# Domination game: effect of edge- and vertex-removal

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### Abstract

The domination game is played on a graph $G$ by two players, named Dominator and Staller. They alternatively select vertices of $G$ such that each chosen vertex enlarges the set of vertices dominated before the move on it. Dominator's goal is that the game is finished as soon as possible, while Staller wants the game to last as long as possible. It is assumed that both play optimally. Game 1 and Game 2 are variants of the game in which Dominator and Staller has the first move, respectively. The game domination number $\gamma_g(G)$, and the Staller-start game domination number $\gamma_g'(G)$, is the number of vertices chosen in Game 1 and Game 2, respectively. It is proved that if $e \in E(G)$, then $|\gamma_g(G) - \gamma_g(G - e)| \leq 2$ and $|\gamma_g'(G) - \gamma_g'(G - e)| \leq 2$, and that each of the possibilities here is realizable by connected graphs $G$ for all values of $\gamma_g(G)$ and $\gamma_g'(G)$ larger than 5. For the remaining small values it is either proved that realizations are not possible or realizing examples are provided. It is also proved that if $v \in V(G)$, then $\gamma_g(G) - \gamma_g(G - v) \leq 2$ and $\gamma_g'(G) - \gamma_g'(G - v) \leq 2$. Possibilities here are again realizable by connected graphs $G$ in almost all the cases, the exceptional values are treated similarly as in the edge-removal case.

**Keywords:** domination game; game domination number; edge-removed subgraph; vertex-removed subgraph

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## 1 Introduction

The domination game is played on an arbitrary graph $G$ by two players, *Dominator* and *Staller*. They are taking turns choosing a vertex from $G$ such that whenever they choose

a vertex, it dominates at least one previously undominated vertex. The game ends when all vertices of $G$ are dominated, so that the set of vertices selected at the end of the game is a dominating set of $G$. The aim of Dominator (Staller) is that the total number of moves played in the game is as small (as large, resp.) as possible. By *Game 1* (*Game 2*) we mean a game in which Dominator (Staller, resp.) has the first move. Assuming that both players play optimally, the *game domination number* $\gamma_g(G)$ (the *Staller-start game domination number* $\gamma'_g(G)$) of a graph $G$, denotes the number of vertices chosen in Game 1 (Game 2, resp.).

Note that the domination game is not a combinatorial game in the strict sense of [5], where the outcome of a game is assumed to be only of the types (lose, win), (tie, tie) and (draw, draw) for the two players.

The domination game was introduced in [1] (with the idea going back to [7]) and explored by now from several points of view. Despite the fact that $\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1$ holds for any graph $G$ (see [1]), the game domination number is essentially different from the domination number. First of all, $\gamma_g(G)$ is generally much more difficult to determine than $\gamma(G)$. Even on simple graphs such as paths and cycles, the problem of determining $\gamma_g$ is non-trivial [9].

As proved in [1, 8], the game domination number and the Staller-start game domination number can differ only by 1: $|\gamma_g(G) - \gamma'_g(G)| \leq 1$. Call a pair of integers $(k, \ell)$ *realizable* if there exists a graph $G$ with $\gamma_g(G) = k$ and $\gamma'_g(G) = \ell$. Some classes of graphs for possible realizable pairs are given in [1, 2, 12]. For the complete answer that all pairs that are potentially realizable can be realized (with relatively simple families of graphs) see [10].

Kinnersley, West, and Zamani [8] conjectured that if $G$ is an isolate-free forest of order $n$ or an isolate-free graph of order $n$, then $\gamma_g(G) \leq 3n/5$. Actually they posed two conjectures, because while the truth for isolate-free graphs clearly implies the truth for isolate-free forests, it is not known whether the converse implication holds. These conjectures are known as the 3/5-conjectures. In [3] large families of trees were constructed that attain the conjectured 3/5-bound and all extremal trees on up to 20 vertices were found; in particular, there are exactly ten trees $T$ on 20 vertices with $\gamma_g(T) = 12$. Further progress on the 3/5-conjecture for forest was very recently done by Bujtás [4] by proving that the 3/5-conjecture holds for the class of forests in which no two leaves are at distance 4.

Clearly, removing an edge from a graph cannot decrease its domination number, that is, $\gamma(G - e) \geq \gamma(G)$. (For an extensive survey on graphs that are domination critical with respect to edge- and vertex-removal see [11].) On the other hand, it was proved in [2] that for any integer $\ell \geq 1$, there exists a graph $G$ and a spanning tree $T$ such that $\gamma_g(T) \leq \gamma_g(G) - \ell$. In this paper we answer the question how much $\gamma_g(G)$ and $\gamma'_g(G)$ can change if an edge is removed from $G$. The answer is given in Theorem 2.1 which is followed by ten subsections in which each of the possibilities indicated by the theorem, is shown to be realizable by connected graphs. We also ask the analogous question for vertex-removal and present the answer in Theorem 3.1. Again, all possibilities can be realized by connected graphs. We conclude the paper with some natural open problems, concerning extensions or generalizations of the results from this paper.

For a vertex subset $S$ of a graph $G$, let $G|S$ denote the graph $G$ in which vertices from $S$ are considered as being already dominated. In particular, if $S = \{x\}$ we write $G|x$. When describing a strategy for a given player, we often use the phrase that the player *follows* the

other player in some (sub)graph. This means that if the last move of the other player was on a (sub)graph $G$, then the next move of the player is a vertex of $G$. In the case that no legal move exists on $G$, the next move of the player will be further defined. For all the other standard notions not defined in this paper see the monograph on graph domination [6].

In the rest of this section we state some known results to be used in the sequel.

**Theorem 1.1** ([8, Lemma 2.1] - Continuation Principle) *Let $G$ be a graph and $A, B \subseteq V(G)$. If $B \subseteq A$, then $\gamma_g(G|A) \leq \gamma_g(G|B)$ and $\gamma'_g(G|A) \leq \gamma'_g(G|B)$.*

**Theorem 1.2** ([1, 8]) *If $G$ is any graph, then $|\gamma_g(G) - \gamma'_g(G)| \leq 1$.*

**Theorem 1.3** ([9]) *If $n \geq 3$, then*

$$\gamma_g(C_n) = \gamma_g(P_n) \;\; = \;\; \begin{cases} \lceil \frac{n}{2} \rceil - 1; & n \equiv 3 \bmod 4, \\ \lceil \frac{n}{2} \rceil; & \text{otherwise}. \end{cases}$$

$$\gamma'_g(P_n) \;\; = \;\; \left\lceil \frac{n}{2} \right\rceil .$$

$$\gamma'_g(C_n) \;\; = \;\; \begin{cases} \lceil \frac{n-1}{2} \rceil - 1; & n \equiv 2 \bmod 4, \\ \lceil \frac{n-1}{2} \rceil; & \text{otherwise}. \end{cases}$$

**Theorem 1.4** [8, Theorem 4.6] *If $F$ is a forest and $S \subseteq V(F)$, then $\gamma_g(F|S) \leq \gamma'_g(F|S)$.*

## 2 Edge removal

**Theorem 2.1** *If $G$ is a graph and $e \in E(G)$, then*

$$|\gamma_g(G) - \gamma_g(G - e)| \leq 2 \qquad \text{and} \qquad |\gamma'_g(G) - \gamma'_g(G - e)| \leq 2 .$$

**Proof.** To prove the bound $\gamma_g(G - e) \leq \gamma_g(G) + 2$ it suffices to show that Dominator has a strategy on $G - e$ such that at most $\gamma_g(G) + 2$ moves are played. His strategy is to play the game on $G - e$ as follows. In parallel to the real game he is playing an *imagined game* on $G$ by copying every move of Staller to this game and responding optimally in $G$. Each response in the imagined game is then copied back to the real game in $G - e$. Let $e = uv$ and consider the following possibilities.

Suppose first that neither Staller nor Dominator play on either of $u$ and $v$ in the course of the real game. This makes all the moves in both games legal and so the imagined game on $G$ lasts no more than $\gamma_g(G)$ moves. (Recall that Dominator plays optimally on $G$ but Staller might not play optimally.) Since the game on $G - e$ uses the same number of moves, we conclude that in this case the number of moves played in the real game is at most $\gamma_g(G)$.

Assume now that at some point of the game, the strategy of Dominator on $G$ is to play a vertex incident with $e$, say $u$, but this move is not legal in the real game. This can happen only in the case when $v$ is the only vertex in $N_G[u]$ not yet dominated. In this case Dominator plays $v$ in the real game and by Theorem 1.1 (that is, by the Continuation Principle), following the same strategy Dominator ensures that the game is finished in no more than $\gamma_g(G)$ moves.

Assume next that in the course of the game one of the players played a vertex incident with $e$, say $u$, and that this is a legal move. This means that, after this move is copied

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into the imagined game on $G$, the vertex $v$ is dominated in this game but may not yet be dominated in the real game. If all the moves are legal in the real game (played on $G - e$), then after at most $\gamma_g(G)$ moves all vertices except maybe $v$ are dominated. Hence the real game finishes in no more than $\gamma_g(G) + 1$ moves. In the other case Staller played a move in which only $v$ was newly dominated, and this is not a legal move in $G$. Let this move of Staller in $G - e$ be the $k$-th move of the game. Note that after this move of Staller, the sets of dominated vertices are the same in both games, denote this set with $D$. Since after the $(k-1)$st move it is Staller's turn in the imagined game, we derive that

$$(k - 1) + \gamma_g'(G|D) \le \gamma_g(G). \tag{1}$$

Indeed, the imagined game (played on $G$) lasts at most $(k - 1) + \gamma_g'(G|D)$ moves. On the other hand, since Staller does not necessarily play optimally in the imagined game, it follows that $\gamma_g(G) \ge (k - 1) + \gamma_g'(G|D)$.

Now Dominator does not copy the move of Staller into the imagined game but simply optimally plays the next moves. Therefore, since the number of moves left to end each of the games is $\gamma_g\left((G - e)|D\right)$, we have:

$$
\begin{aligned}
\gamma_g(G - e) &\le k + \gamma_g\left((G - e)|D\right) \\
&= k + \gamma_g(G|D) \\
&\le k + \gamma_g'(G|D) + 1 \qquad \text{(by Theorem 1.2)} \\
&\le \gamma_g(G) + 2. \qquad \text{(by (1))}
\end{aligned}
$$

We have thus proved that $\gamma_g(G - e) \le \gamma_g(G) + 2$. Note that in the above proof of this inequality it does not matter whether Game 1 or Game 2 is played on $G - e$. Hence analogous arguments also give us $\gamma_g'(G - e) \le \gamma_g'(G) + 2$.

For the rest of the proof let $A = N_G[u]$.

We next want to demonstrate that $\gamma_g(G) \le \gamma_g(G - e) + 2$. The strategy of Dominator on $G$ is to first play on $u$. We get

$$
\begin{aligned}
\gamma_g(G) &\le 1 + \gamma_g'(G|A) \\
&= 1 + \gamma_g'((G - e)|A) \\
&\le 1 + \gamma_g'(G - e) \qquad \text{(by the Continuation Principle)} \\
&\le \gamma_g(G - e) + 2. \qquad \text{(by Theorem 1.2)}
\end{aligned}
$$

Note that the equality in the above computation holds because $u$ and $v$ are both in $A$, hence dominated in $G|A$. Thus the edge $e = uv$ is not relevant for any later move.

To complete the proof we show that $\gamma_g'(G) \le \gamma_g'(G - e) + 2$. Suppose that Staller played first on one of the end vertices of $e$, say $u$. We argue as follows:

$$
\begin{aligned}
\gamma_g'(G) &= 1 + \gamma_g(G|A) \\
&= 1 + \gamma_g((G - e)|A) \\
&\le 1 + \gamma_g(G - e) \qquad \text{(by the Continuation Principle)} \\
&\le \gamma_g'(G - e) + 2. \qquad \text{(by Theorem 1.2)}
\end{aligned}
$$

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Assume now that the first selected vertex $x$ by Staller is neither $u$ nor $v$. In this case Dominator replies with the move on $u$. Now we get

$$
\begin{aligned}
\gamma'_g(G) &= 1 + \gamma_g(G|N[x]) \\
&\leq 2 + \gamma'_g(G|(N[x] \cup A)) \\
&= 2 + \gamma'_g((G - e)|(N[x] \cup A)) \\
&\leq \gamma'_g(G - e) + 2. \qquad \text{(by the Continuation Principle)}
\end{aligned}
$$

$\square$

In the remainder of this section we demonstrate that all possibilities indicated in Theorem 2.1 are realizable by presenting infinite families of connected graphs for each of the cases. Two graphs frequently appear in our constructions, notably $C_6$ and the graph $Z$ from Fig. 1. Recall that $\gamma_g(C_6) = 3 = \gamma_g(C_6|y)$ and $\gamma'_g(C_6) = 2 = \gamma'_g(C_6|y)$, where $y$ is an arbitrary vertex of $C_6$. Note also that $\gamma_g(Z) = 4 = \gamma_g(Z|z)$ and $\gamma'_g(Z) = 3 = \gamma'_g(Z|z)$, where $z$ is the vertex indicated in Fig. 1.
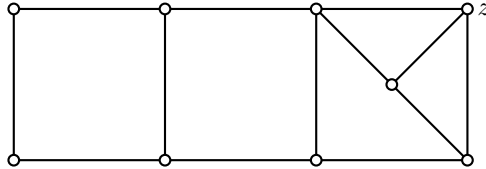


Figure 1: Graph $Z$

## 2.1 $\quad \gamma_g(G) - \gamma_g(G - e) = -2$

**Proposition 2.2** *For any $\ell \geq 3$ there exists a graph $G$ with an edge $e$ such that $\gamma_g(G) = \ell$ and $\gamma_g(G - e) = \ell + 2$.*

**Proof.** We present two different infinite families $U_k$ and $V_k$ realizing odd and even $\ell$, respectively. Let $B$ be the graph isomorphic to $K_{1,4}$ plus an edge, and denote its central vertex by $x$. Let $U_0$ be the graph obtained from the disjoint union of $C_6$ and $B$ by connecting an arbitrary vertex $u$ of the 6-cycle to $x$ in $B$. The graph $U_k$, $k \geq 1$, is obtained from $U_0$ by identifying one end vertex of each of $k$ copies of $P_6$ with $x$, see Fig. 2.
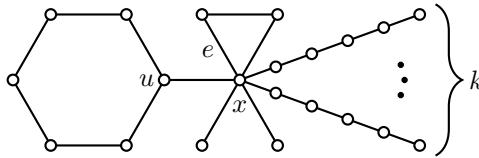


Figure 2: Graphs $U_k$

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We claim that $\gamma_g(U_k) = 2k + 3$ and $\gamma_g(U_k - e) = 2k + 5$, where $e$ is one of the two edges incident to $x$ for which $U_k - e$ remains connected, see Fig. 2 again. By Theorem 2.1 it suffices to prove that $\gamma_g(U_k) \leq 2k + 3$ and $\gamma_g(U_k - e) \geq 2k + 5$.

For the first inequality we present a strategy for Dominator that guarantees at most $2k + 3$ moves are played on $U_k$. Dominator starts by playing $x$. Then, he follows Staller in the 6-cycle and in each of the $k$ attached paths, ensuring two moves in each of the subgraphs. Thus $\gamma_g(U_k) \leq 2k + 3$.

To prove the second inequality, Staller's strategy is to follow Dominator in each of the subgraphs of $U_k$: the 6-cycle, $B - e$, and each of the $k$ attached paths. If Dominator is the first to play in $C_6$, then three moves are played on it. Since each of the $B - e$ and the $k$ attached paths requires at least two moves, the total number of moves is at least $2k + 5$. On the other hand, if Staller is forced to play first on the $C_6$, this means that at least 3 moves were played either on $B - e$ or on some path. (Indeed, if before playing on the $C_6$, exactly two moves would be played on each of the other subgraphs, then $2k + 2$ moves would be played in total and so Dominator would be forced to play first on the $C_6$.) Consequently, if Staller is forced to play first on the $C_6$, then at least $2k + 3$ moves were already played. Hence at least two more moves are required to finish the game. Hence in any case $\gamma_g(U_k - e) \geq 2k + 5$. This concludes the proof for the case when $\ell$ is odd.

For the case when $\ell$ is even we construct a family $V_k$ in a similar way as $U_k$ by replacing the 6-cycle with the graph $Z$ from Fig. 1. More precisely for $V_0$ we take the disjoint union of $Z$ and $B$, and add an edge connecting $z$ and $x$. Then the graph $V_k$, $k \geq 1$, is obtained from $V_0$ by identifying one end vertex of $k$ copies of $P_6$ with $x$. By using parallel arguments to the above case one can derive that $\gamma_g(V_k) = 2k + 4$ and $\gamma_g(V_k - e) = 2k + 6$. □

To round off the subsection we show that when $\ell < 3$, there exists no graph $G$ such that $\gamma_g(G) = \ell$ and $\gamma_g(G - e) = \ell - 2$. Indeed, we have:

- If $\gamma_g(G) = 1$, then $\gamma_g(G - e) \leq 2$.
  If $\gamma_g(G) = 1$, then $G$ has a universal vertex $v$. In $G - e$, Dominator can play $v$ and in this way dominate all but at most one vertex $w$. Hence Staller has to dominate $w$ in any legal move. □

- If $\gamma_g(G) = 2$, then $\gamma_g(G - e) \leq 3$.
  We prove this by contradiction: suppose $\gamma_g(G) = 2$ but $\gamma_g(G - e) = 4$. We propose a strategy of Staller in $G$ that requires at least 3 moves to be played. Suppose Dominator plays on some vertex $d_1$ in $G$, and let $s_1, d_2, s_2$ be the next three optimal moves played in the game on $G - e$. Let $s_1', d_2', s_2'$ be vertices (not necessarily distinct from $s_1, d_2, s_2$) that are newly dominated in $G - e$ when $s_1, d_2, s_2$ are played, respectively. Note that in $G$, if Staller plays on $s_1$, at most one of $d_2'$ and $s_2'$ may be dominated. Thus since $\gamma_g(G) = 2$, the move $s_1$ is not a legal move. Therefore the edge $e$ necessarily connects $s_1'$ either to $d_1$ or to $s_1$. Now if Staller plays on $d_2$, she does not dominate $s_2'$, hence $\gamma_g(G) \geq 3$, a contradiction. □

## 2.2  $\gamma_g(G) - \gamma_g(G - e) = -1$

Let $T$ be a tree with $\gamma_g(T) = \ell$, $\ell \geq 1$, and let $v$ be an optimal first move for Dominator. Let $G_\ell$ be the graph obtained from $T$ by attaching two additional leaves to $v$ and identifying

$v$ with a vertex of a triangle. Note that $|V(G_\ell)| = |V(T)| + 4$. Let $e$ be an edge of the triangle incident with $v$, having $y$ as the other end vertex. By the Continuation Principle, Dominator should not play on any of the new four vertices added to the tree $T$ (since he could play on $v$ instead). Hence $v$ is also an optimal start vertex for Dominator in $G_\ell$, and so $\gamma_g(G_\ell) = \ell$.

At first we prove that $\gamma_g(G_\ell - e) \leq \ell + 1$ by giving a strategy for Dominator. In $G_\ell - e$, Dominator starts by playing $v$, and plays $y$ only if it is the only legal move. If Staller plays on $y$ or its neighbor at the $k$th move, then Dominator continues optimally and the total number of moves is $\gamma_g(T|D) + k$, where $D$ is the set of dominated vertices at this stage in $T$. Applying Theorem 1.4, this is at most $\gamma'_g(T|D) + k = \gamma_g(T) + 1 = \ell + 1$. Otherwise, dominator plays $y$ on his last move and the game also ends after $\ell + 1$ moves. To see that $\gamma_g(G_\ell - e) \geq \ell + 1$ we present Staller's strategy. Whenever Dominator plays in the subgraph of $G_\ell - e$ that corresponds to $T$, she responds in this subgraph as well, by playing as if the game was played in $T$. Note that if Dominator plays the neighbor of $y$, then Staller is the first to play in the remainder of the game with respect to $T$, which thus takes in total at least $\ell$ steps, by using Theorem 1.4. Hence at least $\ell + 1$ moves are played in $G_\ell - e$. If Dominator does not play on the neighbor of $y$ during the game, then in the last move Staller plays on it, which concludes the proof of $\gamma_g(G_\ell - e) = \ell + 1$. $\qquad \square$

## 2.3 $\quad \gamma_g(G) - \gamma_g(G - e) = 0$

From Theorem 1.3 we get that $\gamma_g(C_n) = \gamma_g(C_n - e)$ for any $n \geq 3$. $\qquad \square$

## 2.4 $\quad \gamma_g(G) - \gamma_g(G - e) = 1$

**Proposition 2.3** *For any $\ell \geq 3$ there exists a graph $G$ with an edge $e$ such that $\gamma_g(G) = \ell$ and $\gamma_g(G - e) = \ell - 1$.*

**Proof.** For the case when $\ell = 3$, take the disjoint union of two complete graphs of order at least three and add two edges that form a matching in the resulting graph. Clearly, the game domination number of this graph is 3, while after removing one of the two additional edges the game domination number drops to 2.

For the general case when $\ell \geq 4$, we construct the following family of graphs denoted by $Y_k$, $k \geq 0$. Let $Y_0$ be obtained in the following way. Let $t$ be a vertex of some $C_5$ and $x, x'$ its neighbors. Add a new vertex $y$ connected to both $x$ and $x'$. Finally, attach two leaves to $x$ and one to $t$. For $k \geq 1$, the graph $Y_k$ is obtained from $Y_0$ by identifying the end vertices of $k$ copies of $P_3$ with $x$, see Fig. 3. We claim that $\gamma_g(Y_k) = k + 4$ and $\gamma_g(Y_k - e) = k + 3$, where $e = x'y$ as in Fig. 3.

Note that if Dominator plays his first move on $x$, then only $k + 3$ vertices remain undominated which already yields $\gamma_g(Y_k) \leq k + 4$. Next we present the strategy for Staller which ensures that at least $k + 4$ moves are needed to end the game in $Y_k$.

If Dominator starts on $x$, then Staller responds on $y$. After these two moves, there are still $k + 2$ isolated vertices left undominated, no pair of which has a common neighbor in $Y_k$. Hence $k + 4$ moves must be played in total. Otherwise, if Dominator does not start on $x$, then Staller responds on a leaf adjacent to $x$. It follows that in the subgraph of $Y_k$ that corresponds to $Y_0$ at least 4 moves must be played. In turn at least $k$ moves must be
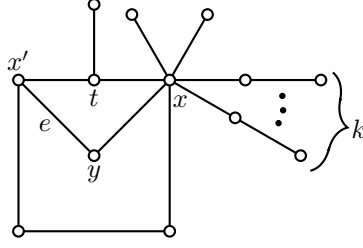
Figure 3: Graphs $Y_k$

played in the $k$ attached paths, thus at least $k + 4$ moves are needed. We conclude that $\gamma_g(Y_k) = k + 4$.

To prove that $\gamma_g(Y_k - e) \leq k + 3$ we first explain the strategy of Dominator. He starts on $x$. If Staller dominates two vertices in the next move, then $k+1$ vertices remain undominated and the bound is ensured. Otherwise, in his second move Dominator dominates two new vertices by playing either $t$ or $x'$. This gives the desired upper bound for $\gamma_g(Y_k - e)$. Finally, we present a strategy for Staller that guarantees at least $k + 3$ moves are played in $Y_k - e$. If Dominator does not play $x$, then the strategy for Staller is the same as above when the game was played in $Y_k$ (in particular, she responds on a leaf adjacent to $x$). Otherwise, if Dominator plays $x$ in his first move, then Staller responds on $t$. Note that $k+1$ isolated vertices, no pair of each has a common neighbor, remain undominated, which yields $\gamma_g(Y_k - e) \geq k + 3$. This concludes the proof. $\qquad \square$

There exists no graph $G$ with $\gamma_g(G) = 2$ and $\gamma_g(G - e) = 1$. Actually, we have the following:

- If $\gamma_g(G) \geq 2$, then $\gamma_g(G - e) \geq 2$. If $\gamma_g(G) \geq 4$, then $\gamma_g(G - e) \geq 3$.
  It is proved in [1] that $\gamma_g(G) \leq 2\gamma(G) - 1$, or, equivalently, $\gamma(G) \geq \frac{\gamma_g(G)+1}{2}$. Together with the fact that the domination number does not decrease by edge removal, this implies

  $$\gamma_g(G - e) \geq \gamma(G - e) \geq \gamma(G) \geq \left\lceil \frac{\gamma_g(G) + 1}{2} \right\rceil \tag{2}$$

  and this yields the desired bounds for $\gamma_g(G) = 2, 3$ or $4$. $\qquad \square$

## 2.5 $\quad \gamma_g(G) - \gamma_g(G - e) = 2$

**Proposition 2.4** *For any $\ell \geq 5$ there exists a graph $G$ with an edge $e$ such that $\gamma_g(G) = \ell$ and $\gamma_g(G - e) = \ell - 2$.*

**Proof.** We present families of graphs $X_k$ and $Q_k$ that realize even and odd values of $\ell$, respectively. The arguments for the first family are given in full detail, while the reasoning for $Q_k$ is analogous.

We construct $X_0$ as follows. Duplicate the vertex $z$ in $Z$ (see Fig. 1) obtaining a new vertex $z'$ with the same closed neighborhood as $z$, and denote the resulting graph by $Z'$. Next, take the disjoint union of $Z'$ with $K_{1,5}$ having $x$ as its center and denote one of its

leaves by $x'$. Finally we get $X_0$ by connecting $z$ with $x$, and $z'$ with $x'$. The graph $X_k$, $k \geq 1$, is obtained from $X_0$ by identifying one end vertex of each of $k$ copies of $P_6$ with $x$, see Fig. 4. We set $e$ to be the edge between $z'$ and $x'$.
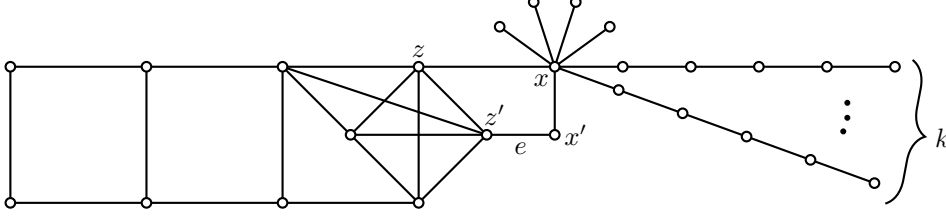


Figure 4: Graphs $X_k$

We claim that $\gamma_g(X_k) = 2k + 6$ and $\gamma_g(X_k - e) = 2k + 4$. By Theorem 1.2 it suffices to present a strategy for Dominator yielding $\gamma_g(X_k - e) \leq 2k + 4$, and a strategy for Staller which gives $\gamma_g(X_k) \geq 2k + 6$. To show the first inequality, Dominator starts the game by playing $x$. Any move of Staller in one of the $k$ attached paths is followed by a move of Dominator in the same path, so that all vertices of this path are dominated. With this strategy Staller is forced to be the first to play in the subgraph that corresponds to $Z'$. Since $\gamma_g'(Z'|z) = 3$, Dominator can ensure that only three moves are played in this subgraph. If at some stage of the game Dominator is forced to play first on an attached path, he plays there such that two moves are played on the path. Altogether we get that $2k + 4$ moves are played in $X_k - e$.

It remains to present a strategy for Staller in $X_k$. Note that we may assume that Dominator never plays on a leaf at $x$ since by the Continuation Principle he would rather play on $x$ itself. Whenever Dominator plays on one of the $k$ attached paths, Staller follows on the same path in such a way that all vertices on the path at distance at least 2 from $x$ are dominated after her move. If Dominator plays one of the vertices $x$ or $x'$, Staller responds with a move on the other vertex from $\{x, x'\}$, if this is possible. Note that this is not possible only in the case when $z'$ and $z$ were dominated before Staller's move. By this strategy, Staller forces Dominator to be the first to play in the subgraph isomorphic to $Z'$. Suppose first that when Dominator starts to play in $Z'$, $x$ and $x'$ have already been played. Since $\gamma_g(Z'|\{z, z'\}) = 4$, at least four moves must be played in $Z'$, hence together with $2k$ moves on the attached paths the total number of moves sums up to at least $2k + 6$. Otherwise, if $x$ and $x'$ have not been played at the time when Dominator starts to play in $Z'$, then Staller responds by playing on one of the leaves attached to $x$. If the next move of Dominator which is not played on one of the attached paths, is also played in $Z'$, then Staller responds by playing on one of the leaves attached to $x$, again. Since $\gamma(Z') = 3$, at this point in the game there are still undominated vertices left in $Z'$ as well as two undominated leaves attached to $x$. Thus at least two more moves are needed (outside of the attached copies of $P_k$), altogether at least $2k + 6$ moves. On the other hand, if the next move of Dominator which is not played on one of the attached paths, is played on $x$, then Staller's next move is in $Z'$ ensuring four moves are played in $Z'$. Again we get that at least $2k + 6$ moves are played in $X_k$ in total which concludes the proof for even $\ell$. Note that if at some stage of the game Staller is forced to play first on an attached path (if Dominator does not play

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optimally in $Z'$ and therefore an odd number of moves greater than four is played in $Z'$), she plays there such that two moves are played on the path.
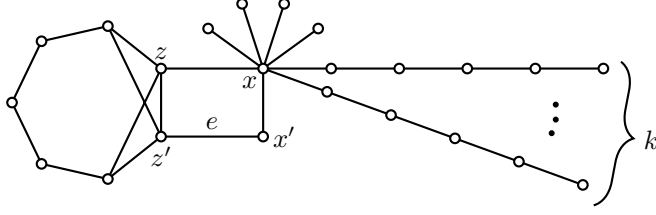


Figure 5: Graphs $Q_k$

The family $Q_k$ which realizes the case when $\ell$ is odd is constructed as follows. Take a copy of $C_6$, denote one of its vertices by $z$, and add a duplicate vertex $z'$ of $z$ so that this two vertices have the same closed neighborhoods. Denote the resulting graph by $Z''$ and take the disjoint union of $Z''$ with $K_{1,5}$ having $x$ as its center, and denote one of its leaves by $x'$. Finally we get $Q_0$ by connecting $z$ with $x$, and $z'$ with $x'$. The graph $Q_k$, $k \geq 1$, is obtained from $Q_0$ by attaching $k$ copies of $P_6$ at their end vertices to $x$, see Fig. 5. We set $e$ to be the edge between $z'$ and $x'$. As noted in the beginning of the proof, the arguments for $\gamma_g(Q_k) = 2k + 5$ and $\gamma_g(Q_k - e) = 2k + 3$ follow similar lines as above. In particular, when Staller is forced to play first on an attached path (if odd number of moves is played in $Z''$), she plays there in such a way that two moves are played on the path. $\qquad\square$

By inequality (2), there exists no graph $G$ such that $\gamma_g(G) = \ell$ and $\gamma_g(G - e) = \ell - 2$ for some edge $e$ when $\ell \leq 4$.

## 2.6 $\quad \gamma'_g(G) - \gamma'_g(G - e) = -2$

Similarly as in Subsection 2.1, one can verify that $\gamma'_g(U_k) = 2k + 4$ and $\gamma'_g(U_k - e) = 2k + 6$ for any $k \geq 0$. Also, $\gamma'_g(V_k) = 2k + 5$ and $\gamma'_g(V_k - e) = 2k + 7$. In particular, note that the optimal first move of Staller is to play on a leaf adjacent to $x$. Hence:

**Proposition 2.5** *For any $\ell \geq 4$ there exists a graph $G$ with an edge $e$ such that $\gamma'_g(G) = \ell$ and $\gamma'_g(G - e) = \ell + 2$.*

Note also that for $\ell < 4$, there are no graphs such that $\gamma'_g(G) = \ell$ and $\gamma'_g(G - e) = \ell + 2$ for some edge $e$ in $G$. Indeed, we have:

- If $\gamma'_g(G) = 1$, then $\gamma'_g(G - e) = 2$.
  Obviously, the only non-trivial graphs $G$ with $\gamma'_g(G) = 1$ are complete graphs, and $\gamma'_g(K_n - e) = 2$. $\qquad\square$

- If $\gamma'_g(G) = 2$, then $\gamma'_g(G - e) \leq 3$.
  Suppose $\gamma'_g(G) = 2$. For any move $s_1$ of Staller in $G$, Dominator has an answer $d_1$ that dominates all of $G$. This move played in $G - e$ would thus dominate all but at most one vertex of $G - e$. The next move of Staller has to dominate that vertex and thus $\gamma'_g(G - e) \leq 3$. $\qquad\square$

- If $\gamma'_g(G) = 3$, then $\gamma'_g(G - e) \leq 4$.

  Suppose $\gamma'_g(G) = 3$. Let $s_1$ be an optimal move of Staller in $G - e$, and let Dominator answer to $s_1$ by the same move as in $G$. Consider an optimal reply $s_2$ of Staller in $G - e$. If $s_2$ is legal in $G$, then $\{s_1, d_1, s_2\}$ is a dominating set of $G$, so it dominates all $G - e$ but at most one vertex, and hence $\gamma'_g(G - e) \leq 4$. If $s_2$ is not legal in $G$, it means that $s_2$ newly dominates only one end of $e$, the other end being $s_1$ or $d_1$. After Staller's move $s_2$ in $G - e$, the set of dominated vertices is then exactly the same as the set of dominated vertices after the two first moves in $G$ and both ends of $e$ are dominated, so any legal move in $G - e$ finishes the game, and again $\gamma'_g(G - e) \leq 4$. $\square$

## 2.7 $\quad \gamma'_g(G) - \gamma'_g(G - e) = -1$

By Theorem 1.3, $\gamma'_g(C_{2\ell+1}) = \ell$ while $\gamma'_g(C_{2\ell+1} - e) = \ell + 1$ for any $\ell \geq 1$. $\hfill\square$

## 2.8 $\quad \gamma'_g(G) - \gamma'_g(G - e) = 0$

Note first that $\gamma'_g(C_4) = \gamma'_g(C_4 - e) = 2$. Let $G$ be the graph obtained from $P_4$ by identifying one of its inner vertices denoted by $u$ with a vertex of a triangle. Then $\gamma'_g(G) = \gamma'_g(G - e) = 3$ where $e$ is the edge of the triangle not incident with $u$. Let $k \geq 4$ and let $G_k$ be the graph obtained from $K_k$ by attaching one leaf to every vertex of $K_k$. Let $e$ be an edge of $G_k$ that lies in the $k$-clique. Then it is straightforward that $\gamma'_g(G_k) = k = \gamma'_g(G_k - e)$. $\hfill\square$

## 2.9 $\quad \gamma'_g(G) - \gamma'_g(G - e) = 1$

Similarly as in Subsection 2.4, one can verify that $\gamma'_g(Y_k) = k + 5$ and $\gamma'_g(Y_k - e) = k + 4$ for any $k \geq 0$. In particular, note that the optimal first move of Staller is to play on a leaf adjacent to $x$. Consider next the graph $H$ obtained from the disjoint union of $K_{1,4}$ and a triangle by joining with an edge the center of $K_{1,4}$ with one vertex of the triangle, and by adding edge $e$ between another vertex of the triangle and a leaf of $K_{1,4}$. Then $\gamma'_g(H) = 4$ and $\gamma'_g(H - e) = 3$. Hence we have:

**Proposition 2.6** *For any $\ell \geq 4$ there exists a graph $G$ with an edge $e$ such that $\gamma'_g(G) = \ell$ and $\gamma'_g(G - e) = \ell - 1$.*

Note that when $\ell < 4$, there are no graphs with $\gamma'_g(G) = \ell$ and $\gamma'_g(G - e) = \ell - 1$ for some edge $e$. Indeed:

- If $G$ is a graph with at least one edge, then $\gamma'_g(G - e) \geq 2$.

  This is clear because $G - e$ is not complete. $\hfill\square$

- If $\gamma'_g(G) = 3$, then $\gamma'_g(G - e) \geq 3$.

  Let $s_1$ be an optimal first move of Staller in $G$. There are no vertices in $G$ that dominate all of $V(G) \setminus N[s_1]$, hence there are none either in $G - e$. Thus the game in $G - e$ requires at least two more moves and $\gamma'_g(G - e) \geq 3$. $\hfill\square$

**2.10** $\quad \gamma'_g(G) - \gamma'_g(G - e) = 2$

Similarly as in Subsection 2.5, one can verify that $\gamma'_g(X_k) = 2k + 7$ and $\gamma'_g(X_k - e) = 2k + 5$ for any $k \geq 0$. Also, $\gamma'_g(Q_k) = 2k + 6$ and $\gamma'_g(Q_k - e) = 2k + 4$. In particular, note that the optimal first move of Staller is to play on a leaf adjacent to $x$. Hence:

**Proposition 2.7** *For any $\ell \geq 6$ there exists a graph $G$ with an edge $e$ such that $\gamma'_g(G) = \ell$ and $\gamma'_g(G - e) = \ell - 2$.*

When $\ell < 6$, there are no graphs with $\gamma'_g(G) = \ell$ and $\gamma'_g(G - e) = \ell - 2$ for some edge $e$. Indeed,

- for $\ell < 4$ we proved the assertion in the previous subsection.

- If $\gamma'_g(G) = 4$, then $\gamma'_g(G - e) \geq 3$.
  Suppose by way of contradiction that $\gamma'_g(G - e) = 2$. In this case to any move $s_1$ of Staller in $G$, Dominator answers by $d_1$ as if in $G - e$. Since $\{s_1, d_1\}$ dominates $G - e$, it also dominates $G$. $\hfill\square$

- If $\gamma'_g(G) = 5$, then $\gamma'_g(G - e) \geq 4$.
  Suppose by way of contradiction that $\gamma'_g(G - e) = 3$. In this case to any move $s_1$ of Staller in $G$, Dominator answers by $d_1$ as if in $G - e$. Now any legal move $s_2$ of Staller in $G - e$ dominates $G - e$ and so dominates $G$. However, Staller may play a move in $G$ that was not legal in $G - e$, but then Dominator answering with $s_2$ finishes the game in at most 4 moves. $\hfill\square$

# 3 Vertex removal

In contrast to the fact that $\gamma_g(G - e) \leq \gamma_g(G) + 2$ holds, the game domination number of a vertex-deleted subgraph of $G$ cannot be bounded above by a function of the game domination number of $G$. This is not surprising because the same phenomenon holds for the usual domination number (and because $\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1$). More explicitly, let $k$ be a non-negative integer and let $H$ be an arbitrary graph with $\gamma_g(H) = k + 1$. If $G$ is the graph obtained from $H$ by adding to it a universal vertex $v$, then $\gamma_g(G) = 1$ and hence $\gamma_g(G - v) - \gamma_g(G) = k$. The same construction works for the Staller-start game domination number.

On the other hand, we prove the following:

**Theorem 3.1** *If $G$ is a graph and $v \in V(G)$, then*

$$\gamma_g(G) - \gamma_g(G - v) \leq 2 \qquad and \qquad \gamma'_g(G) - \gamma'_g(G - v) \leq 2 \,.$$

**Proof.** To prove the first inequality, let Dominator start on $v$ when Game 1 is played in $G$. We get

$$
\begin{aligned}
\gamma_g(G) &\leq 1 + \gamma'_g(G | N[v]) \\
&= 1 + \gamma'_g((G - v) | (N[v] - \{v\})) \\
&\leq 1 + \gamma'_g(G - v) \qquad \text{(by the Continuation Principle)} \\
&\leq \gamma_g(G - v) + 2 \,. \qquad \text{(by Theorem 1.2)}
\end{aligned}
$$

In Game 2 we consider two cases. In the first case Staller plays $v$ in her first move. We get that

$$
\begin{aligned}
\gamma_g'(G) &= 1 + \gamma_g(G|N[v]) \\
&= 1 + \gamma_g((G-v)|(N[v]-\{v\})) \\
&\leq 1 + \gamma_g(G-v) \qquad \text{(by the Continuation Principle)} \\
&\leq \gamma_g'(G-v) + 2. \qquad \text{(by Theorem 1.2)}
\end{aligned}
$$

In the second case Staller chooses a vertex $x$, $x \neq v$. When Dominator responds by playing $v$, we get

$$
\begin{aligned}
\gamma_g'(G) &= 1 + \gamma_g(G|N[x]) \\
&\leq 2 + \gamma_g'(G|(N[x] \cup N[v])) \\
&= 2 + \gamma_g'((G-v)|(N[x] \cup N[v] - \{v\})) \\
&\leq \gamma_g'(G-v) + 2. \qquad \text{(by the Continuation Principle)}
\end{aligned}
$$

$\square$

We have already observed, that $\gamma_g(G-v) - \gamma_g(G)$ as well as $\gamma_g'(G-v) - \gamma_g'(G)$ can be arbitrarily small. In the rest of this section we construct infinite families of (connected) graphs demonstrating that for any $t \in \{0,1,2\}$ and any integer $\ell \geq 5$ (or smaller—depending of the case), there exists a graph $G$ with $\gamma_g(G) = \ell$ (resp. $\gamma_g'(G) = \ell$) and $\gamma_g(G-v) - \gamma_g(G) = t$ (resp. $\gamma_g'(G-v) - \gamma_g'(G) = t$).

## 3.1 $\gamma_g(G) - \gamma_g(G-v) = 0$

Let $\ell$ be a positive integer and let $G'$ be an arbitrary graph with $\gamma_g(G') = \ell$. Let $x$ be an optimal start vertex for Dominator and let $G$ be the graph obtained from $G'$ by attaching a leaf $v$ to $x$ (actually, we could attach any number of leaves). We claim that $\gamma_g(G) = \gamma_g(G-v) = \ell$. Clearly, $\gamma_g(G-v) = \ell$ since $G - v = G'$. By the Continuation Principle, Dominator would start the game rather on $x$ than on $v$. But then $x$ is an optimal start vertex for Dominator also in $G$, hence $\gamma_g(G) = \ell$. $\square$

## 3.2 $\gamma_g(G) - \gamma_g(G-v) = 1$

To see that for any integer $\ell \geq 2$ there exists a graph $G$ such that $\gamma_g(G) = \ell$ and $\gamma_g(G-v) = \ell - 1$ for some $v \in V(G)$ it suffices to notice that the sequence $\big(\gamma_g(P_n)\big)_{n \geq 1}$ is unbounded, non-decreasing and $\gamma_g(P_{n+1}) - \gamma_g(P_n) \leq 1$ for any $n$. $\square$

## 3.3 $\gamma_g(G) - \gamma_g(G-v) = 2$

**Proposition 3.2** *For any $\ell \geq 5$ there exists a graph $G$ with a vertex $v$ such that $\gamma_g(G) = \ell$ and $\gamma_g(G-v) = \ell - 2$.*

**Proof.** We start the proof for even $\ell$ by presenting the following family $Z_k$, $k \geq 0$. Let $S$ be the graph obtained from $K_{1,3}$ with $x$ as its center in which one edge is subdivided. Denote

the vertex that is not in $N_S[x]$ by $v$. Then $Z_0$ is obtained from the disjoint union of $Z$ and $S$ by connecting $x$ and $z$ with an edge. See Fig. 6 where $Z_0$ is encircled by a dashed curve. The graph $Z_k$, $k \geq 1$, is obtained from $Z_0$ by identifying the end vertex of $k$ copies of $P_6$ with $x$, see Fig. 6 again.
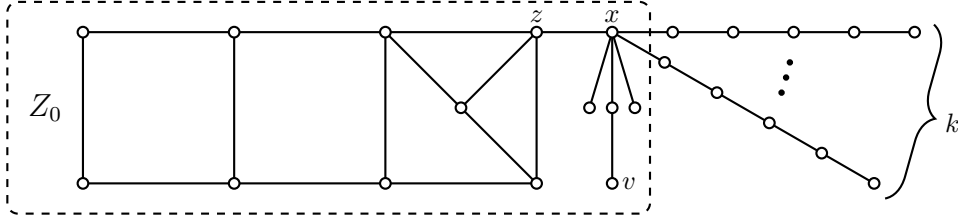


Figure 6: Graphs $Z_k$

We claim that $\gamma_g(Z_k) = 2k + 6$ and $\gamma_g(Z_k - v) = 2k + 4$. By Theorem 3.1 it suffices to show that $\gamma_g(Z_k) \geq 2k + 6$ and $\gamma_g(Z_k - v) \leq 2k + 4$.

To prove the first assertion consider the following strategy of Staller. We first observe that on each of the $k$ attached paths at least two vertices different from $x$ must be played, hence at least $2k$ vertices in total. Moreover, at least two vertices must be played in the subgraph of $Z_k$ that corresponds to $S$. If exactly $2k + 2$ moves are played on this part, Staller can force Dominator to be the first one to play in the subgraph isomorphic to $Z$. In this case four moves are played in this subgraph and hence at least $2k + 6$ moves in total. Otherwise, if Staller is forced to play first in $Z$, then at least $2k + 3$ moves were played on the rest of $Z_k$. Since at least three moves must be played in $Z$ (note that $\gamma(Z) = 3$) again at least $2k + 6$ moves are played on $Z_k$.

To prove that $\gamma_g(Z_k - v) \leq 2k + 4$ consider the strategy of Dominator to play first on $x$. Later, if Staller plays on $Z$, Dominator replies with an (optimal) move on $Z$, and if Staller plays on an attached path, Dominator replies with a move on the path. If at some stage of the game Dominator is forced to play first on an attached path, he plays there such that two moves are played on the path. Note that this strategy of Dominator ensures that he is not the first to play on $Z$. In this way Dominator ensures that three moves are played on $Z$ and two moves on each of the attached paths. Hence in total $2k + 4$ moves are played. This proves the proposition in the case when $\ell$ is even.

We use a similar construction to prove the result for odd $\ell$. In the construction of $Z_k$ replace $Z$ by $C_6$, denoting any of its vertices by $z$. Let the resulting graph be denoted by $W_k$. We claim that $\gamma_g(W_k) = 2k + 5$ and $\gamma_g(W_k - v) = 2k + 3$. Note that $\gamma_g(C_6) = 3 = \gamma_g(C_6|z)$ and $\gamma_g'(C_6) = 2 = \gamma_g'(C_6|z)$. We argue that this is indeed the case with arguments parallel to those that we used for the graphs $Z_k$. $\qquad \square$

Note that there does not exist a graph $G$ such that $\gamma_g(G) = 4$ and $\gamma_g(G - v) = 2$ for a vertex $v$. Indeed after the first optimal move of Dominator in $G - v$, the set $C$ of undominated vertices induces a complete subgraph of $G - v$, and any vertex in $G - C$ that is adjacent to a vertex of $C$ dominates the entire $C$. In $G$, Dominator can start by playing on the same vertex so that only vertices of $C \cup \{v\}$ are left undominated. Clearly at most

two more moves are played in $G$, hence $\gamma_g(G) \leq 3$. It is also easy to see that there does not exist a graph $H$ with a vertex $v$ such that $\gamma_g(H) = 3$ and $\gamma_g(H - v) = 1$.

### 3.4 $\quad \gamma_g'(G) - \gamma_g'(G - v) = 0$

Let $\ell$ be a positive integer and let $G$ be the graph obtained from $K_{\ell+2}$ by attaching a leaf to $\ell$ of its vertices. $G$ is thus of order $2\ell + 2$. Let $v$ be one of the two vertices of the clique that has no leaf attached. It is not difficult to see that $\gamma_g'(G) = \gamma_g'(G - v) = \ell + 1$. $\qquad \square$

### 3.5 $\quad \gamma_g'(G) - \gamma_g'(G - v) = 1$

One can use paths in the same way as in Subsection 3.2.

### 3.6 $\quad \gamma_g'(G) - \gamma_g'(G - v) = 2$

Similarly as in Subsection 3.3, one can verify that $\gamma_g'(Z_k) = 2k + 7$ and $\gamma_g'(Z_k - v) = 2k + 5$ for any $k \geq 0$. Also, $\gamma_g'(W_k) = 2k + 6$ and $\gamma_g'(W_k - v) = 2k + 4$. In particular, note that the optimal first move of Staller is to play on a leaf adjacent to $x$.

The graph $H$ obtained from attaching a leaf $v$ to any vertex of $C_6$ provides $\gamma_g'(H) = 4$ and $\gamma_g'(H - v) = 2$. Similarly, for the graph $H'$ obtained from $Z$ by attaching a leaf $v$ to the vertex $z$ we get that $\gamma_g'(H') = 5$ and $\gamma_g'(H' - v) = 3$. Hence we have the following.

**Proposition 3.3** *For any $\ell \geq 4$ there exists a graph $G$ with a vertex $v$ such that $\gamma_g'(G) = \ell$ and $\gamma_g'(G - v) = \ell - 2$.*

## 4 Concluding remarks

We conclude the paper by two problems that arise from the results of this paper.

**Problem 4.1** *Given a positive integer $k$, can one find a general upper and lower bound for $\gamma_g(G) - \gamma_g(G_k)$ where $G_k$ is obtained from a graph $G$ by deletion of $k$ edges from $G$?*

An interesting instance of Problem 4.1 is the question whether $|\gamma_g(G) - \gamma_g(G - \{e, e'\})|$ can be 3 or 4.

**Problem 4.2** *Which of the subsets of $\{-2, -1, 0, 1, 2\}$ can be realized as*

$$\{\gamma_g(G) - \gamma_g(G - e) : \ e \in E(G)\}$$

*within the family of all (respectively connected) graphs $G$?*

In particular, does there exist a graph $G$ with edges denoted by $e_{-2}, e_{-1}, e_0, e_1, e_2$ such that $\gamma_g(G) - \gamma_g(G - e_i) = i$ for all $i$?

In addition, one can ask for a characterization of certain subfamilies of graphs with respect to the above properties. For instance, following domination terminology a possible question is to characterize the graphs that are game domination edge-critical. That is, for which $G$ we have $\{\gamma_g(G) - \gamma_g(G - e) : \ e \in E(G)\} \subseteq \{-2, -1\}$?

# Acknowledgments

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