Cover-incomparability graphs of posets

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Abstract

Cover-incomparability graphs (C-I graphs, for short) are introduced, whose edge-set is the union of edge-sets of the incomparability and the cover graph of a poset. Posets whose C-I graphs are chordal (resp. distance-hereditary, Ptolemaic) are characterized in terms of forbidden isometric subposets, and a general approach for studying C-I graphs is proposed. Several open problems are also stated.

Keywords: poset, underlying graph, transit function, chordal graph, distance-hereditary graph, claw.

1 Introduction

There are three standard ways in which one can associate a graph to a given poset P. In all the cases the vertex sets of the associated graphs consist of the points of P. In the *cover graph* of P points x and y are adjacent if either x covers y or y covers x. Points x and y are adjacent in the *compararability graph* if they are comparable in P. The *incomparability graph* is the complement of the comparability graph. For more information on the interrelation between posets and graphs see the survey paper [16] as well as [15]. We also refer to [8] where two additional graphs are associated to a poset.

In this paper we introduce a new graph that can be associated to a poset P, we call it the cover-incomparability graph (C-I graph) of P. This is the graph in which the edge set is the union of the edge sets of the corresponding cover graph and the corresponding incomparability graph. Note that this is the only nontrivial way to construct a new associated graph as unions and/or intersections of the edge sets of the three standard associated graphs. Our motivation for C-I graphs comes from the theory of transit functions that can in particular be studied on posets.

The notion of transit functions was introduced by Mulder about ten years ago and finally written up in [12]. The central idea of this concept is to generalize the interval function of a graph [11], and to study how to move around in discrete structures. Instances of this theory include the all-paths transit function [3] and the induced path transit function [4, 10]. For a survey on path transit functions on graphs we refer to [5].

The study of transit functions on posets has been initiated in [9] where in particular the standard poset transit function, the meet/join semilattice transit function, and the lattice transit function are introduced and studied. It turns out that the underlying graph of the standard transit function on a poset P is just the C-I graph of P, hence our main motivation. We hope, however, that C-I graphs will be useful in some other context of poset theory.

We proceed as follows. In the rest of this section some definitions on posets and graphs are recalled. In the subsequent section the cover-incomparability graphs are introduced and their basic properties observed. It is also proved that a given class of posets has a characterization with forbidden isometric subposets, provided that their C-I graphs belong to a class of graphs having a forbidden induced subgraphs characterization. In the rest of the paper we give several explicit such characterizations. In Section 3 we give a forbidden isometric subposet characterization of posets whose C-I graphs are chordal, while in Section 4 similar theorems for posets whose C-I graphs are distance-hereditary and Ptolemaic are given. In Section 5 we introduce a relation $\prec$ on C-I graphs that relates two graphs with respect to the corresponding forbidden isometric subposet characterizations. Finally in the last section several questions are posed.

Let $P = (V, \leq)$ be a poset. If $u \leq v$ but $u \neq v$, then we write $u < v$. If u and v are in V, then v *covers* u in P if $u < v$ and there is no w in V with $u < w < v$. If $u \leq v$ we will sometimes say that u is *below* v, and that v is *above* u. Let V' be a nonempty subset of V. Then there is a natural poset $Q = (V', \leq')$, where $u \leq' v$ if and only if $u \leq v$ for any $u, v \in V'$. The poset Q is called a *subposet* of P and its notation is simplified to $Q = (V', \leq)$. If, in addition, together with any two comparable elements u and v of Q, a chain of shortest length between u and v of P is also in Q, we say that Q is an *isometric subposet*. For the purposes of this paper we will say that P is called *Q-free* if P has no **isometric** subposet isomorphic to Q. For other definitions on posets and related concepts we refer to [6].

A graph G is *chordal* if it does not contain induced cycles of length at least 4. A *distance-hereditary graph* is a connected graph in which every induced path is a shortest path. Hence the distance-hereditary graphs are the graphs in which the geodesic convexity and the induced path convexity coincide. These graphs were characterized by Howorka [7] as the connected graphs without induced *long cycles* (cycles of length greater than four), the house, the domino, and the 3-fan as induced subgraphs, see Fig. 1.

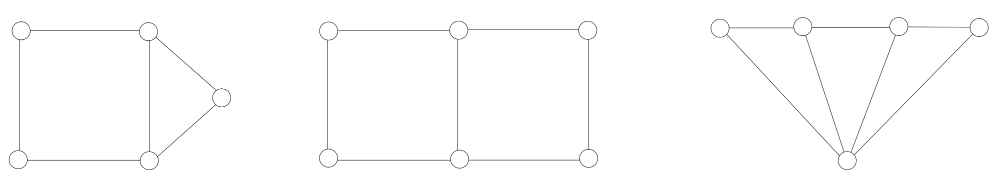

Figure 1: House, domino, and 3-fan

Finally, *Ptolemaic graphs* are distance-hereditary graphs without induced 4-cycles. In other words, Ptolemaic graphs are chordal distance-hereditary graphs.

2 Cover-incomparability graphs

A *transit function* on a non empty set V is a function $T : V \times V \to 2^V$ satisfying the following transit axioms:

(t1) $u \in T(u, v)$ for any u and $v \in V$.

(t2) $T(u, v) = T(v, u)$ for all u and $v \in V$.
(t3) $T(u, u) = \{u\}$ for all $u \in V$.

The *underlying graph* G_T of a transit function T on a set V is the graph with vertex set V, where distinct u and v in V are joined by an edge if $\mid T(u, v) \mid= 2$.

For a poset $P = (V, \leq)$, the *standard poset transit function* $T_P : V \times V \to 2^V$ is defined in the following way:

(i) If x and y are incomparable, then $T_P(x, y) = \{x, y\}$.
(ii) If $x \leq y$, then $T_P(x, y) = \{z \mid x \leq z \leq y\}$.
(iii) If $y \leq x$, then $T_P(x, y) = \{z \mid y \leq z \leq x\}$.

Clearly, T_P satisfies (t1)-(t3). In other words, T_P is a transit function.

Note that the underlying graph G_{T_P} of T is obtained from the cover graph of P by adding an edge between any pair of incomparable elements of P. Thus the edges of G_{T_P} are the union of the edges of the cover graph of P and the incomparability graph of P. Hence we say that G_{T_P} is the *cover-incomparability graph* (*C-I graph*) of P.

For instance, if P is a linear order, then the C-I graph is the cover graph of P and if P is an antichain then its C-I graph is the incomparability graph of P. The n-cube Q_n is the cover graph of the usual inclusion defined on subsets of an n-set. Since two subsets are incomparable if none is contained in the other, the C-I graph is obtained from Q_n by adding edges between each pair of vertices $b_1 \ldots b_n$ and $c_1 \ldots c_n$ for which there exist indices i and j such that $b_i = c_j = 0$ and $b_j = c_i = 1$.

We now collect some simple observations about C-I graphs that will be often implicitly used in the rest of the paper.

Lemma 2.1 *Let P be a poset. Then*

(i) the C-I graph of P is connected;
(ii) points of P that are independent in the C-I graph of P lie on a common chain;
(iii) an antichain of P corresponds to a complete subgraph in the C-I graph of P.

Recall that a poset P is *dual* to a poset Q if for any $x, y \in P$ the following holds: $x \leq y$ in P if and only if $y \leq x$ in Q. Then we have the following simple observation.

Lemma 2.2 *Let Q be the dual poset of P. Then G_{T_P} is isomorphic to G_{T_Q}.*

By this lemma we infer that in order to characterize a class of underlying graphs of transit functions of posets in terms of forbidden isometric subposets, in the list of forbidden subposets all subposets will appear in dual pairs (by agreement that in self-dual subposets the dual-pair consists of one poset). To shorten our presentation we shall only list one of the posets of every dual-pair.

We next state a general theorem that led us to the investigations in the rest of the paper. For it we need:

Lemma 2.3 *Let Q be an isometric subposet of a poset P. Then G_{T_Q} is isomorphic to the subgraph of G_{T_P} induced by the points of Q.*

Proof. Let H be the subgraph of G_{T_P} induced by the points of Q. Let u and v be arbitrary points of Q. Suppose u and v are adjacent in H. Note that this happens if and only if either u covers v (or vice versa) in P or u and v are incomparable in P. If u covers v in P, then u covers v also in Q, and so u and v are adjacent in G_{T_Q}. If they are incomparable in P, they are also incomparable in Q, and so again they are adjacent in G_{T_Q}. Now, suppose that u and v are not adjacent in H. Then $u \leq v$ but v does not cover u. Since Q is an isometric subposet of P, there exists a point w in Q, $w \neq u, v$, such that $u \leq w \leq v$, and so u and v are also not adjacent in G_{T_Q}. $\square$

We point out that Lemma 2.3 need not hold if Q is a subposet that is not isometric.

Theorem 2.4 *Let $\mathcal{G}$ be a class of graphs with a forbidden induced subgraphs characterization. Let*

$$\mathcal{P} = \{P \mid P \text{ is a poset with } G_{T_P} \in \mathcal{G}\}\,.$$

Then $\mathcal{P}$ has a forbidden isometric subposets characterization.

Proof. Let G be a forbidden induced subgraph for the class $\mathcal{G}$. Let $P \in \mathcal{P}$, then G is not an induced subgraph of G_{T_P}. By Lemma 2.3, P does not contain any isometric subposet P' that yields G in G_{T_P}. (Note that there may be no such subposet). Hence any such subposet P' is a forbidden subposet of P. Repeating the argument for all the forbidden subgraphs for $\mathcal{G}$ we find a list of forbidden isometric subposets $\{P_i\}_{i\in I}$ for $\mathcal{P}$.

We claim that $\mathcal{P}$ is characterized by forbidden isometric subposets $\{P_i\}_{i\in I}$. Let $P \in \mathcal{P}$. Then P contains no isometric subposet P_i, for otherwise G_{T_P} would contain a forbidden induced subgraph by Lemma 2.3. Conversely, suppose that P contains no isometric subposet P_i. Then by the construction, G_{T_P} contains no forbidden subgraph for $\mathcal{G}$. It follows that G_{T_P} is from $\mathcal{G}$ and hence P is from $\mathcal{P}$. $\square$

Note that in Theorem 2.4 $\{G_{T_P} \mid P \in \mathcal{P}\}$ will in general be a proper subclass of $\mathcal{G}$.

Theorem 2.4 leads us to the following question. For a given class of graphs $\mathcal{G}$ that has forbidden induced subgraphs characterization, determine the list of forbidden (isometric) subposets $\mathcal{P}$. This is the question that we follow in the next two sections.

3 Posets whose C-I graphs are chordal

In this section we prove the following theorem.

Theorem 3.1 *Let P be a poset. Then G_{T_P} is chordal if and only if P is P_1-, P_2- and P_3-free; see Fig. 2 (all points in the figure are pairwise distinct).*

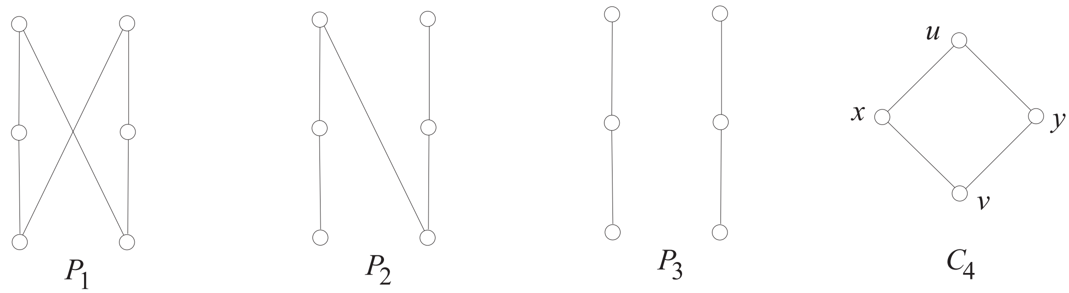

Figure 2: Forbidden subposets for C_4

The proof will be given in two steps, we first consider 4-cycles in C-I graphs and then proceed with longer cycles.

Lemma 3.2 *Let P be a poset. Then G_{T_P} contains an induced 4-cycle if and only if P contains one of the posets P_1, P_2 and P_3 as an isometric subposet.*

Proof. Suppose P contains one of P_1, P_2 or P_3 as an isometric subposet. Then, using Lemma 2.3, the vertices u, x, v, y (see Fig. 2) induce a 4-cycle in G_{T_P}.

Conversely, suppose that G_{T_P} contains an induced 4-cycle u, x, v, y, u as shown in Fig. 2. Let $S = \{u, x, v, y\}$.

If u, x, v, y lie on only one chain then in G_{T_P} they induce a path which is a contradiction. Hence there exist at least two chains on which vertices from S lie, and suppose that one of the four points, say u, is below the other three points. Since $deg_{C_4}(u) = 2$ we infer that u is covered in P by both x and y. Hence x and y are incomparable in P, and so they are adjacent in G_{T_P}. This yields the triangle in G_{T_P}, a contradiction. Using the duality argument we derive that no point is above all other points from S.

From the above we derive there are two chains S_1, S_2 in P on which points from S lie, and no point is comparable to all other points from S. Then there exist two minimal elements with respect to S (that is, no point from S is below these two elements), each of the two minimal elements lying on one of the chains. (There cannot be three minimal elements with respect to S because in G_{T_P} they would form a triangle.) Without loss of generality we may assume that $u \in S_1$ is one of the minimal elements with respect to S. We derive that the minimal element with respect to S that lies in S_2 is either x or y, say x. Since u is adjacent to y, either u is covered by y or u and y are incomparable.

First, let u and y be incomparable, so y necessarily lies in S_2. Since x and y lie on the same chain, y lies on S_2 above x, and y does not cover x. Since u and v are nonadjacent, v lies above u on S_1, and v does not cover u. Let $u = u_1, \ldots, u_k = v$ be the chain between u and v on S_1, and $x = x_1, \ldots, x_m = y$ be the chain between x and y on S_2. Since u is not comparable to y, also u_{k-2}, u_{k-1} and v are not below any x_i (including y). Since x and v are adjacent in G_{T_P}, either they are incomparable in P or v covers x. In each case, we deduce that $x_2, \ldots, x_{m-1}$ and y are not below any u_i. Hence x_2 and x_3 are incomparable with u_{k-2}, u_{k-1} and v. We infer that $u_{k-2}, u_{k-1}, v, x, x_2, x_3$ induce one of the posets P_2 or P_3 as an isometric subposet (depending on whether x and v are comparable or not), see Fig. 2.

Secondly, let u be covered by y. So $y \in S_1$. Suppose that v is in S_1. Then y and u are below v. Since x is minimal and x and y lie on the same chain in P (because they are not adjacent in G_{T_P}), x is below y, and so v covers also x. This is not possible, since we established earlier that no point from S can be above all other three points from S. We find that $v \in S_2$. Now, x and v adjacent in G_{T_P}, implies that v covers x. It is clear that u and x are incomparable and also that v and y are incomparable (by the same argument). Since u and v are not adjacent in G_{T_P}, there is a chain of length at least 2 between u and v, and similarly we get for x and y. If both chains are of length exactly two we infer that the poset P_1 from Fig. 2 is an isometric subposet. If one of the lengths of these two chains is greater than 2, we obtain in a similar way as above, one of the subposets P_2 or P_3. $\square$

Lemma 3.3 *For a poset P, G_{T_P} has no induced long cycles.*

Proof. Suppose G_{T_P} contains an induced n-cycle $C = v_1, v_2, \ldots, v_n, v_1$, $n \geq 5$. Let P' be the subposet of P formed by $v_1, v_2, \ldots, v_n$. Then P' is not a chain for then $v_1, v_2, \ldots, v_n$ form a path in G_{T_P} contrary to our assumption. We distinguish two cases.

Case 1: P' contains an antichain of length 3.
Without loss of generality we assume that v_1, v_2, v_3 are three points in an antichain of length 3. Clearly they form a triangle in G_{T_P} and so C has a chord.

Case 2: P' has no antichain of length 3 or more.
Let $\{v_i, v_j\}$, $i \neq j$, be an antichain of P'. Then if we consider a third point say v_k different from v_i, v_j, it will not be incomparable with both v_i and v_j because otherwise $\{v_i, v_j, v_k\}$ form an antichain of length 3. So v_k is comparable with v_i or v_j. Suppose $v_i < v_k$. Let w be the vertex covering v_i, where $w \leq v_k$. Then we have the following subcases relating w and v_j.

Subcase 2.1: w and v_j incomparable.
Obviously $\{v_i, w, v_j\}$ form a triangle in G_{T_P}. Hence C has a chord, a contradiction.

Subcase 2.2: w covers v_j.
Again $\{v_i, w, v_j\}$ form a triangle in G_{T_P}, thus C has a chord.

Subcase 2.3: $v_j < w$.
Let z be such that $v_j < z < w$, where z covers v_j. Then v_i, v_j, z form a triangle in G_{T_P}, again yielding a chord in C which concludes the proof of this case and hence the theorem. $\square$

Theorem 3.1 now follows by combining Lemma 3.2 and Lemma 3.3.

4 Posets with distance-hereditary C-I graphs

The main result of this section is the following:

Theorem 4.1 *Let P be a poset. Then G_{T_P} is distance-hereditary if and only if P is Q_1-, Q_2-, Q_3-, Q_4- and Q_5-free; see Fig. 3.*

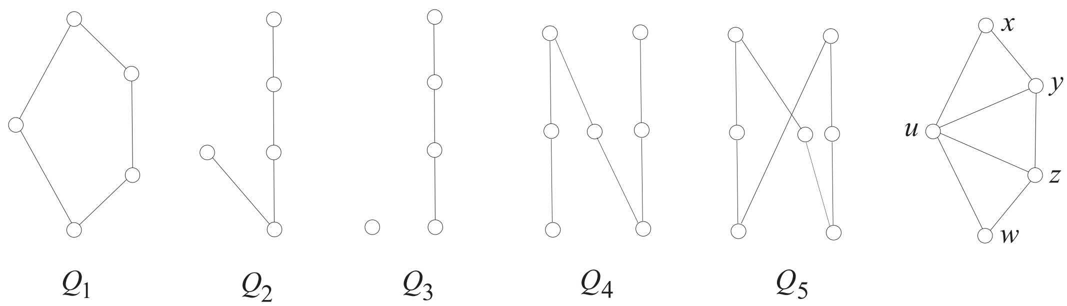

Figure 3: Forbidden subposets for 3-fan

Note that by the agreement after Lemma 2.2 we do not list dual posets in Fig. 3 (otherwise there would be another forbidden subposet – the dual poset of Q_2).

Combining Theorem 4.1 with Lemma 3.2 we obtain:

Corollary 4.2 *Let P be a poset. Then G_{T_P} is Ptolemaic if and only if P is P_1-, P_2-, P_3-, Q_1-, Q_2-, Q_3-, Q_4- and Q_5-free; see Fig. 2 and Fig. 3.*

As we have already mentioned, distance-hereditary graphs are the connected graphs without long cycles, the 3-fan, the house, and the domino. Since the long cycles were treated in the previous section, we consider in the rest of the section the effect of the remaining forbidden subgraphs in G_{T_P} to P.

Lemma 4.3 *Let P be a poset. Then G_{T_P} contains an induced* 3*-fan if and only if P contains one of the Q_1, Q_2, Q_3, Q_4, or Q_5 as an isometric subposet; see Fig. 3.*

Proof. Suppose P contains an isometric subposet isomorphic to Q_1, Q_2, Q_3, Q_4, or Q_5. In the cases Q_1, Q_2, and Q_3 it is clear (having in mind Lemma 2.3) that G_{T_P} contains an induced 3-fan. If Q_4 or Q_5 are subposets we get the same conclusion by considering the left point of the middle level as vertex u and the points in the bottom level and the top level as vertices x, y, z, and w.

For the converse suppose G_{T_P} has an induced 3-fan as shown in Fig. 3. Then w, z, y, x is an induced path in G_{T_P}. We distinguish three possibilities.

Case 1. All points w, z, y, x lie on one chain.
We may then assume without loss of generality that $w < z < y < x$ where x covers y, y covers z, and z covers w. As u is adjacent to x, y, z and w in G_{T_P}, it follows

that u is in the following relation with the other 4 points: either incomparable with a point, covers a point, or it is covered by a point.

Note that y and z must both be incomparable with u, otherwise we easily infer that one of the four vertices is not adjacent to u in G_{T_P}. For instance, if y is covered by u then z and w are not adjacent to u in G_{T_P}. Similarly one verifies that x is not covered by u and u is not covered by w. Hence, the remaining three cases are: u is incomparable with all four points (yielding subposet Q_3), u covers w and is incomparable to other three points (yielding subposet Q_2), and u covers w, u is covered by x and is incomparable to y and z (yielding subposet Q_1). Note that the fourth case which we excluded yields a subposet dual to Q_2.

Case 2. Three of the points w, z, y, x lie on one chain, but not all four.
Suppose three points that lie on a chain correspond to a path P_3 in G_{T_P}. Without loss of generality, we may assume that these are w, z and y, and that $w < z$ and $w < y$. Then, clearly, z covers w and y covers z. Since x and z are not adjacent in G_{T_P}, they must be comparable, but not in a covering relation. We infer that $z < x$, otherwise x and y would not be adjacent in G_{T_P}. Then w, z and two points on the chain between z and x form an isometric subposet – chain on 4 points. Together with u they form one of the posets Q_1, Q_2 or Q_3.

Suppose three points that lie on a chain do not correspond to a path P_3 in G_{T_P}. Without loss of generality, we may assume that these are w, y and x. Since w is not adjacent in G_{T_P} with any of x and y, we derive that there is another point on a chain between w and the pair x, y. Again we are in the situation of Case 1, and obtain one of Q_1, Q_2 or Q_3 as a subposet.

Case 3. No three points of w, z, y, x lie on one chain.
Note first that in this case x and y are incomparable. Indeed, if they would be comparable, then, as w is comparable with both x and y, we get that x, y, and w would lie on a common chain. Since w is not adjacent to x and to y in G_{T_P}, it must be comparable with both x and y. We may assume without loss of generality that $w < x$ and $w < y$. We also note that there is a point on a chain S_1 (respectively S_2) between w and x (respectively w and y). Similarly, since x is not adjacent to z in G_{T_P}, x is comparable with z and hence $z < x$ by the condition of Case 3. In addition, there is a point on a chain S_3 between z and x. If any of the chains S_i has four points, we are in the situation of Case 1 again. On the other hand, if all S_i have only three points, we obtain a Q_4 or a Q_5 as an isometric subposet, depending on whether y and z are comparable. (Note that if they are comparable, y necessarily covers z.) □

In finding forbidden subposets for G_{T_P} to be house-free and domino-free, it is useful to start with subposets P_1, P_2, and P_3 that yield an induced C_4 in G_{T_P} as obtained by Lemma 3.2. Starting from this, a simple case analysis (that we leave to the reader) gives the following two results.

Lemma 4.4 *Let P be a poset. Then G_{T_P} contains an induced house if and only if P contains one of R_1, R_2, R_3, R_4 or R_5 as an isometric subposet; see Fig. 4.*

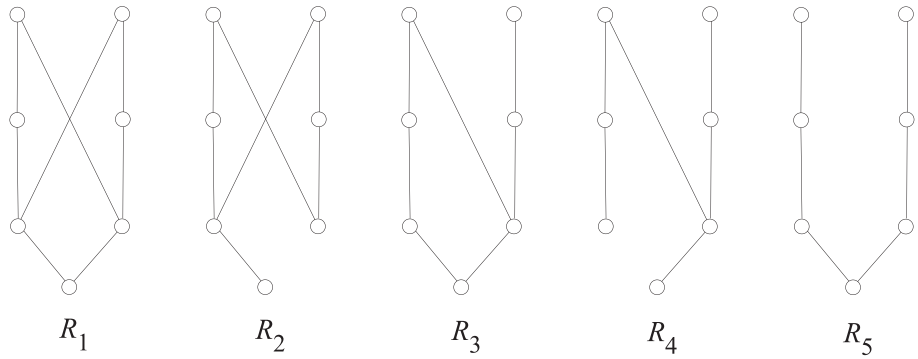

Figure 4: Forbidden subposets for house

Lemma 4.5 *Let P be a poset. Then G_{T_P} contains an induced domino if and only if P contains one of D_1, D_2, D_3, D_4, D_5, D_6 or D_7 as an isometric subposet; see Fig. 5.*

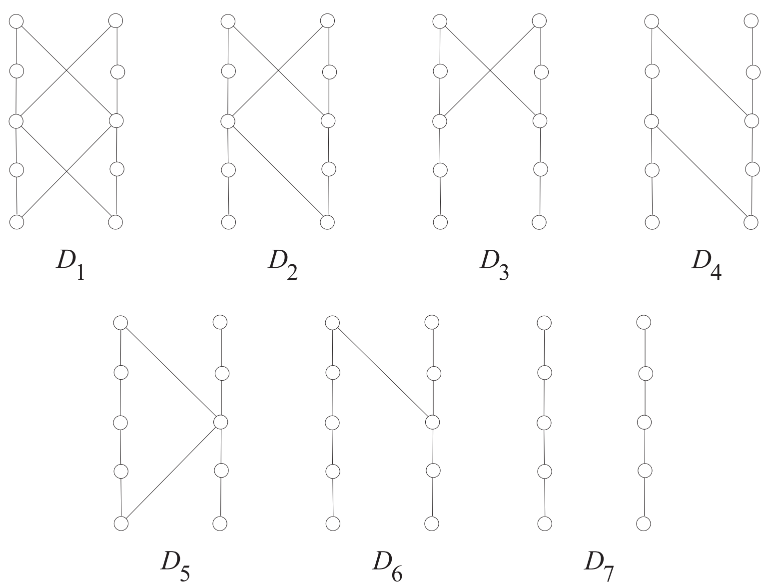

Figure 5: Forbidden subposets for domino

From Lemma 3.3 we know that G_{T_P} contains no induced long cycles. Observe now that each of the posets R_1-R_5 and D_1-D_7 contains one of the posets Q_1, Q_2, and Q_3 as an isometric subposet. Therefore, Theorem 4.1 follows from Lemmas 3.3, 4.3, 4.4 and 4.5.

5 Relation $\prec$

In this section we introduce a relation $\prec$ on graphs that is derived from the connection between posets and their C-I graphs. The motivation for this concept arises from the following result, and its corollaries. (Recall that a *claw* is the graph isomorphic to $K_{1,3}$, see Fig. 6 where it is depicted on the right-hand side.)

Proposition 5.1 *Let P be a poset. Then G_{T_P} contains an induced claw if and only if P contains one of S_1, S_2 or S_3 as an isometric subposet; see Fig. 6.*

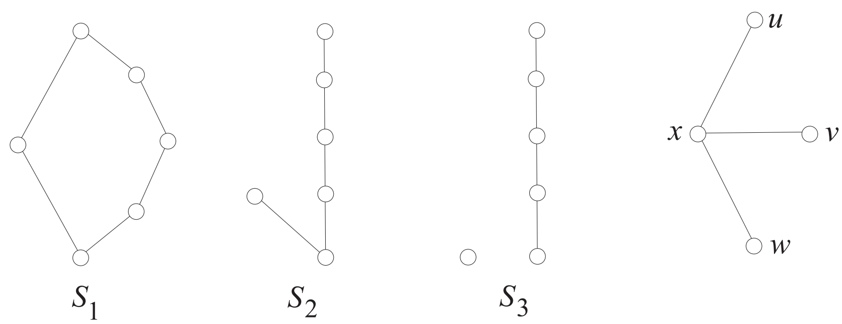

Figure 6: Forbidden subposets for claw

Proof. It is clear that if P has isometric subposets isomorphic to S_1, S_2 or S_3, then G_{T_P} contains an induced claw.

For the converse suppose that G_{T_P} contains an induced claw, and denote by x the central vertex and by u, v, w the other vertices of the claw. As u, v, w form an independent set in G_{T_P} we may assume without loss of generality that $u < v < w$, and it is clear that u, v, w are not pairwise covering each other.

First, suppose that x is not comparable in P to any of the points u, v, w. Then, by the above, we find that P has an isometric subposet isomorphic to S_3.

Second, let x be comparable to at least one of the points u, v, w. Clearly, x cannot be comparable to v (the middle point) since then x would not be adjacent to one of u, w in G_{T_P}. Suppose that x is comparable with exactly one of the three points, and first let this be u. Obviously then $u < x$, and in addition, since x and u are adjacent in G_{T_P}, x covers u. We clearly get S_2 as an isometric subposet. The case when x is comparable only to w yields as a subposet the dual of S_2. The final case is that x is comparable to both u and w. Note that the chain between u and w contains at least 5 points (u, v, w and two "buffer points" between u and v and v and w). If the chain has exactly 5 points, we get S_1 as an isometric subposet. If it has more than 5 points, then both S_2 and its dual can be easily found as isometric subposets. $\square$

The key observation from Proposition 5.1 and Lemma 4.3 is the following: each forbidden poset that appears in Proposition 5.1 includes as an isometric subposet some of the posets from Lemma 4.3 (that characterizes posets with 3-fan-free C-I graphs). Thus the following result follows.

Corollary 5.2 *If P is a poset such that G_{T_P} is 3-fan-free, then G_{T_P} is also claw-free.*

Similarly, one can readily check that forbidden posets R_i (that are used in the characterization of posets with house-free C-I graphs) contain as a subposet one of the posets Q_1, Q_2 or Q_3 or their duals from Fig. 3. Hence:

Corollary 5.3 *If P is a poset such that G_{T_P} is claw-free, then G_{T_P} is also house-free.*

Clearly, claw-free graphs are also domino-free. Hence knowing these relations between poset families, defined by forbidden subposets, one can immediately characterize posets with distance-hereditary (Ptolemaic) C-I graphs as posets with 3-fan-free (C_4-free and 3-fan-free) C-I graphs. As we already know from the direct proofs from previous sections, the forbidden list of subposets for distance-hereditary C-I graphs is the same as for the 3-fan-free C-I graphs. The following relation between graphs is thus natural.

Let H_1 and H_2 be graphs that can appear as induced subgraphs of some C-I graphs. That is, there exist posets P_i, $i = 1, 2$, such that $G_{T_{P_i}}$ contains H_i as an induced subgraph. Let $\mathcal{D}_i$ denote the set of forbidden isometric subposets by which the family of posets whose C-I graphs are H_i-free are characterized. Then we write $H_1 \prec H_2$ if for any poset $B_2 \in \mathcal{D}_2$ there exists a poset $B_1 \in \mathcal{D}_1$ such that B_1 is an isometric subposet of B_2.

For instance, our results show that $F_3 \prec K_{1,3}$ (where F_3 stands for the 3-fan), $K_{1,3} \prec H$, $K_{1,3} \prec D$ (where H stands for the house, and D for the domino).

It is clear that $\prec$ is a reflexive and transitive relation on the family of all C-I graphs (hence also $F_3 \prec H$ etc.). But the relation $\prec$, is not antisymmetric, because the forbidden subposets for 4-fan and claw are the same. This can be checked as follows. Since $K_{1,3}$ is an induced subgraph of the 4-fan, one direction is clear. For the converse relation, just observe that the forbidden subposets for claw all yield the 4-fan, and so also 4-fan $\prec K_{1,3}$. Thus the relation $\prec$ need not be a partially ordered relation in general. It is clear that if H_1 is an induced subgraph of H_2, then $H_1 \prec H_2$. We believe that if the classes of C-I graphs of posets will be investigated in more detail, the relation $\prec$ will need to be further explored.

6 Concluding remarks

Two natural questions can be posed for any class $\mathcal{G}$ of graphs that is characterized by forbidden induced subgraphs. The first one is to determine the list of forbidden subposets so that the C-I graphs are from the class. This question was answered in the paper for several well-known classes. Another question is to characterize the graphs $\{G_{T_P} \mid P \in \mathcal{P}\}$ among the graphs from $\mathcal{G}$ and this question was not addressed in this paper. So we pose it as a problem:

Question 6.1 *Which chordal (distance-hereditary, Ptolemaic) graphs are C-I graphs?*

The list of classes in the above question can, of course, be extended. Moreover, the following related question is also interesting.

Question 6.2 *Which graphs are C-I graphs?*

The question could be also posed in a different form as a construction or algorithmic problem. Recall that the recognition problem for cover graphs of posets is NP-complete [13, 14], whereas the recognition problem for incomparability graphs is polynomial, cf. [2]. It might be an intriguing problem whether the same holds for C-I graphs of posets as well.

Question 6.3 *Can C-I graphs be recognized in polynomial time? In addition, do the C-I graphs themselves possess a forbidden subgraphs characterization?*

Concerning the relation $\prec$ between induced subgraphs of C-I graphs many questions can be posed. It would be interesting to find some general structural approach by which $\prec$ between some C-I graphs could be determined more easily (for instance, it is already clear that a graph H_1 obtained by deletion of some vertices from a graph H_2 is in relation $\prec$ with H_2). We repeat the following question from the previous section.

Question 6.4 *For which family of C-I graphs, is the relation $\prec$ a partial order on the family of all induced subgraphs?*

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References

[1] H.-J. Bandelt and H. M. Mulder, Distance-hereditary graphs, J. Combin. Th. Ser. B 41 (1986) 182–208.

[2] A. Brandstädt, V.B. Le and J. Spinrad, *Graph Classes*, SIAM, Philadelphia, 1999.

[3] M. Changat, S. Klavžar and H. M. Mulder, The all-paths transit function of a graph, Czech. Math. J. 51(126) (2001) 439–448.

[4] M. Changat and J. Mathews, Induced path transit function, monotone and Peano axioms, Discrete Math. 286 (2004) 185–194.

[5] M. Changat, H. M. Mulder and G. Sierksma, Convexities related to path properties on graphs, Discrete Math. 290 (2005) 117–131.

[6] G. Grätzer, *General Lattice Theory, Second Edition*, Birkhauser, Basel, 1998.

[7] E. Howorka, A characterization of distance-hereditary graphs, Quart. J. Math. Oxford Ser. 2, 28 (1977) 417–420.

[8] F. Larrión, M.A. Pizaña and R. Villarroel-Flores, Posets, clique graphs and their homotopy type, European J. Combin. 29 (2008) 334–342.

[9] A. Mathews and J. Mathews, Transit functions on posets and lattices. In: Convexity in Discrete Structures (M. Changat, S. Klavžar, H.M. Mulder, A. Vijayakumar, eds.), Lecture Notes Ser. 5, Ramanujan Math. Soc. (2008) 105–116.

[10] M. A. Morgana and H. M. Mulder, The induced path convexity, betweenness and svelte graphs, Discrete Math. 254 (2002) 349–370.

[11] H. M. Mulder, *The Interval Function of a Graph*, Mathematisch Centrum, Amsterdam, 1980.

[12] H. M. Mulder, Transit functions on graphs (and posets). In: Convexity in Discrete Structures (M. Changat, S. Klavžar, H.M. Mulder, A. Vijayakumar, eds.), Lecture Notes Ser. 5, Ramanujan Math. Soc. (2008) 117–130.

[13] J. Nešetřil and V. Rödl, Complexity of diagrams, Order 3 (1987) 321–330.

[14] J. Nešetřil and V. Rödl, More on the complexity of cover graphs, Comment. Math. Univ. Carolin. 36 (1995) 269–278.

[15] I. Rival, The role of graphs in the theory of ordered sets and its applications, J. Symbolic Logic 57 (1992) 269–271.

[16] W. T. Trotter, Graphs and partially ordered sets: recent results and new directions, Congr. Numer. 116 (1996) 253–278.