# Domination game and an imagination strategy*

Boštjan Brešar[†]
Faculty of Natural Sciences and Mathematics
University of Maribor, Slovenia
bostjan.bresar@uni-mb.si

Sandi Klavžar[†]
Faculty of Mathematics and Physics
University of Ljubljana, Slovenia
and
Faculty of Natural Sciences and Mathematics
University of Maribor, Slovenia
sandi.klavzar@fmf.uni-lj.si

Douglas F. Rall
Department of Mathematics, Furman University
Greenville, SC, USA
doug.rall@furman.edu

**Abstract**

The domination game played on a graph $G$ consists of two players, Dominator and Staller who alternate taking turns choosing a vertex from $G$ such that whenever a vertex is chosen by either player, at least one additional vertex is dominated. Dominator wishes to dominate the graph in as few steps as possible and Staller wishes to delay the process as much as possible. The game domination number $\gamma_g(G)$ is the number of vertices chosen when Dominator starts the game and the Staller-start game domination number $\gamma_g'(G)$ when Staller starts the game. An imagination strategy is developed as a general tool for proving results on the domination game. We show that for any graph $G$, $\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1$, and that all possible values can be realized. It is proved that for any graph $G$, $\gamma_g(G) - 1 \leq \gamma_g'(G) \leq \gamma_g(G) + 2$, and that most of the possibilities for mutual values of $\gamma_g(G)$ and $\gamma_g'(G)$ can be realized. A connection with Vizing's conjecture is established, and a lower bound on the

*The paper was initiated during the visit of the third author to the IMFM and completed during the visit of the first two authors at Furman University supported in part by the Slovenia-USA bilateral grant BI-US/08-10-004 and the Wylie Enrichment Fund of Furman University.

[†]Supported by the Ministry of Science of Slovenia under the grants P1-0297 and J1-2043. The author is also with the Institute of Mathematics, Physics and Mechanics, Jadranska 19, 1000 Ljubljana.

game domination number of an arbitrary Cartesian product is proved. Several problems and conjectures are also stated.

**Keywords:** domination, domination game, game domination number, Vizing's conjecture

**AMS subject classification (2010)**: 05C57, 91A43, 05C69, 05C76

# 1 Introduction

The game chromatic number was described for the first time in 1981 [11] and remained more or less unnoticed for many years. However, in the last several years the topic has received an astonishing amount of attention, see, for instance, [3, 8, 16, 20, 21] and the appealing survey [5]. We point out that very recently Kierstead and Kostochka [15] found an application to a graph packing problem of game coloring number of a graph, a concept closely related to game chromatic number. We also add that different variations of the game chromatic number were studied in [2, 4, 6, 17, 18, 19]; for a comprehensive bibliography on the game chromatic number and related topics see [10].

It seems natural to study an analogous game with respect to domination, the problem brought to our attention by Mike Henning [12]. In fact, we find it quite surprising that this game has been ignored so far and hope to initiate its research here. Just like in the coloring variant, we have one player who wishes to dominate a graph in as few steps as possible and another player who wishes to delay the process as much as possible. As far as we know, the papers [1, 9] are until now the only papers dealing with game domination; however, their concept is different from ours.

We describe two games played on a finite graph $G = (V, E)$. In Game 1 two players, Dominator and Staller, alternate—with Dominator going first—taking turns choosing a vertex from $G$. We let $d_1, d_2, \ldots$ denote the sequence of vertices chosen by Dominator and $s_1, s_2, \ldots$ the sequence chosen by Staller. (Note that in some cases it will be convenient to speak about the 0-th step of some player.) These vertices must be chosen in such a way that whenever a vertex is chosen by either player, at least one additional vertex of the graph $G$ is dominated that was not dominated by the vertices previously chosen. That is, for each $i$,

- $N[d_i] \setminus \cup_{j=1}^{i-1} N[\{d_j, s_j\}] \neq \emptyset$; and
- $N[s_i] \setminus \left(\cup_{j=1}^{i-1} N[\{d_j, s_j\}] \cup N[d_i]\right) \neq \emptyset$.

In Game 2 the players alternate choosing vertices as in Game 1, except that Staller begins. In this game we denote the two sequences of vertices by $s'_1, s'_2, \ldots$ and $d'_1, d'_2, \ldots$. As in Game 1, we also require that each chosen vertex strictly enlarges the closed neighborhood of the set of chosen vertices.

Since the graph $G$ is finite, each of these games will end in some finite number of moves regardless of how the vertices are chosen. In both of the games Dominator chooses vertices using a strategy that will force the game to end in the fewest number of moves, and Staller uses a strategy that will prolong the game as long as possible. We define the *game domination number* of $G$ to be the total number of vertices chosen when Dominator and Staller play Game 1 on graph $G$ using optimal strategies, and we denote this value by $\gamma_g(G)$. The *Staller-start game domination number* of $G$, denoted by $\gamma_g'(G)$, is the cardinality of the set of vertices chosen when Game 2 is played on $G$.

When Game 1 is played on a graph $G$, the set of vertices chosen by Dominator and Staller together is a dominating set of $G$. Thus we easily have the bound $\gamma_g(G) \geq \gamma(G)$ for every $G$. On the other hand, Dominator can order the vertices in a minimum dominating set $A = \{d_1, d_2, \ldots, d_{\gamma(G)}\}$ and play according to this list. Because of the way that Staller selects vertices, it is possible that Dominator may not be able to use some of the vertices in this list. However, when Dominator exhausts the sequence in $A$ the graph is dominated, and hence no more moves are legal. Together these prove the following bounds relating the ordinary domination number and the game domination number.

**Theorem 1** *For any graph $G$,*

$$\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1\,.$$

We note that there is no general upper bound on the game chromatic number of a graph in terms of a function of its chromatic number. Nevertheless in establishing results of this paper we found more inspiration in game coloring theory than in domination theory.

In the next section we introduce the main idea to be used in our proofs and prove several related lemmas. The approach uses the so-called imagination strategy and was (in the context of the coloring games) invented in [3] and further developed in [20]. In Section 3 we investigate the relation between Game 1 and Game 2. We prove that for any graph $G$, $\gamma_g(G) - 1 \leq \gamma_g'(G) \leq \gamma_g(G) + 2$ and study possible pairs $(k, \ell)$ such that there exists a graph $G$ with $\gamma_g(G) = k$ and $\gamma_g'(G) = \ell$. Then, in Section 4, we show that for any positive integer $k$ and any nonnegative integer $r \leq k - 1$ there exists a graph $G$ such that $\gamma(G) = k$ and $\gamma_g(G) = k + r$. In Section 5 we establish a connection between the game domination number and Vizing's conjecture and in particular prove a lower bound on the game domination number of an arbitrary Cartesian product. We conclude the paper with several remarks and open problems.

## 2 Imagination strategy

In this section we present the central concept of this paper and demonstrate how the corresponding proof techniques work. The main idea is that one of the players imagines another appropriate game and plays in it according to an optimal strategy. The imagined game is dependent on a concrete situation and most of the moves are copied between the imagined and the real game. We call this approach the *imagination strategy*.

Let us explain it more precisely. When proving bounds on some type of game domination number of a graph, one of the players (either Dominator or Staller) imagines another game is played at the same time, usually on a copy of the same graph. For the imagined game the optimal number of moves $k$ is known, and hence if Dominator (resp. Staller) is the player who imagines the parallel game, then he has a strategy that ensures the imagined game takes at most (at least) $k$ moves. The basic procedure of his strategy in the real game is simply to copy each move of the opponent to the imagined game, respond in the imagined game by an optimal move, and finally copy back this move to the real game. Two problems are possible: some of his moves that are legal in the imagined game need not be legal in the real game, and some of the moves of the opponent in the real game need not be legal in the imagined game. This is, of course, the main problem, and such cases are handled in different ways with respect to a given situation. The overall aim is to ensure that the number of moves in the real game is bounded by the number of moves in the imagined game (usually these numbers are the same or differ by at most one), which gives the bound on the corresponding game domination number of the graph.

In the rest of the section we will prove several auxiliary results to be used in the rest of the paper, each of which introduces a variant of the original games.

First consider the game in which Dominator starts, and Staller is allowed, but not obligated, to skip exactly one move. That is, at some point in the game, instead of picking a vertex, Staller may decide to pass, and it is Dominator's turn again. Afterwards the vertices are picked alternatingly again until the end. The number of moves in such a game, where both players are playing optimally, is denoted by $\gamma_g^{sp}(G)$. We call this game the *Staller-pass game*.

**Lemma 2** *For any graph $G$, $\gamma_g^{sp}(G) \leq \gamma_g(G) + 1$.*

**Proof.** Let the players play the Staller-pass game. The strategy of Dominator is that he will be imagining another game is being played at the same time—an ordinary domination game—and he will be playing it according to an optimal strategy (hence the length of this game will not be greater than $\gamma_g(G)$). So two games are played at the same time: the real (Staller-pass) game, and the (ordinary) game, imagined by Dominator. In the first part of the game Dominator will just copy each move of Staller to his imagined game, and respond in the imagined game with an optimal move from the ordinary domination game. Each of these moves of Dominator is then also copied to the real game. This will continue until Staller decides to pass a move in the real game (if he decides that at all; but if he does not decide to pass any move, then the two games are the same and thus both have $\gamma_g(G)$ moves). Up to that point the moves in both games are the same, and they form the sequence

$d_1, s_1, \ldots, d_{k-1}, s_{k-1}, d_k$. Then in the real game it is Dominator's turn, but in the imagined game it is Staller's turn. Hence in the imagined game Dominator imagines that Staller made a (legal) move, say $s_k$, and he responds in this game in an optimal way by picking, say $d_{k+1}$. Then he also picks the same vertex in the real game. So the current sequence of the real game is $d_1, s_1, \ldots, d_{k-1}, s_{k-1}, d_k, d_{k+1}$. The game next continues in the same way as before. Note that all moves of Dominator are chosen (in an optimal way) with respect to the imagined game, so all his moves will be legal also in the real game. On the other hand, Staller chooses his moves with respect to the real game, but in the imagined game the copy of this move may not be legal. Suppose that some move of Staller, say $s_m$, is not legal in the imagined game. Note that this can only happen for $m > k$. Denote by

$$N[C] = \bigcup_{i=1}^{m} N[d_i] \cup \bigcup_{\substack{i=1 \\ i \neq k}}^{m-1} N[s_i]\,.$$

Note that the move of Staller is illegal in the imagined game precisely when

$$N[s_m] \setminus N[C] \subseteq N[s_k] \setminus N[C]\,.$$

Indeed, the move $s_m$ is legal in the real game, so its closed neighborhood is not included in $N[C]$, and it must be that all vertices from $N[s_m] \setminus N[C]$ (and such vertices exist) are in the neighborhood of $s_k$.

We distinguish two cases. First suppose that $N[s_m] \setminus N[C]$ is a proper subset of $N[s_k] \setminus N[C]$. Then Dominator makes a non standard move—he picks $s_k$ in the real game—and he does nothing in the imagined game. Note that $s_k$ is a legal move in the real game, and after that move of Dominator the sets of dominated vertices are the same in both games, and it is Staller's turn in both games. Moreover, the number of moves in the real game is one more than the number of moves in the game imagined by Dominator. Since Dominator can play until the end by the same strategy as in the beginning of the game (since Staller already used his pass), the total number of moves in the real (Staller-pass) game is at most $\gamma_g(G) + 1$.

The second possibility is that $N[s_m] \setminus N[C] = N[s_k] \setminus N[C]$. Hence, at that time (after Staller's move $s_m$) both games already have the same sets of dominated vertices. Hence, instead of this move Dominator imagines another (legal) move of Staller in the imagined game. He again responds to that move optimally, and copies the same move in the real game where it is his turn. (Note that, at that time, the number of moves in the real game is one less than the number of moves in the imagined game.) The game continues in the same way as in the beginning, until either the imagined game is ended, or some move of Staller is again not legal in the imagined game. In the latter case again one of the two cases appear, which Dominator can resolve in the way we have explained, and the game goes on. In the former case (which eventually must happen since the graph is finite), when the imagined game is finished, the real game may not be finished, since one of the vertices (the last imagined move of Staller) was not picked in the real game. As soon as it is Dominator's turn in the real game, he picks that vertex and the game ends. In the worst case, the imagined game ended with the move of Dominator. Hence the real game ends after at most two additional moves (the move of Staller and the final move of Dominator). Since before that the real game had one less move, we get that the (real) Staller-pass game has at most one move more than the imagined game, yielding the inequality of the lemma. □

**Lemma 3** *For any graph $G$, $\gamma_g'(G) \leq \gamma_g^{sp}(G) + 1$.*

**Proof.** Let the players play Game 2 on $G$. This time Dominator will be imagining a Staller-pass domination game started by Dominator. Dominator will be playing the imagined game according to an optimal strategy, hence the number of moves in this game will not be greater than $\gamma_g^{sp}(G)$. Let us denote the first move in the real game by $s_0$. In the imagined game, this move is ignored, and it is Dominator's turn to play. His first move $d_1$ is played according to his strategy in the Staller-pass game, by which he can ensure that there are at most $\gamma_g^{sp}(G)$ moves altogether. In the first part of the game each of his moves is copied to the real game, hence $d_1$ is also his first move in the real game (unless already the second part of the game began, which we will explain soon). Staller responds in the real game by $s_1$, and this move is copied also to the imagined game, and so on. In the first part of the game the sequence of moves in the real game is $s_0, d_1, s_1, \ldots, d_{k-1}, s_{k-1}$, and in the imagined game it is $d_1, s_1, \ldots, d_{k-1}, s_{k-1}$. Note that every move which is legal in the real game is also legal in the imagined game, hence all moves of Staller will result in a legal move of Staller in the imagined game. On the other hand, moves of Dominator are copied to the real game, and it can happen at some point that such a move, say $d_k$, where $k \geq 1$, is not legal in the real game. The first part of the game is then ended, and we come to the second part. If this situation never happens during the game, there is clearly nothing to prove, since then the real game is ended in at most $\gamma_g^{sp}(G) + 1$ steps, as we claim.

Set

$$N[C] = \bigcup_{i=1}^{k-1} (N[s_i] \cup N[d_i])$$

and similarly as in the proof of Lemma 2 note that the move $d_k$ is illegal in the real game precisely when $N[d_k] \setminus N[C] \subseteq N[s_0] \setminus N[C]$.

First suppose that $N[d_k] \setminus N[C]$ is a proper subset of $N[s_0] \setminus N[C]$. Then after Dominator plays $d_k$ in the imagined game, he imagines Staller plays $s_0$ in this game (which is legal in this case). In the second case when $N[d_k] \setminus N[C] = N[s_0] \setminus N[C]$, Dominator imagines that Staller skips a move. Recall that the Staller-pass game is imagined, hence this is the first and the only time a move of the Staller is skipped. In both cases, after that point the set of vertices that are dominated coincide in both games, and it is Dominator's turn in both games. Hence the game is played as in the beginning (Dominator following the optimal strategy of the Staller-pass game), but now there are no problems with legality of his moves anymore. When the imagined game ends, also the real game ends. In the first case, the number of moves in the imagined game is one more than in the real game, while in the second case, the number of moves in both games is equal. □

We next consider the game, called the *Dominator-pass game*, in which Dominator is allowed to pass a move. In this game Dominator starts (unless he decides to pass already the first move), and then he is allowed to pass one move (analogously as Staller in the Staller-pass game). Afterwards the vertices are picked alternatingly again until the end. The number of moves in such a game, where both players are playing optimally, is denoted by $\gamma_g^{dp}(G)$.

**Lemma 4** *For any graph $G$, $\gamma_g^{dp}(G) \geq \gamma_g(G) - 1$.*

**Proof.** Let the players play the Dominator-pass game. Since our goal is a lower bound, we will take the position of Staller and present a strategy for him so that the game will not last less than $\gamma_g(G) - 1$ moves. The strategy of Staller is that he will be imagining another game is being played at the same time—an ordinary domination game—and he will be playing it according to an optimal strategy (hence the length of this game will not be less than $\gamma_g(G)$). So two games are played at the same time: the real (Dominator-pass) game, and the (ordinary) game, imagined by Staller. In the first part of the game Staller will just copy each move of Dominator to his imagined game, and respond in the imagined game with an optimal move from the ordinary domination game. Each of these moves of Staller is then also copied to the real game. This will continue until Dominator decides to pass a move in the real game (which can happen already at the beginning). Up to that point the moves in both games are the same, and they form the sequence $d_1, s_1, \ldots, d_{k-1}, s_{k-1}$ (which could also be the empty sequence). Then in the real game it is Staller's turn, but in the imagined game it is Dominator's turn. Hence in the imagined game Staller imagines that Dominator made a (legal) move, say $d_k$, and he responds in this game in an optimal way by picking, say $s_k$. Then he also picks the same vertex in the real game. So the current sequence of the real game is $d_1, s_1, \ldots, d_{k-1}, s_{k-1}, s_k$. The game next continues in the same way as before, so that the real game is represented by the sequence $d_1, s_1, \ldots, d_{k-1}, s_{k-1}, s_k, d_{k+1}, s_{k+1}, \ldots$. Note that all moves of Staller are chosen (in an optimal way) with respect to the imagined game, so all his moves will be legal also in the real game. On the other hand, Dominator chooses his moves with respect to the real game, but in the imagined game the copy of this move may not be legal. Suppose that some move of Dominator, say $d_m$, is not legal in the imagined game for some $m > k$. Denote by

$$N[C] = \bigcup_{\substack{i=1 \\ i \neq k}}^{m-1} N[d_i] \cup \bigcup_{i=1}^{m-1} N[s_i] \,.$$

Note that the move of Dominator is illegal in the imagined game precisely when

$$N[d_m] \setminus N[C] \subseteq N[d_k] \setminus N[C]\,.$$

Indeed, the move $d_m$ is legal in the real game, so its closed neighborhood is not included in $N[C]$, and it must be that all vertices from $N[d_m] \setminus N[C]$ (and such vertices exist) are in the neighborhood of $d_k$.

We distinguish two cases. First suppose that $N[d_m] \setminus N[C]$ is a proper subset of $N[d_k] \setminus N[C]$. Then Staller makes a nonstandard move—he picks $d_k$ in the real game—and he does nothing in the imagined game. Note that $d_k$ is a legal move in the real game, and after that move of Staller the sets of dominated vertices are the same in both games, and it is Dominator's turn in both games. Moreover, the number of moves in the real game is the same as the number of moves in the imagined game. Since Staller can play until the end by the same (optimal) strategy as in the beginning of the game, the total number of moves in the real (Dominator -pass) game is at least $\gamma_g(G)$.

The second possibility is that $N[d_m] \setminus N[C] = N[d_k] \setminus N[C]$. Then at that time (after Dominator's move $d_m$) both games have the same sets of dominated vertices. Hence, instead of this move Staller imagines another (legal) move of Dominator in the imagined game. He again responds to that move optimally, and copies the same move in the real game where it is his turn. (Note that, at that time, the number of moves in the real game is again one less than the number of moves in the imagined game.) The game continues in the same way as in the beginning, until either the imagined game is ended, or some move of Dominator is again not legal in the imagined game. In the latter case again one of the two cases appears, which Staller can resolve in the way we have explained, and the game goes on. Note that the real game cannot be finished before the imagined game, and at every time the imagined game (ordinary Game 1) has at most one move more than the real game. Since the imagined game uses at least $\gamma_g(G)$ moves, the real game ends in at least $\gamma_g(G) - 1$ moves. □

We will use Lemma 4 in the proof of Theorem 6. In order to prove Theorem 11 we will need to consider the game in which Dominator is allowed to pass as many moves as he wants. We call this game *Dominator-pass-k game* where $k$ is the number of moves Dominator is allowed to pass. (Clearly, Dominator-pass-1 game is just the Dominator-pass game and Dominator-pass-0 game is Game 1.) The number of moves in such a game, where both players are playing optimally, is denoted by $\gamma_g^{dp(k)}(G)$.

Note that each pass of Dominator may result in prolonging the imagined game (which is an ordinary Game 1) by one more with respect to the real game (which is *Dominator-pass-k game*). All other details of comparison between the real and the imagined game can be checked by following the lines of the proof of Lemma 4. We thus infer the following result (as a corollary of the former proof).

**Corollary 5** *For any graph $G$, $\gamma_g^{dp(k)}(G) \geq \gamma_g(G) - k$.*

## 3 Dominator-start versus Staller-start game

In this section we compare the game domination number with the Staller-start game domination number. We first note that, a bit surprisingly, for some graphs the latter invariant is strictly smaller than the game domination number. For example, $\gamma_g'(C_6) = 2$ while $\gamma_g(C_6) = 3$. When Dominator moves first on the 6-cycle, Staller responds by choosing a neighbor of $d_1$ and thus forces the game to last a total of 3 moves. However, when Game 2 is played on $C_6$, Dominator can always choose $b_1'$ which together with $s_1'$ forms a minimum dominating set of $C_6$. On the other hand, the two invariants are not far apart. More precisely, in this section we prove the following.

**Theorem 6** *For any graph $G$,*

$$\gamma_g(G) - 1 \leq \gamma_g'(G) \leq \gamma_g(G) + 2.$$

**Proof.** From Lemmas 2 and 3 the upper bound of Theorem 6 follows immediately. In playing the Dominator-pass game on $G$, Dominator is allowed to pass his first move. When this occurs the two players are in fact playing Game 2 on $G$, and this game ends in $\gamma_g'(G)$ moves. Since it could be better for Dominator to save his pass for later in the game it follows that $\gamma_g^{dp}(G) \leq \gamma_g'(G)$. The lower bound now is an immediate consequence of Lemma 4. □

For a pair of positive integers $k$ and $\ell$ we say the pair $(k, \ell)$ is *realizable* if there exists a graph $G$ such that $\gamma_g(G) = k$ and $\gamma_g'(G) = \ell$. From Theorem 6 we know that $k - 1 \leq \ell \leq k + 2$ holds for a realizable pair $(k, \ell)$.

**Proposition 7** *For any positive integer $k$, each of the pairs $(k, k)$, $(k, k+1)$ and $(2k+1, 2k)$ is realizable.*

**Proof.** For a positive integer $k$, let $P_k'$ denote the tree of order $2k$ obtained from the path of order $k$ by adding a vertex of degree 1 adjacent to each vertex of $P_k$. It is straightforward to show that if either Game 1 or Game 2 is played on $P_k'$, then the game will last exactly $k$ steps. Thus, $\gamma_g(P_k') = k = \gamma_g'(P_k')$, and thus $P_k'$ realizes $(k, k)$.

Let $T$ be the tree constructed from the path of order 4 by adding two new leaves adjacent to one of the vertices of degree 2. For an integer $k \geq 2$, let $F_k$ be the graph $T \cup (k-2)P_2$. When Game 1 is played on $F_k$, Dominator can begin by selecting the vertex of degree 4 in $T$. It is clear that exactly $k - 1$ more moves will be made in this game, and since $\gamma(F_k) = k$, Theorem 1 implies that $\gamma_g(F_k) = k$. However, Staller can select a leaf adjacent to the vertex of degree 4 in $T$ on his first move in Game 2. Dominator can now force the game to end in at most $k$, and no fewer, additional steps. It is easy to check that Staller can do no better in his first move, and hence $\gamma_g'(F_k) = k+1$. Therefore, $F_k$ realizes $(k, k+1)$. The star $K_{1,2}$ realizes $(1, 2)$, and so each pair $(k, k+1)$ can be realized by some graph.

For $k \geq 1$, let $G_k = C_6 \cup (2k-2)P_2$. In Game 1 played on $G_k$, Dominator can be forced to be the first player to select a vertex from $C_6$. By then selecting a vertex on the 6-cycle adjacent to the one chosen by Dominator, Staller can make the game last until exactly $2k+1$ vertices are chosen in total. Dominator would never allow more than 3 vertices to be chosen from the cycle, and since exactly one vertex must be selected from each $P_2$, it follows that $\gamma_g(G_k) = 2k+1$. On the other hand, Staller can be required to select the first vertex from the 6-cycle in a play of Game 2 on $G_k$. By then selecting the vertex on the 6-cycle diametrically opposite the one chosen by Staller, Dominator can force the game to end in exactly $2k$ moves. Hence $\gamma_g'(G_k) \leq 2k$ and by Theorem 6 and the above fact that $\gamma_g(G_k) = 2k+1$ we conclude that $\gamma_g'(G_k) = 2k$. Thus $G_k$ realizes $(2k+1, 2k)$. □

Suppose $G$ is a graph that realizes $(k, k+2)$ for some $k$. Then it is easy to see that $G \cup C_4$ realizes $(k+2, k+4)$. In Game 1 (resp. Game 2) Dominator (Staller) plays first on $G$, and all of his subsequent moves are also in $G$, except when his opponent plays on $C_4$, in which case he responds in $C_4$ as well. In fact, the strategies on $G \cup C_4$ only show that $\gamma_g(G \cup C_4) \leq k+2$ and $\gamma_g'(G \cup C_4) \geq k+4$, but we know by Theorem 6 that $\gamma_g'(G \cup C_4) - \gamma_g(G \cup C_4)$ is not greater than 2 which implies that $(k+2, k+4)$ is really the pair realized by $G \cup C_4$.

From the definitions it is clear that no graph realizes $(2, 1)$. We think that also for $k \geq 2$, there is no graph $G$ such that $\gamma_g(G) = 2k$ and $\gamma_g'(G) = 2k-1$. It is also obvious that $(1, 3)$ cannot be a realized pair, since graphs with $\gamma_g(G) = 1$ are precisely those with $\gamma(G) = 1$. In addition let us show that $(2, 4)$ cannot be realized.

Let $\gamma_g(G) = 2$ and $d_1, s_1$ be a sequence of moves in Game 1. Then the set $A = V(G) \setminus N[d_1]$ induces a complete graph, otherwise Staller could enforce the game last longer by playing on a vertex in $A$ that is not adjacent to some other vertex of $A$. Moreover, by the same argument, any vertex that is adjacent to a vertex of $A$ is adjacent to all vertices of $A$. Now, let Game 2 be played on $G$, and let $s_1'$ be the first move of Staller. If $s_1'$ is such a move that $d_1' = d_1$ is not legal, this means that $N[d_1]$ is already dominated. But then $\gamma_g'(G) = 2$ since Dominator can end the game by playing on a vertex of $A$. Hence, we may assume that $d_1$ is legal, and so after that only a subset of $A$ remains undominated, which implies $\gamma_g'(G) \leq 3$.

## 4 Realization of game domination numbers

We have observed in Theorem 1 that $\gamma_g(G)$ lies between $\gamma(G)$ and $2\gamma(G)-1$. In this section we prove that all possible values for $\gamma_g$ are eventually realizable. As motivating examples we note that $\gamma_g(P) = 2\gamma(P) - 1 = 5$, where $P$ is the Petersen graph, and that $\gamma_g(\widetilde{G}) = \gamma(\widetilde{G})$, where $G$ is an arbitrary graph and $\widetilde{G}$ is the graph (called *corona with base* $G$) obtained from $G$ by attaching a leaf to each vertex.

If $H$ is a graph, then we say that $G$ is a *generalized corona* with *base* $H$ if $G$ is constructed from $H$ by adding at least one leaf as a neighbor of each vertex of $H$.

**Lemma 8** *Let $G$ be a generalized corona. Then when Game 1 is played on $G$ there is an optimal strategy for Dominator in which he selects only base vertices.*

**Proof.** Players are playing Game 1 on generalized corona $G$ with base $H$. Dominator will imagine another game is played at the same time on the graph $G$, and he will play it according to an optimal strategy that forces the game to end in $\gamma_g(G)$ steps. In this imagined game it is possible that Dominator will make some of his moves on leaves. We need to show that in the real game there will be no more moves than in the imagined game. Whenever it is legal, Dominator will copy a move he makes in the imagined game to the real game, and copy the move made in response by Staller from the real to the imagined game. When in the imagined game an optimal move of Dominator is on a leaf $d_k$, then instead of copying the leaf to the real game, Dominator plays on the base vertex, $d'_k$, adjacent to that leaf. (Note that this is a legal move, since the leaf $d_k$ was not dominated before that.) Hence the sequences of moves are $d_1, s_1, \ldots, d_{k-1}, s_{k-1}, d_k$ in the imagined game and $d_1, s_1, \ldots, d_{k-1}, s_{k-1}, d'_k$ in the real game. The game continues in the same way, and whenever a leaf $d_i$ is played by Dominator in the imagined game, its neighboring base vertex $d'_i$ is picked in the real game. All moves of Staller (made in the real game) are legal when copied to the imagined game since he is playing on the graph (of the real game) in which all vertices, that are dominated in the graph of the imagined game, are dominated. Suppose that in the strategy of Dominator (that is used in the imagined game) he plays on $d'_k$ or one of the leaves attached to it (or some other base vertex whose leaf was previously chosen in the game). In this case Dominator imagines another move of Staller (on a leaf), and responds with the neighboring base vertex. His response is then copied also to the real game. Note that also after these moves, the set of vertices dominated in the real game contains the set of vertices dominated in the imagined game. Hence when the imagined game is finished the real game is also finished, and the number of moves in the real game is less than or equal to the number of moves in the imagined game. $\square$

The above lemma tells us also that we may assume that in a generalized corona Dominator plays on base vertices. More precisely, given a generalized corona $G$, there is a game in $G$ with $\gamma_g(G)$ moves in which Dominator plays only on base vertices and Staller uses an optimal strategy.

**Lemma 9** *Let $G$ be a generalized corona and let $x$ be a vertex in $G$ of degree 1. Then*

$$\gamma_g(G - x) \leq \gamma_g(G) \leq \gamma_g(G - x) + 1.$$

**Proof.** Let $G' = G - x$. We first prove that $\gamma_g(G') \leq \gamma_g(G)$. Suppose Game 1 is played on $G'$. As in the proof of Lemma 8, Dominator will imagine another game is played at the same time on the graph $G$, and he will play it according to a strategy that forces the game to end in at most $\gamma_g(G)$ steps. He will copy each of his moves from the imagined game to the real game, and copy each of Staller's moves from the real to the imagined game. Since $x$ cannot be selected by Staller (as he plays only on $G'$), we can assume that $x$ will not be selected during the imagined game played on $G$ because Dominator chooses only base vertices of $G$. Hence the moves in both games are the same all the time, and once the game on $G$ is finished, the game on $G'$ is also trivially finished.

We next prove that $\gamma_g(G) \leq \gamma_g(G') + 1$. Suppose Game 1 is played on $G$. Similarly as before, Dominator will imagine another game is played at the same time on the graph $G'$, and he will play it according to a strategy that forces the game to end in at most $\gamma_g(G')$ steps. He will copy each of his moves from the imagined game to the real game, and copy each of Staller's moves from the real to the imagined game, as long as this is possible. Note that at some point in the game Staller can choose $x$ which cannot be copied in the imagined game. In that case (when $x$ is chosen) Dominator imagines another move of Staller, which he may assume is a leaf $s_i$, in the imagined game, and responds according to his optimal strategy (which is always copied to the real game). If Staller later makes a move that is illegal in the imagined game (by picking the leaf $s_i$ or by picking the base neighbor of $s_i$ whose only undominated neighbor at that point is $s_i$), then Dominator again imagines another move of Staller in the imagined game, and so on. At the end of the imagined game all vertices are also dominated in the real game, except perhaps one leaf. Hence, the last player to move must dominate that leaf and the game ends. The number of steps is at most one more in the real game. In the case when $x$ is not chosen, the games have the same sequence of moves, according to the definition of the imagined game, and so after the imagined game is ended, $x$ is the only vertex that could be left undominated in the real game. In that case, $x$ must be dominated at the final step no matter whose turn it is.

In both cases the total number of vertices selected in the real game is at most one more than in the imagined game. Thus, $\gamma_g(G) \leq \gamma_g(G') + 1$. □

We can now prove the announced result of this section.

**Theorem 10** *For any $k \geq 1$ and any $r \in \{0, 1, \ldots, k-1\}$ there exists a graph $G$ such that $\gamma(G) = k$ and $\gamma_g(G) = k + r$.*

**Proof.** Consider an arbitrary graph $G$ with $\gamma(G) = 1$ to see that the statement holds for $k = 1$. Select now $k \geq 2$ and fix it. Let $G_k$ be the generalized corona with base $K_k$ and exactly one leaf attached everywhere. Then it is clear that $\gamma(G_k) = \gamma_g(G_k) = k$. Let $H_k$ be the generalized corona with base $K_k$ and enough, say $k$, leaves attached everywhere. Again, $\gamma(H_k) = k$. By Lemma 8, we may assume that when Game 1 is played on $H_k$, Dominator selects only base vertices. If Staller plays only on leaves of $H_k$, then no matter how they play, before Dominator selects the last base vertex, Staller has at least one leaf available to select. It follows that Dominator needs $k$ steps to finish the game and consequently, $\gamma_g(H_k) = 2k - 1$.

The generalized corona $H_k$ can be obtained from the corona $G_k$ by attaching leaves one by one until all the base vertices have $k$ leaves attached. In this way there is a sequence of graphs

$$G_k = X_0, X_1, \ldots, X_{k(k-1)} = H_k\,,$$

such that $\gamma_g(G_k) = k$, $\gamma_g(H_k) = 2k - 1$, and $\gamma_g(X_{i-1}) \leq \gamma_g(X_i) \leq \gamma_g(X_{i-1}) + 1$ for any $1 \leq i \leq k(k-1)$. Therefore, for any $r \in \{0, 1, \ldots, k-1\}$ there exists an index $j$, such that $\gamma_g(X_j) = k + r$. Since $\gamma(X_j) = k$, the proof is complete. □

We note that in the above proof a smaller number of leaves in $H_k$ would also do the job. If $2^s \leq k < 2^{s+1}$, then attaching $t$ leaves at each vertex of $K_k$, where $t > \log_2 k$, already suffices.

## 5 Relations to Vizing's conjecture

In this section we connect $\gamma_g$ with Vizing's conjecture, the main open problem in graph domination. This conjecture states that for all graphs $G$ and $H$,

$$\gamma(G \,\Box\, H) \geq \gamma(G)\gamma(H)\,,$$

where $G \,\Box\, H$ denotes the Cartesian product of graphs $G$ and $H$ [13]. (Recall that the *Cartesian product* $G \,\Box\, H$ of graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$ where vertices $(g, h)$ and $(g', h')$ are adjacent if and only if either $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$.) One of the finest results so far, related to this problem, is due to Clark and Suen [7] who proved that for all graphs $G$ and $H$,

$$\gamma(G \,\Box\, H) \geq \frac{1}{2}\gamma(G)\gamma(H).$$

We note that if Vizing's conjecture is true, then, for every pair of graphs $G$ and $H$, we can use Theorem 1 to get the following inequalities:

$$\gamma_g(G \,\Box\, H) \geq \gamma(G \,\Box\, H) \geq \gamma(G)\gamma(H) \geq \frac{1}{4}\gamma_g(G)\gamma_g(H)\,.$$

Thus, if one could find graphs $G$ and $H$ such that $\gamma_g(G)\gamma_g(H) > 4\,\gamma_g(G \,\Box\, H)$, then this shows that Vizing's conjecture is false.

Another connection between the game domination number and the inequality in Vizing's conjecture is the following. Suppose there is a constant $c > 0$ such that $c\,\gamma_g(G \,\Box\, H) \geq \gamma_g(G)\gamma_g(H)$ for any connected graphs $G$ and $H$. Then

$$c \cdot 2\gamma(G \,\Box\, H) \geq c \cdot \gamma_g(G \,\Box\, H) \geq \gamma_g(G)\gamma_g(H) \geq \gamma(G)\gamma(H)\,.$$

That is, for all such pairs $G$ and $H$,

$$\gamma(G \,\Box\, H) \geq \frac{1}{2c}\gamma(G)\gamma(H)\,.$$

Note that the bound for $c = 1$, if correct, generalizes the bound established by Clark and Suen. The $c = 1$ bound for connected graphs (that we find natural and plausible),

$$\gamma_g(G \,\Box\, H) \geq \gamma_g(G)\gamma_g(H)\,, \tag{1}$$

if true, is best possible. For instance, by Theorem 1 we infer that $\gamma_g(C_5 \,\Box\, C_5)$ is at most 9, and thus (1) would imply $\gamma_g(C_5 \,\Box\, C_5) = 9$.

We next prove a weaker form of (1), a result similar to the well-known bound on the ordinary domination number of the Cartesian product (see [14]). A subset $X$ of vertices is called a 2-*packing* if any two vertices from $X$ have disjoint closed neighborhoods. The maximum cardinality of a 2-packing in $G$ is denoted $\rho(G)$.

**Theorem 11** *Let $G$ and $H$ be arbitrary graphs. Then*

$$\gamma_g(G \,\Box\, H) \geq \rho(G)(\gamma_g(H) - 1) + 1\,.$$

**Proof.** Let $\{v_1, v_2, \ldots, v_\rho\}$ be a maximum 2-packing in $G$. In $G \,\Box\, H$ let $H_i = \{v_i\} \times V(H)$ and let $\overline{H}_i = N[v_i] \times V(H)$. Dominator and Staller play Game 1 on $G \,\Box\, H$.

In order to prove the theorem we need to describe an appropriate strategy of Staller. The basis of his strategy is that he will simultaneously play $\rho = \rho(G)$ separate games on $\overline{H}_1, \ldots, \overline{H}_\rho$. Any move of Dominator in $\overline{H}_j$ is answered by Staller in the same $\overline{H}_j$, as long as there is such a (legal) move. Moreover, as long as possible Staller proceeds as follows. He makes an optimal choice of a vertex from $H_j$ as if Dominator's choice had been made from the $H$-fiber $H_j$. (That is, if the vertex chosen by Dominator from within $\overline{H}_j$ is $(x, h)$ for some $x \in N[v_j]$, then Staller responds by choosing $(v_j, h')$ where $h'$ is Staller's optimal response to Dominator choosing $h$ when Game 1 is played on $H$.) When this is no longer possible, $V(H)$ is the union of the closed neighborhoods of second coordinates of the vertices chosen from $\overline{H}_j$; we say that Stage 1 of the game played on $\overline{H}_j$ has finished. Note that after Stage 1 has finished there can be additional moves of both players in $\overline{H}_j$.

Stage 1 of each of the games played on $\overline{H}_1, \ldots, \overline{H}_\rho$ can be considered as a Dominator-pass-$k$ game for some $k \geq 0$. Now, every pass of Dominator that happens in any of these games is a consequence of Dominator making the last (possible) move in some $\overline{H}_r$ (since after such a move, Staller must play in another $\overline{H}_s$). Note in addition that when Game 1 ends on $G \,\Box\, H$ and the last move was played in $\overline{H}_r$, then this does not lead to a Dominator pass. Denoting by $k_j$ the corresponding number of passes made in $\overline{H}_j$, it follows that $\sum_{j=1}^{\rho} k_j \leq \rho - 1$.

By Corollary 5 the number of moves played in $\overline{H}_j$ is at least $\gamma_g(H) - k_j$. We conclude that the total number of moves is at least

$$\sum_{j=1}^{\rho}(\gamma_g(H) - k_j) = \rho\gamma_g(H) - \sum_{j=1}^{\rho} k_j \geq \rho\gamma_g(H) - \rho + 1\,.$$

□

The bound in Theorem 11 is tight as can be seen when $G$ is totally disconnected and $H$ is any graph with $\gamma_g(H) = 2\ell + 1 = \gamma_g'(H) + 1$ (e.g., $H = C_6$).

# 6 Concluding remarks

It is well-known that the chromatic analog of $\gamma_g$ is not monotone, even with respect to spanning subgraphs; a typical case is the complete bipartite graph $K_{n,n}$ whose game chromatic number is 3, while its spanning subgraph, obtained by removing a perfect matching has game chromatic number $n$. Somewhat surprisingly, $\gamma_g$ does not behave on spanning subgraphs in a manner consistent with the ordinary domination number $\gamma$. For example, if $H$ is the connected spanning subgraph of $K_2 \,\square\, K_3$ obtained by deleting two $K_2$-fiber's edges, then

$$\gamma_g(K_2 \,\square\, K_3) = 3 > 2 = \gamma_g(H)\,.$$

We continue with a question that could be very useful in proving results about the game domination number.

**Problem 1** *Let $G$ be a graph, and let $A, B$ be subsets of $V(G)$ such that $A \subset B$. Suppose that Game 1 (or Game 2) is played on two copies of $G$. On the first copy of $G$ the set of vertices that are already dominated is $A$, and on the second copy it is $B$. In both games the same player (Dominator or Staller) has the next move. Is it then true that the number of moves until the end on the first copy is at least as big as the number of moves on the second copy?*

We suspect that the above problem has an affirmative answer, but could find no argument.

Note that the bound in Lemma 2 is sharp (for example on $G = K_{1,2} \cup C_6$), but for Lemma 3 we do not know. As we have seen there exist graphs $G$ for which $\gamma_g'(G) = \gamma_g(G) + 1$. We don't know whether the upper bound of Theorem 6 can be lowered by 1 and hence pose:

**Conjecture 1** *Pairs $(k, k+2)$ for $k \geq 1$ are not realizable. That is, there exists no graph $G$, such that $\gamma_g(G) = k$ and $\gamma_g'(G) = k + 2$.*

We also suspect that the remaining pairs that satisfy the conditions of Theorem 6 are not realizable. More precisely:

**Conjecture 2** *Pairs* $(2k, 2k-1)$ *for* $k \geq 1$ *are not realizable. That is, there exists no graph* $G$*, such that* $\gamma_g(G) = 2k$ *and* $\gamma_g'(G) = 2k-1$*.*

It would be interesting to determine the game domination number for some well-known classes of graphs such as trees or chordal graphs. In particular, we pose the question whether there is a polynomial strategy for Dominator and Staller on an arbitrary tree. Note that the greedy Staller strategy to choose a leaf when one is legal is not always optimal as can be seen by considering $P_5$.

It is well-known that the domination number of a connected graph is at most half of its order. This is not the case with the game domination number since $\gamma_g(P_{4m+1}) = 2m+1$. We conclude the paper by asking what is the maximum ratio between the game domination number of a connected graph and its order.

# References


[1] N. Alon, J. Balogh, B. Bollobás, T. Szabó, Game domination number, Discrete Math. 256 (2002) 23–33.

[2] S. D. Andres, The incidence game chromatic number, Discrete Appl. Math. 157 (2009) 1980–1987.

[3] T. Bartnicki, B. Brešar, J. Grytczuk, M. Kovše, Z. Miechowicz, I. Peterin, Game chromatic number of Cartesian product graphs, Electron. J. Combin. 15 (2008) #R72, 13 pp.

[4] T. Bartnicki, J. Grytczuk, H. A. Kierstead, The game of arboricity, Discrete Math. 308 (2008) 1388–1393.

[5] T. Bartnicki, J. Grytczuk, H. A. Kierstead, X. Zhu, The map-coloring game, Amer. Math. Monthly 114 (2007) 793–803.

[6] A. Beveridge, T. Bohman, A. Frieze, O. Pikhurko, Game chromatic index of graphs with given restrictions on degrees, Theoret. Comput. Sci. 407 (2008) 242–249.

[7] W. E. Clark, S. Suen, An inequality related to Vizing's conjecture, Electron. J. Combin. 7 (2000) #N4, 3 pp.

[8] T. Dinski, X. Zhu, Game chromatic number of graphs, Discrete Math. 196 (1999) 109–115.

[9] O. Favaron, H. Karami, R. Khoeilar, S. M. Sheikholeslami, L. Volkmann, Proof of a conjecture on game domination, J. Graph Theory, in press, doi:10.1002/jgt.20459.

[10] A. Fraenkel, Combinatorial games: selected bibliography with a succinct gourmet introduction, Electron. J. Combin. (January 1, 2009) DS2, 88pp.

[11] M. Gardner, Mathematical games, Scientific American 244 (1981) 18–26.

[12] M. Henning, personal communication, 2003.

[13] W. Imrich, S. Klavžar, D. F. Rall, *Topics in Graph Theory: Graphs and Their Cartesian Product*, A K Peters, Wellesley, MA, 2008.

[14] M. S. Jacobson and L. F. Kinch, On the domination of the products of graphs II: trees, J. Graph Theory 10 (1986) 97–106.

[15] H. A. Kierstead, A. V. Kostochka, Efficient graph packing via game colouring, Combin. Probab. Comput. 18 (2009) 765–774.

[16] H. A. Kierstead, T. Trotter, Planar graph coloring with an uncooperative partner, J. Graph Theory 18 (1994) 569–584.

[17] H. A. Kierstead, T. Trotter, Competitive colorings of oriented graphs, Electron. J. Combin. 8 (2001) #R12, 15pp.

[18] J. Y. Kim, The incidence game chromatic number of paths and subgraphs of wheels, Discrete Appl. Math., in press, doi:10.1016/j.dam.2010.01.001.

[19] J. Nešetřil, E. Sopena, On the oriented game chromatic number, Electron. J. Combin. 8 (2001) R#14, 13pp.

[20] X. Zhu, Game coloring the Cartesian product of graphs, J. Graph Theory 59 (2008) 261–278.

[21] X. Zhu, Refined activation strategy for the marking game, J. Combin. Theory ser. B 98 (2008) 1–18.