Infinite families of circular and Möbius ladders that are total domination game critical

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Abstract

Let $\gamma_{\rm tg}(G)$ denote the game total domination number of a graph G and let $G|v$ mean that a vertex v of G is declared to be already totally dominated. A graph G is total domination game critical if $\gamma_{\rm tg}(G|v) < \gamma_{\rm tg}(G)$ holds for every vertex v in G. If $\gamma_{\rm tg}(G) = k$, then G is further called k-$\gamma_{\rm tg}$-critical. In this paper we prove that the circular ladder $C_{4k} \,\square\, K_2$ is $4k$-$\gamma_{\rm tg}$-critical and that the Möbius ladder ${\rm ML}_{2k}$ is $2k$-$\gamma_{\rm tg}$-critical.

Keywords: total domination game; game total domination number; critical graph; circular ladder; Möbius ladder

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1 Introduction

Motivated with the domination game which was first studied in 2010 in [3] (see also [1, 4, 8, 13, 18–21]), the total version of the game was introduced in 2015 in [9]. The total domination game instantly received lots of attention [2, 6, 7, 11, 14, 15].

Very recently critical graphs with respect to the total domination game were introduced in [12]. (See [5] for the related concept on the domination game.) The main results of [12] are (i) a characterization of critical cycles, (ii) a characterization of critical paths, and (iii) a characterization of critical graphs among the graphs with the total game domination number equal to 3. In this paper we add to this list an infinite family of circular ladders and an infinite family of Möbius ladders. In the rest of the introduction we recall the concepts needed, while in Sections 2 and 3 we state and prove our main results, respectively.

A vertex u in a graph G *totally dominates* a vertex v if u is adjacent to v in G. Let G be a graph and consider two players, called *Dominator* and *Staller*, that take turns choosing a vertex from G. Each vertex chosen must totally dominate at least one vertex not yet totally dominated. Such a chosen vertex is a *legal move*. The game ends when no legal move is available. Dominator wishes to minimize the number of moves and Staller wishes to maximize it. When Dominator has the first move we speak about a *D-game*. The *game total domination number*, $\gamma_{\rm tg}(G)$, of G is the number of moves played in the D-game when both players play optimally.

A *partially total dominated graph* is a graph together with a declaration that some vertices are already totally dominated. Given a graph G and a subset S of vertices of G, we denote by $G|S$ the partially total dominated graph in which the vertices of S in G are already totally dominated. If $S = \{v\}$ for some vertex v in G, we write $G|v$. We use $\gamma_{\rm tg}(G|S)$ to denote the number of moves remaining in the D-game on $G|S$. The graph G is *total domination game critical*, $\gamma_{\rm tg}$-critical for short, if $\gamma_{\rm tg}(G) > \gamma_{\rm tg}(G|v)$ holds for every $v \in V(G)$. If G is $\gamma_{\rm tg}$-critical and $\gamma_{\rm tg}(G) = k$, then G is *k-$\gamma_{\rm tg}$-critical*.

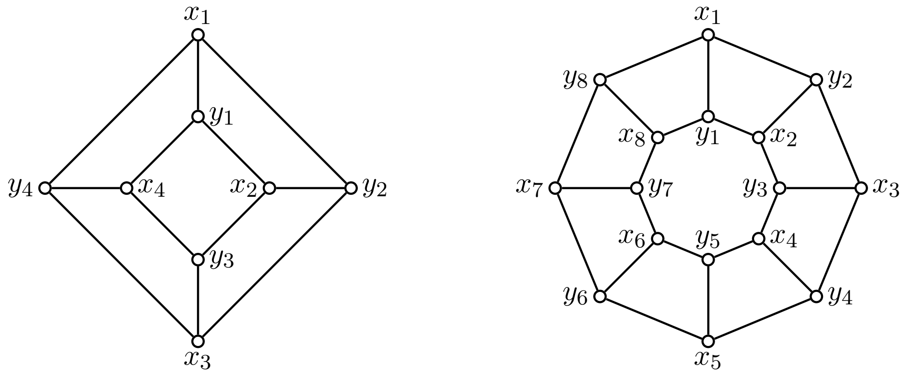
Finally, the *Cartesian product* $G \,\square\, H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H)$, vertices (g, h) and (g', h') being adjacent if either $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$ [10]. We use the notation $[k] = \{1, \ldots, k\}$. For notation and graph theory terminology not defined herein, we in general follow [10, 16].

2 Circular ladder graphs

The *circular ladder graph* CL_n of order $2n$ is the Cartesian product of a cycle C_n on $n \geq 3$ vertices and a path P_2 on two vertices; that is, $\mathrm{CL}_n = C_n \,\square\, K_2$ (cf. [17]). We note that CL_n is bipartite if and only if n is even. The circular ladders CL_4 and CL_8 are illustrated in Figs. 1(a) and 1(b), respectively.

Theorem 1 *If $k \geq 1$, then the circular ladder CL_{4k} is $4k$-$\gamma_{\rm tg}$-critical.*

Proof. Let $X = \{x_1, x_2, \ldots, x_{4k}\}$ and $Y = \{y_1, y_2, \ldots, y_{4k}\}$ be the two partite sets of CL_{4k}, where $x_1y_2x_3y_4 \ldots x_{4k-1}y_{4k}x_1$ and $y_1x_2y_3x_4 \ldots y_{4k-1}x_{4k}y_1$ are the two disjoint copies of the cycle C_{4k} used to construct $\mathrm{CL}_{4k} = C_{4k} \,\square\, K_2$ and where $x_iy_i \in E(\mathrm{CL}_{4k})$ for $i \in [4k]$, cf. Fig. 1. Thus, the three neighbors of x_i in CL_{4k} are y_{i-1}, y_i, y_{i+1}, and the three neighbors of y_i in CL_{4k} are x_{i-1}, x_i, x_{i+1}, where addition



(a) CL_4 (b) CL_8

Figure 1: The circular ladders CL_4 and CL_8.

is taken modulo $4k$. We call the two vertices x_i and x_{i+1} (respectively, y_i and y_{i+1}) consecutive vertices in X (respectively, Y), where addition is again taken modulo $4k$.

We proceed further with the following three claims.

Claim 1.1 *$\gamma_{\rm tg}(\mathrm{CL}_{4k}) \ge 4k$.*

Proof. In order to prove that $\gamma_{\rm tg}(\mathrm{CL}_{4k}) \ge 4k$ we describe a strategy for Staller that guarantees that at least $4k$ moves are played in order to complete the game. For this purpose, we introduce the following notation. In the partially total dominated circular ladder CL_{4k}, we define an *X-run* in CL_{4k} to be a maximal sequence of consecutive vertices all of which belong to X that contains at least three vertices, none of which have yet been played, but does not contain all vertices of X. A *Y-run* is defined analogously. We call a *run* an X-run or a Y-run.

Suppose it is Staller's turn to play. Since we are in a D-game, we can assume that at least one vertex has already been played in the game, and that the game is not yet complete. Staller's strategy is to play on the extremity of a run, if one exists; otherwise, she plays any legal move. We show that Staller's strategy guarantees that she totally dominates exactly one new vertex on all, except possibly one, of her moves. We consider two possibilities, depending on whether the circular ladder CL_{4k} contains a run or not.

Suppose the circular ladder CL_{4k} contains a run. Renaming X and Y we may assume CL_{4k} contains an X-run, i.e., a maximal sequence $s\colon x_i x_{i+1} x_{i+2} \dots x_{i+r}$, where $2 \le r < |X| - 1$, of consecutive vertices in X none of which have yet been played. Staller follows her strategy and plays on one of the run extremities x_i or x_{i+r}, say x_i. Since s is a maximal sequence, we note that the vertex x_{i-1} has already been played, implying that Staller totally dominate only one new vertex, namely the vertex y_{i+1}. Hence whenever Staller plays on the extremity of a run, she ensures that only one new vertex is totally dominated when playing her move.

Suppose that the circular ladder CL_{4k} contains no run (and it is Staller's turn to play). We show that this can occur at most once in the game. Since the game is not yet complete, Staller plays any legal move, say the vertex x_i. If the vertex y_{i-1} is totally dominated by her move, then neither x_{i-2} nor x_{i-1} have yet been played. If the vertex y_i is totally dominated by her move, then neither x_{i-1} nor x_{i+1} have yet been played. If the vertex y_{i+1} is totally dominated by her move, then neither x_{i+1} nor x_{i+2} have yet been played. In all cases, there are three consecutive vertices in X, none of which have been played immediately before Staller played her move x_i. If at least one vertex of X has already been played before Staller plays her move x_i, then this would imply the existence of an X-run, contrary to our supposition. Hence, no vertex of X is played before Staller plays her move x_i; that is, no vertex in Y is totally dominated before she played her move. As we are in a D-game, this implies that Dominator's first move played a vertex in Y. Further if there is a legal move for Staller to play from the set Y, then analogous arguments as before show that this implies the existence of a Y-run, a contradiction. Therefore, immediately before Staller played her move x_i, there is no legal move in Y, implying that all vertices in X are totally dominated. As observed earlier, no vertex in Y is totally dominated immediately before Staller played her move x_i.

Thus, we have shown that Staller can always plays on the extremity of a run, thereby ensuring that only one new vertex is totally dominated on each such move, except possibly for exactly one of her moves, which occurs when either all vertices of X are totally dominated and no vertex of Y is totally dominated or all vertices of Y are totally dominated and no vertex of X is totally dominated. Further, if Staller plays such a move that is not the extremity of a run, she totally dominates three new vertices.

Suppose that every move of Staller totally dominates exactly one new vertex. In this case, every move of Staller together with the previous move of Dominator, combined totally dominate at most four new vertices (at most three from Dominator's move, and exactly one from Staller's move). Thus, after $4k-1$ moves have been played ($2k$ moves by Dominator and $2k-1$ moves by Staller), at most $4(2k-1)+3 = 8k-1$ vertices have been totally dominated, and at least additional move is needed to complete the game. Hence, we may assume that during the game Staller plays exactly one move that is not the extremity of a run, for otherwise at least $4k$ moves are played to complete the game.

Renaming sets X and Y if necessary, we may assume that such a move of Staller plays a vertex in Y. As observed earlier, this is the first vertex in Y played in the game, implying that immediately before she plays her move, all vertices of Y are totally dominated and no vertex of X is totally dominated.

Let m be the number of moves played immediately before Staller plays this vertex in Y. We note that all these m moves belong to X and totally dominate only vertices in Y. Further, m is odd since Dominator played the last vertex in X. Thus, the first $(m+1)/2$ vertices played by Dominator all belong to X and the first $(m-1)/2$ vertices played by Staller all belong to X. Each such move of Dominator totally dominates at most three new vertices, while each such move of Staller totally dominates exactly one new vertex. Thus, the number of vertices totally dominated upon completion of the first m moves is exactly $|Y| = 4k$ and is at most $3(m+1)/2 + (m-1)/2 = 2m+1$, implying that $m \geq (4k-1)/2$. Since m is an odd integer, this implies that $m \geq 2k+1$.

Let r be the number of moves needed to complete the game. We note that all these r moves belong to Y. As observed earlier, Staller makes the first move in Y and her move totally dominates three vertices. All subsequent moves of Staller totally dominate exactly one new vertex. If r is even, then Dominator plays the final move in the game, and both Dominator and Staller play $r/2$ vertices from Y. In this case, the number of vertices totally dominated by these r moves is exactly $|X| = 4k$ and is at most $3r/2 + 3 + (r/2 - 1) = 2r + 2$, implying that $r \geq 2k$. If r is odd, then Staller plays the final move in the game. Thus, Dominator plays $(r-1)/2$ vertices from Y and Staller plays $(r+1)/2$ vertices from Y. In this case, the number of vertices totally dominated by these r moves is exactly $|X| = 4k$ and is at most $3(r-1)/2 + 3 + (r-1)/2 = 2r + 1$, implying that $r \geq 2k+1$. Thus, if r is even, then $r \geq 2k$, while if r is odd, then $r \geq 2k+1$. As observed earlier, $m \geq 2k+1$. Hence in this case when Staller plays exactly one move that is not the extremity of a run, the total number of moves needed to complete the game is $m + r \geq 4k+1$. We have therefore shown Staller has a strategy that guarantees that at least $4k$ moves are played in order to complete the game. Therefore, $\gamma_{\rm tg}({\rm CL}_{4k}) \geq 4k$. This completes the proof of the claim. (□)

Claim 1.2 $\gamma_{\rm tg}({\rm CL}_{4k}) \leq 4k$.

Proof. In order to prove that $\gamma_{\rm tg}({\rm CL}_{4k}) \leq 4k$ we describe a strategy for Dominator that guarantees that at most $4k$ moves are played to complete the game. At each point in the game, we denote by U the set of vertices that are not playable, that is vertices already played or vertices not played but all of whose neighbors are already totally dominated.

Observe that each move of the game adds at least one vertex to the set U, namely the vertex played. Dominator starts with any move, adding only one vertex to U. Thereafter, the following claim shows that he has a strategy that guarantees that after each of Staller's moves he can play a move which together with her move adds at least four vertices to U.

Claim 1.2.1 *Dominator has a strategy that guarantees that after each of Staller's moves, his answer together with her move add at least four vertices to U.*

Proof. Consider a move of Staller on some vertex x_i or y_i, say x_i, for some $i \in [4k]$. Her move totally dominates at least one new vertex, namely one of the vertices y_{i-1}, y_i or y_{i+1}.

Suppose that Staller's move totally dominates the vertex y_{i+1}. Thus, before Staller's move, neither the vertex x_{i+1} nor the vertex x_{i+2} has been played. Dominator responds to Staller's move x_i by playing the vertex x_{i+3} if it has not yet been played, thereby totally dominating at least one new vertex, namely the vertex y_{i+2}, and ensuring that the four vertices x_i, x_{i+1}, x_{i+2} and x_{i+3} are added to U after these two moves. On the other hand, suppose that the vertex x_{i+3} has already been played (before Staller plays the vertex x_i). This implies that immediately Staller plays her move x_i, the three vertices x_i, x_{i+1}, and x_{i+2} are added to U. In this case, Dominator plays any legal move in response to Staller's move x_i, thereby adding at least one vertex to U, and thus ensuring that at least four vertices get added to U after these two moves.

If Staller's move totally dominates the vertex y_{i-1}, then analogously as in the case when she totally dominates the vertex y_{i+1}, Dominator can ensure that at least four vertices get added to U after her move and his answer to her move. Suppose therefore that the vertex y_i is the only new vertex totally dominated by Staller's move. Thus in this case, before Staller plays the vertex x_i neither the vertex x_{i-1} nor the vertex x_{i+1} has been played, but both vertices x_{i-2} and x_{i+2} were already played. This implies that immediately Staller plays her move x_i, the three vertices x_{i-1}, x_i, and x_{i+1} are added to U. So Dominator can play any legal move, adding at least one vertex to U, and thus ensuring that at least four vertices get added to U after these two moves. (□)

Claim 1.2.2 *The final move of the game adds at least three vertices to U.*

Proof. Suppose the vertex x_i is the final move played in the game for some $i \in [4k]$. This move totally dominates at least one new vertex, namely one of the vertices y_{i-1}, y_i or y_{i+1}. Further, after this move is played the game is complete and therefore no legal moves remains. If the final move totally dominates the vertex y_{i-1}, then before this move is played, neither the vertex x_{i-2} nor the vertex x_{i-1} has been played, and implying that the final move adds the three vertices x_{i-2}, x_{i-1} and x_i to U. If the final move totally dominates the vertex y_i, then this move adds the three vertices x_{i-1}, x_i, and x_{i+1} to U. If the final move totally dominates the vertex y_{i+1}, then this moves adds the three vertices x_i, x_{i+1}, and x_{i+2} to U. In all cases, the final move adds at least three vertices to U. (□)

We now return to the proof of Claim 1.2, and compute the bounds that Dominator's strategy gives us. Let m be the number of moves played when the game finishes. As observed earlier, the first move of Dominator adds only one vertex to U. Consider first the case when m is even. In this case, Staller made the final move and with her last move, she adds at least three new vertices to U, by Claim 1.2.2. By Claim 1.2.1, each move of Staller played before her final move, together with Dominator's response to her move, adds at least four vertices to U. Thus, we have $8k = |U| \ge 1 + \frac{4}{2}(m-2) + 3 = 2m$, and so $m \le 4k$. Consider next the case when m is odd. In this case, Dominator made the final move. By Claim 1.2.1, each move of Staller, together with Dominator's response to her move, adds at least four vertices to U. Thus, we have $8k = |U| \ge 1 + \frac{4}{2}(m-1) = 2m - 1$, and so $m \le 4k + \frac{1}{2}$. Since m is odd, this implies that $m \le 4k - 1$. Thus, in both cases, $m \le 4k$, and so $\gamma_{\rm tg}({\rm CL}_{4k}) = m \le 4k$. This completes the proof of Claim 1.2. (□)

Claim 1.3 $\gamma_{\rm tg}({\rm CL}_{4k}|v) \le 4k - 1$ *for every vertex* v *in* ${\rm CL}_{4k}$.

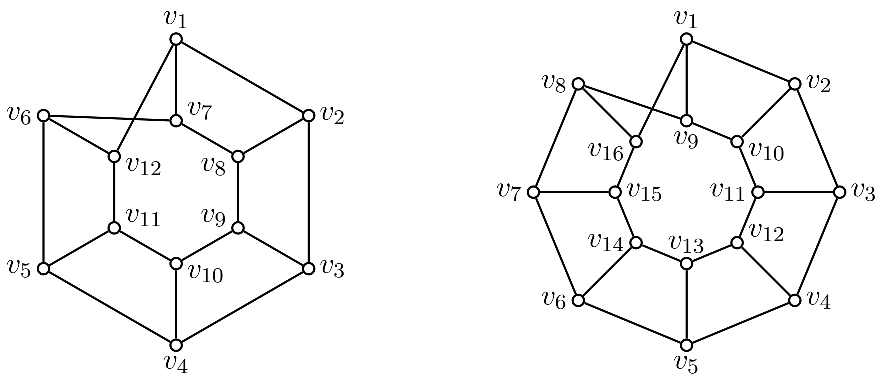
Proof. Let v be an arbitrary vertex of ${\rm CL}_{4k}$. We may assume that $v = x_i$ for some $i \in [4k]$. We describe a strategy for Dominator that guarantees that at most $4k-1$ moves are played to complete the game played in ${\rm CL}_{4k}|v$. Following the notation of Claim 1.2, we denote by U the set of vertices that are not playable. Dominator plays as his first move in the game played in ${\rm CL}_{4k}|v$ the vertex y_{i+2}, thereby adding two vertices to U, namely the vertices y_{i+1} and y_{i+2}. Thereafter, he adopts his strategy explained in the proof of Claim 1.2.1, which guarantees that after each of Staller's moves, his answer together with her move add at least four vertices to U.

We now compute the bounds that Dominator's strategy gives us. Let m be the number of moves played when the game finishes. As observed earlier, the first move of Dominator adds two vertices to U. Consider first the case when m is even. In this case, we have $8k = |U| \ge 2 + \frac{4}{2}(m-2) + 3 = 2m + 1$, and so $m \le 4k - \frac{1}{2}$. Thus, since m is even, this implies that $m \le 4k - 2$. Consider next the case when m is odd. In this case, Dominator made the final move and $8k = |U| \ge 2 + \frac{4}{2}(m-1) = 2m$, and so $m \le 4k$. Since m is odd, this implies that $m \le 4k - 1$. Thus, in both cases, $m \le 4k - 1$, and so $\gamma_{\rm tg}({\rm CL}_{4k}|v) = m \le 4k - 1$. This completes the proof of Claim 1.3. (□)

By Claim 1.1 and 1.2, we deduce that $\gamma_{\rm tg}({\rm CL}_{4k}) = 4k$. By Claim 1.3, $\gamma_{\rm tg}({\rm CL}_{4k}|v) \le 4k - 1$ for every vertex v in ${\rm CL}_{4k}$. Thus, the graph ${\rm CL}_{4k}$ is $4k$-$\gamma_{\rm tg}$-critical. This completes the proof of Theorem 1. □

3 Möbius ladders

For $n \geq 2$, the Möbius ladder ML_n is a cubic circulant graph of order $2n$, formed from a $2n$-cycle by adding n edges (called "rungs") joining opposite pairs of vertices in the cycle (cf. [17]). The Möbius ladder ML_2 is the complete graph K_4. The Möbius ladders ML_6 and ML_8 are illustrated in Fig. 2(a) and 2(b), respectively. The Möbius ladder ML_n is bipartite if and only if n is odd.



(a) ML_6 (b) ML_8

Figure 2: The Möbius ladders ML_6 and ML_8.

Theorem 2 *If $k \geq 1$, then the Möbius ladder ML_{2k} is $2k$-$\gamma_{\rm tg}$-critical.*

Proof. When $k = 1$, the Möbius ladder ML_{2k} is the complete graph $\mathrm{ML}_2 \cong K_4$, which is 2-$\gamma_{\rm tg}$-critical. Hence, in what follows we may assume that $k \geq 2$. Let ML_{2k} be obtained from the $4k$-cycle $v_1 v_2 \ldots v_{4k} v_1$ by adding the $2k$ edges $v_i v_{i+2k}$ for $i \in [2k]$, where addition is taken modulo $4k$, cf. Fig. 2. We note that ML_{2k} is not bipartite.

We first show that $\gamma_{\rm tg}(\mathrm{ML}_{2k}) \geq 2k$. Staller adopts exactly her strategy used in the proof of Claim 1.1 for the circular ladder. We note that this strategy is a local strategy, and adapts readily to the Möbius ladder. More precisely, we define two vertices to be consecutive vertices of ML_{2k} if they have two common neighbors. A *run* in the partially total dominated Möbius ladder ML_{2k} is a maximal sequence of (distinct) consecutive vertices that contains at least three vertices, none of which have yet been played, but does not contain all vertices of ML_{2k}. For example, if v_{i+2k-1} is already played for some $i \in [2k]$ but none of v_i, v_{i+2k+1}, v_{i+2} have yet been played, then v_i, v_{i+2k+1}, v_{i+2} are the first three vertices of a run. This is illustrated in Fig. 3.

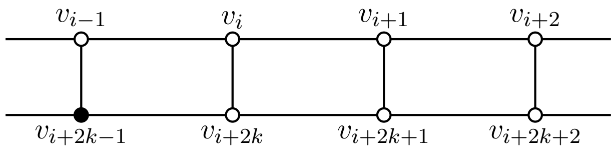


Figure 3: A subgraph of ML_{2k}.

Staller's adopts her identical strategy as in proof of Claim 1.1 and plays on the extremity of a run, if one exists; otherwise, she plays any legal move. Since the Möbius ladder ML_{2k} is in this case not bipartite, an analogous proof as in Claim 1.1 shows that Staller's strategy guarantees that she totally dominates exactly one new vertex on all of her moves. (This differs slightly from the case of the circular ladder which is a bipartite graph, since then it is possible that Staller's strategy guarantees that she totally dominates exactly one new vertex on all, except for possibly one, of her moves.) Thus, every move of Staller together with the previous move of Dominator, combined totally dominate at most four new vertices (at most three from Dominator's move, and exactly one from Staller's move). Hence, after $2k-1$ moves have been played (k moves by Dominator and $k-1$ moves by Staller), at most $4(k-1)+3 = 4k-1$ vertices have been totally dominated, and at least one additional move is needed to complete the game, implying that at least $2k$ moves are played in order to complete the game. Therefore, $\gamma_{\rm tg}(\mathrm{ML}_{2k}) \ge 2k$.

We show next that $\gamma_{\rm tg}(\mathrm{ML}_{2k}) \le 2k$. Dominator adopts his identical strategy used in the proof of Claim 1.2 for the circular ladder. Once again, we note that this strategy is a local strategy, and adapts readily to the Möbius ladder. An identical proof as in Claim 1.2.1 shows that his strategy guarantees that after each of Staller's moves, his answer together with her move add at least four vertices to U. An identical proof as in Claim 1.2.2 shows that the final move of the game adds at least three vertices to U.

We now compute the bounds that Dominator's strategy gives us. Let m be the number of moves played when the game finishes. As observed earlier, the first move of Dominator adds only one vertex to U. Consider first the case when m is even. In this case, Staller made the final move and with her last move, she adds at least three new vertices to U. Each of Staller's previous moves, together with Dominator's response to her move, adds at least four vertices to U. Thus, we have $4k = |U| \ge 1 + \frac{4}{2}(m-2) + 3 = 2m$, and so $m \le 2k$. Consider next the case when m is odd. In this case, Dominator made the final move, implying that $4k = |U| \ge 1 + \frac{4}{2}(m-1) = 2m-1$, and so $m \le 2k + \frac{1}{2}$. Since m is odd, this implies that $m \le 2k-1$. Thus, in both cases, $m \le 2k$, and so $\gamma_{\rm tg}(\mathrm{ML}_{2k}) = m \le 2k$. This shows that $\gamma_{\rm tg}(\mathrm{ML}_{2k}) \le 2k$. As shown earlier, $\gamma_{\rm tg}(\mathrm{ML}_{2k}) \ge 2k$. Consequently, $\gamma_{\rm tg}(\mathrm{ML}_{2k}) = 2k$.

An identical proof as in Claim 1.3 shows that $\gamma_{\rm tg}(\mathrm{ML}_{2k}|v) \le 2k-1$ for every vertex v in ML_{2k}. Thus, the graph ML_{2k} is $2k$-$\gamma_{\rm tg}$-critical. This completes the proof of Theorem 2. $\Box$

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