $G$  the vertices of  $H$  induce a disjoint union of complete graphs. Since  $\text{diam}(G) = 2$ , the vertices of these complete subgraphs

clearly form an in-transitive, distance-constant partition. Hence by Theorem 3.1,  $\text{gp}(G) \geq \eta(G)$ . Therefore,  $\text{gp}(G) \geq \max\{\omega(G), \eta(G)\}$ .

Let now  $S$  be a set of vertices in a general position in  $G$ . Then by Theorem 3.1 the components of  $G[S]$  are complete subgraphs, the vertices of which form an in-transitive, distance-constant partition of  $S$ . If there is only one such component, then  $|S| \leq \omega(G)$ , and if there are at least two components, then  $|S| \leq \eta(G)$ . Hence,  $\text{gp}(G) \leq \max\{\omega(G), \eta(G)\}$ .  $\square$

If  $P$  is the Petersen graph, then  $\omega(P) = 2$  and  $\eta(P) = 6$ , hence by Theorem 4.1 we have  $\text{gp}(P) = 6 = \eta(P)$ . Let further  $G_{n,k}$ , be the graph obtained from  $K_n$  and one more vertex that is adjacent to  $k+1$  vertices of  $K_n$ , where  $n \geq 3$  and  $1 \leq k+1 < n$ . Then  $\omega(G) = n$  and  $\eta(G) = n - k$ , so that  $\text{gp}(G_{n,k}) = n = \omega(G)$ . These examples show that the values from the maximum in Theorem 4.1 are independent.

Cographs form an important class of graphs that is still extensively investigated, [1, 19] is a selected couple of recent studies. Recall that  $G$  is a *cograph* if  $G$  contains no path  $P_4$  as an induced subgraph. These graphs were independently introduced several times and can be characterized in many different ways, see [5]. In particular, a graph is a cograph iff it can be obtained from  $K_1$  by means of the disjoint union and join of graphs. Note that this implies that every connected cograph of order at least 2 is the join of at least two smaller connected cographs. This implies that  $\text{diam}(G) \leq 2$  holds for any connected cograph  $G$ .

**Theorem 4.2** *If  $G$  be a connected cograph, then  $\text{gp}(G) = \max\{\omega(G), \alpha(G)\}$ .*

**Proof.** If  $G = K_n$ , then  $\text{gp}(K_n) = n = \max\{\omega(K_n), \alpha(K_n)\}$ . Hence assume in the rest that  $G$  is a connected cograph with  $\text{diam}(G) = 2$ .

We claim that  $\eta(G) = \alpha(G)$  and proceed by induction on the order of  $G$ . The assertion is clear if  $n(G) = 3$  (in other words, for  $G = P_3$ ). Assume now that  $G$  is a connected cograph with  $\text{diam}(G) = 2$  and  $n(G) \geq 4$ . Then  $G = G_1 + \dots + G_k$ , where  $k \geq 2$  and  $G_i$ ,  $i \in [k]$ , are connected cographs. Since for arbitrary graphs  $X$  and  $Y$  we have  $\alpha(X + Y) = \max\{\alpha(X), \alpha(Y)\}$  and  $\eta(X + Y) = \max\{\eta(X), \eta(Y)\}$ , we get, by the induction assumption, that

$$\eta(G) = \max\{\eta(G_1), \dots, \eta(G_k)\} = \max\{\alpha(G_1), \dots, \alpha(G_k)\} = \alpha(G).$$

The result now follows from Theorem 4.1.  $\square$

If  $G$  is a cograph, then  $\alpha(G)$  and  $\omega(G)$  can be determined in polynomial time, cf. [5, 15]. Hence Theorem 4.2 implies that the general position problem is polynomial on connected cographs. Since the general position function of a graph is clearly additive on its components, the general position problem is thus polynomial on all cographs.

Suppose that  $G$  is not complete and that it contains at least one universal vertex. Then  $\text{diam}(G) = 2$  and Theorem 4.1 applies. For this situation we get:

**Corollary 4.3** *Let  $G$  be a non-complete graph,  $U \neq \emptyset$  the set of its universal vertices, and let  $U' = V(G) - U$ . Then*

$$\text{gp}(G) = \max\{|U| + \omega(G[U']), \eta(G[U'])\}.$$

**Proof.** Since  $U$  contains universal vertices, every largest complete subgraph of  $G$  contains  $U$ . Hence  $\omega(G) = |U| + \omega(G[U'])$ . In  $\overline{G}$  every vertex from  $U$  is isolated. Hence every induced complete multipartite subgraph of  $\overline{G}$  with at least two parts contains only vertices from  $U'$ . Thus  $\eta(G) = \eta(G[U'])$ .  $\square$

Every graph  $G$  can be represented as the graph obtained from  $K_{n(G)}$  by removing appropriate edges. To present examples how Corollary 4.3 can be applied, let us use the notation  $K_n - E(H)$  for the graph obtained from  $K_n$  in which we consider  $H$  as its subgraph, and then deleting the edges of  $H$  from  $K_n$ . Then we have the following formulas that can be easily deduced from Corollary 4.3, where  $W_k$  denotes the wheel graph of order  $k$ , that is, the graph obtained from  $C_{k-1}$  by adding an additional vertex and making it adjacent to all the vertices of  $C_{k-1}$ .

- $\text{gp}(K_n - E(K_k)) = \max\{k, n - k + 1\}$ , where  $2 \leq k < n$ .
- $\text{gp}(K_n - E(K_{1,k})) = \max\{k + 1, n - 1\}$ , where  $2 \leq k < n$ .
- $\text{gp}(K_n - E(P_k)) = \max\{3, n - k + \lceil \frac{k}{2} \rceil\}$ , where  $3 \leq k < n$ .
- $\text{gp}(K_n - E(K_{r,s})) = \max\{r + s, n - r\}$ , where  $2 \leq r \leq s$  and  $r + s < n$ .
- $\text{gp}(K_n - E(W_k)) = \max\{3, n - k + \lfloor \frac{k-1}{2} \rfloor\}$ , where  $5 \leq k < n$ .
- $\text{gp}(K_n - E(C_k)) = \begin{cases} \max\{3, n - k + \lfloor \frac{k}{2} \rfloor\}, & 5 \leq k < n; \\ \max\{4, n - 2\}, & k = 4. \end{cases}$

## 5 Bipartite graphs and their complements

For bipartite graphs we have the following result.

**Theorem 5.1** *If  $G$  is a connected, bipartite graph on at least 3 vertices, then  $\text{gp}(G) \leq \alpha(G)$ . Moreover, if  $\text{diam}(G) \in \{2, 3\}$ , then  $\text{gp}(G) = \alpha(G)$ .*

**Proof.** If  $G$  is a connected graph, then  $\text{gp}(G) = 2$  if and only if  $G = P_n$  ( $n \geq 2$ ) or  $G = C_4$ , see [20, Theorem 2.10]. Thus  $\text{gp}(G) \leq \alpha(G)$  holds in these cases. (In the case of  $P_n$  the inequality holds because we have assumed that  $n \geq 3$  and so  $\alpha(P_n) \geq 2$ .) In the rest we may thus assume that  $G$  is a connected, bipartite graph that is neither a path nor  $C_4$ , so  $\text{gp}(G) \geq 3$ .

Let  $S$  be a gp-set of  $G$  and let  $S_1, \dots, S_k$  be the components of  $G[S]$ . As  $\text{gp}(G) \geq 3$  and  $G$  is bipartite, Theorem 3.1 implies that  $k \geq 2$ . Also, since  $G$  is bipartite,  $|S_i| \in [2]$  for  $i \in [k]$ . We claim that actually  $|S_i| = 1$  for every  $i \in [k]$ . Suppose on the contrary that, w.l.o.g.,  $S_1 = \{u, v\}$ . Let  $w \in S_2$ . Then, since  $uw \in E(G)$  and  $G$  is bipartite,  $|d(u, w) - d(v, w)| = 1$ , which means that either  $v$  lies on a  $u, w$ -geodesic or  $u$  lies on a  $v, w$ -geodesic. This contradiction proves the claim, that is,  $S$  is an independent set. We conclude that  $\text{gp}(G) \leq \alpha(G)$ .

Assume now that  $\text{diam}(G) = 2$ . The complement of an independent set  $I$  of a graph  $G$  induces the complete graph  $K_{|I|}$  which is an instance of a complete multipartite graph. Hence  $\alpha(G) \leq \eta(G)$  and Theorem 4.1 implies that  $\alpha(G) \leq \text{gp}(G)$  holds for a graph  $G$  of diameter 2.

Assume finally that  $\text{diam}(G) = 3$ . Then we recall from [12, Corollary 4.3] that every independent set is a general position set. Hence  $\alpha(G) \leq \text{gp}(G)$  holds also in this case.  $\square$

If  $G$  is bipartite,  $\text{gp}(G)$  can be arbitrary smaller than  $\alpha(G)$ . Consider first the paths  $P_n$ ,  $n \geq 2$ , for which we have  $\text{gp}(P_n) = 2$  and  $\alpha(P_n) = \lceil n/2 \rceil$ . We also note that none of the two assertions of Theorem 5.1 need hold if  $G$  is not bipartite. To see this, consider again the Petersen graph  $P$ . (Of course,  $\text{diam}(P) = 2$ .) As already noticed,  $\text{gp}(P) = 6$ , while  $\alpha(P) = 4$ . For a corresponding example of diameter 3 just add a pendant vertex to  $P$ .

We now turn our attention to complements of bipartite graphs for which some preparation is needed. If  $G = (V(G), E(G))$  is a bipartite graph and  $V(G) = A \cup B$  is its bipartition, then we will write  $G$  as triple  $(A, B, E(G))$ . If  $G = (A, B, E(G))$  is a bipartite graph, then let  $M_G$  be the set of vertices of largest possible degree, more precisely,

$$M_G = \{u \in A : \deg(u) = |B|\} \cup \{u \in B : \deg(u) = |A|\}.$$

Let  $\psi(G)$  be the maximum order of an induced complete bipartite subgraph of  $G$ . Note that if  $G$  is a bipartite, but not complete bipartite, then  $\text{diam}(\overline{G}) \leq 3$ . Now we can formulate:

**Theorem 5.2** *If  $G = (A, B; E(G))$  is a bipartite graph, then*

$$\text{gp}(\overline{G}) = \begin{cases} n(G), & \text{diam}(\overline{G}) \in \{1, \infty\}; \\ \max\{\alpha(G), \psi(G)\}, & \text{diam}(\overline{G}) = 2; \\ \max\{\alpha(G), \psi(G \setminus (M_G \cap A)), \psi(G \setminus (M_G \cap B)), |M_G|\}, & \text{diam}(\overline{G}) = 3. \end{cases}$$



**Proof.** Let  $G = (A, B; E(G))$  be a bipartite graph. Then  $\overline{G}$  is disconnected iff  $G$  is complete bipartite. In this case we have  $\text{diam}(\overline{G}) = \infty$  and  $\overline{G}$  is a disjoint union of  $K_{|A|}$  and  $K_{|B|}$ . Therefore,  $\text{gp}(\overline{G}) = |A| + |B| = n(G)$ . Further,  $\text{diam}(\overline{G}) = 1$  iff  $G$  is edge-less, hence again  $\text{gp}(\overline{G}) = n(G)$ . If  $\text{diam}(\overline{G}) = 2$ , then by Theorem 4.1 we have  $\text{gp}(\overline{G}) = \max\{\omega(\overline{G}), \eta(\overline{G})\}$ . Since  $\omega(\overline{G}) = \alpha(G)$  and  $\eta(\overline{G}) = \psi(G)$ , the assertion for the diameter 2 follows.

In the rest we may thus assume  $|A| \geq 2$ ,  $|B| \geq 2$ , and  $\text{diam}(\overline{G}) = 3$ . Note that if  $u \in M_G \cap A$ , then  $u$  has no neighbor in  $B$  and if  $u \in M_G \cap B$ , then  $u$  has no neighbor in  $A$ . Consequently, in  $\overline{G}$  two vertices are at distance 3 if and only if one lies in  $M_G \cap A$  and the other in  $M_G \cap B$ . Since  $\text{diam}(\overline{G}) = 3$  it follows that  $M_G \cap A \neq \emptyset$  and  $M_G \cap B \neq \emptyset$ .

Consider a set  $T$  in general position in  $\overline{G}$  and set  $T_A = T \cap A$ ,  $T_B = T \cap B$ . If  $T$  has at least one vertex in  $M_G \cap A$ , say  $x$ , and at least one vertex in  $M_G \cap B$ , say  $y$ , then every vertex from  $(A \cup B) \setminus M_G$  lies on a  $x, y$ -geodesic. Therefore,  $T \subseteq M_G$ . This means that  $|T| \leq |M_G|$ . Suppose next that  $T \cap (M_G \cap A) = \emptyset$ . If there is an edge between a vertex from  $T_A$  and a vertex from  $T_B$ , then  $T$  must induce a clique and hence  $|T| \leq \omega(\overline{G}) = \alpha(G)$ . Otherwise, in view of Theorem 3.1, the vertices from  $T_A$  and from  $T_B$  are pairwise at distance 2. But then  $T$  induces a complete bipartite graph in  $G \setminus (M_G \cap A)$  and therefore  $|T| \leq \psi(G \setminus (M_G \cap A))$ . Analogously, if  $T \cap (M_G \cap B) = \emptyset$  then we get that  $|T| \leq \alpha(G)$  or  $|T| \leq \psi(G \setminus (M_G \cap B))$ . In summary,

$$\text{gp}(\overline{G}) \leq \max\{\alpha(G), \psi(G \setminus (M_G \cap A)), \psi(G \setminus (M_G \cap B)), |M_G|\}.$$

On the other hand, we clearly have  $\text{gp}(\overline{G}) \geq \omega(\overline{G}) = \alpha(G)$ . Note next that each vertex from  $M_G$  is simplicial in  $\overline{G}$  and consequently  $\text{gp}(\overline{G}) \geq |M_G|$ . Finally, an induced complete bipartite graph in  $G \setminus (M_G \cap A)$  as well as in  $G \setminus (M_G \cap B)$  corresponds to a disjoint union of cliques in  $\overline{G}$  which form an in-transitive, distance constant partition (with constant 2). Hence we also have  $\text{gp}(\overline{G}) \geq \psi(G \setminus (M_G \cap A))$  and  $\text{gp}(\overline{G}) \geq \psi(G \setminus (M_G \cap B))$ .  $\square$

If  $n \geq 5$ , then  $\text{diam}(\overline{P}_n) = 2$  and for  $n \geq 7$  we have  $\psi(P_n) = 3 < \lceil n/2 \rceil = \alpha(P_n)$ . Let next  $G_n$  be a bipartite graph with the bipartition  $A = \{x_1, \dots, x_n, a_1, a_2\}$  and  $B = \{y_1, \dots, y_n, b_1, b_2\}$ , where vertices  $(A \cup B) \setminus \{a_1, a_2, b_1, b_2\}$  induce a complete bipartite graph  $K_{n,n}$ , and the remaining edges of  $G_n$  are  $a_1y_1$ ,  $a_2y_2$ ,  $b_1x_1$ , and  $b_2x_2$ . For  $n \geq 3$  we have  $\psi(G_n) = 2n > n + 2 = \alpha(G_n)$ . As  $\text{diam}(\overline{G}_n) = 2$ , these two examples demonstrate that the values in Theorem 5.2 are independent in the case  $\text{diam}(\overline{G}) = 2$ .

Let  $H(n, m, s, t)$ ,  $n, m, s, t \geq 2$ , be a bipartite graph with the bipartition  $A = A_1 \cup A_2 \cup A_3$  and  $B = B_1 \cup B_2 \cup B_3$ , where  $|A_1| = n$ ,  $|B_1| = m$ ,  $|A_2| = |B_3| = s$ , and  $|A_3| = |B_2| = t$ . The vertices in  $(A_1 \cup A_2) \cup (B_1 \cup B_2)$  induce  $K_{n+s, m+t}$ , the

vertices in  $A_2 \cup B_3$  induce  $K_{s,s}$  and the vertices in  $A_3 \cup B_2$  induce  $K_{t,t}$ . These are all the edges of  $H(n, m, s, t)$ . Assume that  $n \leq m$  and set  $H = H(n, m, s, t)$ . Then  $M_H \cap A = A_2$  and  $M_H \cap B = B_2$  and we have:

- $|M_H| = s + t$ ,
- $\alpha(H) = m + s + t$ ,
- $\psi(H \setminus A_2) = \max\{m + n + t, 2t\}$ , and
- $\psi(H \setminus B_2) = \max\{m + n + s, 2s\}$ .

It is now clear that the parameters  $n$ ,  $s$ , and  $t$  can be selected such that exactly one of  $\alpha(H)$ ,  $\psi(H \setminus A_2)$ , and  $\psi(H \setminus B_2)$  is strictly larger than the other two (as well as bigger than  $|M_H|$ ). Note finally that  $\text{diam}(\overline{H}) = 3$ .

To see that  $|M_G|$  can be strictly larger than the other three terms from Theorem 5.2 when  $\text{diam}(\overline{G}) = 3$ , consider the edge deleted complete bipartite graph  $K = K_{n,n} - e$ . Then  $\text{diam}(\overline{K}) = 3$ , and  $|M_K| = 2n - 2$ .

In the rest we present the general position number for some natural families of bipartite complements.

In [20, Theorem 2.5] and in [12, Corollary 3.7] it was independently observed that the gp-number of a tree  $T$  is the number of its leaves. (Actually, the set of leaves is the unique gp-set of  $T$ .) For the complements of trees we have:

**Corollary 5.3** *It  $T$  is a tree, then  $\text{gp}(\overline{T}) = \max\{\alpha(T), \Delta(T) + 1\}$ .*

**Proof.** Let  $T = (A, B; E(T))$ .

If  $\text{diam}(T) \leq 2$ , then it is clear that  $T$  is a star. Hence  $\text{diam}(\overline{T}) = \infty$ . By Theorem 5.2 we thus have  $\text{gp}(\overline{T}) = n(T) = \Delta(T) + 1$ .

If  $\text{diam}(T) = 3$ , then it is straightforward to see that  $T$  is isomorphic to a double star. Therefore,  $|M_T \cap A| = 1$  and  $|M_T \cap B| = 1$ . Thus  $|M_T| = 2$ ,  $\alpha(T) = |n(T)| - 2$ , and  $\psi(T \setminus (M_T \cap A)) = |A|$ ;  $\psi(T \setminus (M_T \cap B)) = |B|$ . Since  $|n(T)| \geq 4$  and  $|A| \geq 2$  and  $|B| \geq 2$ , we have  $\text{gp}(\overline{T}) = |n(T)| - 2 = \alpha(T)$ .

Let finally  $\text{diam}(T) \geq 4$ . Then from [18, Lemma 2.2] we deduce that  $\text{diam}(\overline{T}) = 2$ . Since  $T$  has no cycles, we have that  $\psi(T)$  is the order of a maximum induced star, that is,  $\psi(T) = \Delta(T) + 1$ . Thus  $\text{gp}(\overline{T}) = \max\{\alpha(T), \Delta(T) + 1\}$ .  $\square$

The *Cartesian product*  $G \square H$  of graphs  $G$  and  $H$  is defined as follows.  $V(G \square H) = V(G) \times V(H)$ . As for the edges,  $(g, h)$  and  $(g', h')$  are adjacent if (i)  $g = g'$  and  $hh' \in E(H)$ , or (ii)  $h = h'$  and  $gg' \in E(G)$ . (See the book [8].) In [13, Theorem 3.1] it was proved that  $\text{gp}(P_\infty \square P_\infty) = 4$ , where  $P_\infty$  is the two-way infinite path. It follows from this result that if  $n, m \geq 3$ , then  $\text{gp}(P_n \square P_m) = 4$  as well. For the complements of these grids we have:

**Corollary 5.4** *If  $n, m \geq 2$ , then*

$$gp(\overline{P_n \square P_m}) = \begin{cases} 4, & n = m = 2; \\ \lceil \frac{n}{2} \rceil \lceil \frac{m}{2} \rceil + \lfloor \frac{n}{2} \rfloor \lfloor \frac{m}{2} \rfloor, & \text{otherwise.} \end{cases}$$

**Proof.**  $\overline{P_2 \square P_2} = \overline{C_4}$ , hence the assertion holds for  $n = m = 2$ .  $\overline{P_2 \square P_3}$  is the graph obtained from the 6-cycle  $v_1v_2 \dots v_6$  by inserting into it the edges  $v_1v_3$  and  $v_4v_6$ . The assertion then follows immediately.

Suppose in the rest that  $n, m \geq 3$  and set  $G = P_n \square P_m$ . Applying [18, Lemma 2.2] once more we get that  $\text{diam}(\overline{G}) = 2$ . Hence by Theorem 5.2 we see that  $gp(G) = \max\{\alpha(G), \psi(G)\}$ . Since the only induced complete bipartite subgraphs in  $P_n \square P_m$  are isomorphic to  $K_{2,2}$  or  $K_{1,r}$ ,  $r \in [4]$ , we get  $gp(G) = \alpha(G)$ . The conclusion of the theorem now follows because  $\alpha(P_n \square P_m) = \lceil \frac{n}{2} \rceil \lceil \frac{m}{2} \rceil + \lfloor \frac{n}{2} \rfloor \lfloor \frac{m}{2} \rfloor$ , a result that can be deduced from [10, Theorem 4.2].  $\square$

Using parallel arguments as in the proof of Corollary 5.4 we also get the general position number of the complements of hypercubes.

**Corollary 5.5** *If  $k \geq 3$ , then  $gp(\overline{Q_k}) = 2^{k-1}$ .*

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