Total version of the domination game

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Abstract

In this paper, we continue the study of the domination game in graphs introduced by Brešar, Klavžar, and Rall [SIAM J. Discrete Math. 24 (2010) 979–991]. We study the total version of the domination game and show that these two versions differ significantly. We present a key lemma, known as the Total Continuation Principle, to compare the Dominator-start total domination game and the Staller-start total domination game. Relationships between the game total domination number and the total domination number, as well as between the game total domination number and the domination number, are established.

Key words: Domination game; Total domination number.

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1 Introduction

In this paper, we continue the study of the domination game in graphs introduced in [1]. The game played on a graph G consists of two players, *Dominator* and *Staller*, who take turns choosing a vertex from G. Each vertex chosen must dominate at least one vertex not dominated by the vertices previously chosen. The game ends when the set of vertices chosen becomes a dominating set in G. Dominator wishes to dominate the graph as fast as possible and Staller wishes to delay the process as much as possible. The *game domination number* (resp. *Staller-start game domination* *number*), $\gamma_g(G)$ (resp. $\gamma_g'(G)$), of G is the number of vertices chosen when Dominator (resp. Staller) starts the game and both players play optimally.

Let us recall some results proved about this game. First of all, it was proved in [1, 8] that the game domination number and the Staller-start game domination number are always close together; more precisely, $|\gamma_g(G)-\gamma_g'(G)| \le 1$ holds for every graph G. A key lemma needed to give a short proof of this result is the so-called *Continuation Principle* proved in [8]. We will not state it here but instead will prove a completely parallel assertion, named the *Total Continuation Principle*, for the total domination game; see Lemma 3.1. Call a pair of integers (k, ℓ) *realizable* if there exists a graph G with $\gamma_g(G) = k$ and $\gamma_g'(G) = \ell$. Realizable pairs were studied in [1, 2, 11]; for the complete answer (with relatively simple families of graphs) see [10]. Several exact values for the game domination number were established in [9], including paths, cycles (the problem is non-trivial even on these graphs!) and several grid-like graphs. The main message of the paper [2] is that the game domination number can behave intrinsically different from the usual domination number. Kinnersley, West, and Zamani [8] conjectured that if G is an isolate-free forest of order n or an isolate-free graph of order n, then $\gamma_g(G) \le 3n/5$. Progress on these two 3/5-*conjectures* was made in [3] by constructing large families of trees that attain the conjectured 3/5-bound and by finding all extremal trees on up to 20 vertices. Bujtás [4] proved the 3/5-*conjecture* for the class of forests in which no two leaves are at distance 4 apart.

In this paper we introduce and study the total version of the domination game and in particular compare it with the usual domination game. The *total domination game*, played on a graph G again consists of two players called *Dominator* and *Staller* who take turns choosing a vertex from G. In this version of the game, each vertex chosen must totally dominate at least one vertex not totally dominated by the vertices previously chosen. We say that a move v in the total domination game is *legal* if the played vertex v totally dominates at least one new vertex. The game ends when the set of vertices chosen becomes a total dominating set in G. Dominator wishes to totally dominate the graph as fast as possible and Staller wishes to delay the process as much as possible. The *game total domination number*, $\gamma_{tg}(G)$, of G is the number of vertices chosen when Dominator starts the game and both players play optimally. The *Staller-start game total domination number*, $\gamma_{tg}'(G)$, of G is the number of vertices chosen when Staller starts the game and both players play optimally. For simplicity, we shall refer to the Dominator-start total domination game and the Staller-start total domination game as *Game 1* and *Game 2*, respectively.

We proceed as follows. In Section 2, we introduce our graph theory terminology and notation. In Section 3, we compare the total domination Game 1 and Game 2 and present our key lemma, named the *Total Continuation Principle*, which shows that the number of moves in Game 1 and Game 2 when played optimally can differ by at most one. A relationship between the game total domination number and the total domination number is established in Section 4, as is a relation between the game domination number and the game total domination number. Corollary 4.2 and the examples afterwards demonstrate that the latter two invariants can differ significantly. In Section 5 a relationship between the game total domination number and the domination number is established. We close in Section 6 by investigating the game total domination number and the Staller-start game total domination number for paths and cycles.

2 Preliminaries

For notation and graph theory terminology not defined herein, we in general follow [6]. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *open neighborhood* of a vertex $v \in V(G)$ is $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = \{v\} \cup N_G(v)$. For a set $S \subseteq V(G)$, its *open neighborhood* is the set $N_G(S) = \bigcup_{v \in S} N_G(v)$ and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$. The degree of v is $d_G(v) = |N_G(v)|$. If the graph G is clear from the context, we simply write V, E, $N(v)$ $N[v]$, $N(S)$, $N[S]$ and $d(v)$ rather than $V(G)$, $E(G)$, $N_G(v)$, $N_G[v]$, $N_G(S)$, $N_G[S]$ and $d_G(v)$, respectively. A *universal vertex* is a vertex adjacent to every other vertex.

For subsets $X, Y \subseteq V$, we denote the subgraph induced by X by $G[X]$ and we denote the set of edges that join a vertex of X and a vertex of Y by $[X, Y]$. Thus, $|[X, Y]|$ is the number of edges with one end in X and the other end in Y. In particular, $|[X, X]| = m(G[X])$. If there is no edge in $[X, Y]$, we say that $[X, Y]$ is *empty*.

Let G be a graph with vertex set V and with no isolated vertex. A *dominating set* of G is a set S of vertices of G such that every vertex in $V \setminus S$ is adjacent to a vertex in S. Thus a set $S \subseteq V$ is a dominating set in G if $N[S] = V$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. If X and Y are subsets of vertices in G, then the set X *dominates* the set Y in G if $Y \subseteq N[X]$. In particular, if X dominates V, then X is a dominating set in G. A *total dominating set*, abbreviated TD-set, of G is a set S of vertices of G such that every vertex is adjacent to a vertex in S. Thus a set $S \subseteq V$ is a TD-set in G if $N(S) = V$. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. A vertex u *totally dominates* a vertex v if $v \in N(u)$. If X and Y are subsets of vertices in G, then the set X *totally dominates* the set Y in G if $Y \subseteq N(X)$. In particular, if X totally dominates V, then X is a TD-set in G. For more information on the total domination in graphs see the recent book [7].

Since an isolated vertex in a graph cannot be totally dominated by definition, *all graphs considered will be without isolated vertices.*

3 Game 1 versus Game 2

In this section we prove that just as for the usual domination game, also for the total domination game the number of moves in Game 1 and Game 2 when played optimally can differ by at most one. To do so, we will mimic the idea of the proof of the Continuation Principle from [8], for which we need the following concept.

A *partially total dominated graph* is a graph together with a declaration that some vertices are already totally dominated; that is, they need not be totally dominated in the rest of the game. If G is a graph and $A \subseteq V(G)$ is such a set, we will denote with G_A this partially total dominated graph. Moreover, $\gamma_{tg}(G_A)$ and $\gamma'_{tg}(G_A)$ are the minimum number of moves needed to finish the game on G_A when Dominator or Staller starts, respectively.

Lemma 3.1 (Total Continuation Principle) *Let G be a graph, $A, B \subseteq V(G)$, and let G_A and G_B be the corresponding partially total dominated graphs. If $B \subseteq A$, then $\gamma_{tg}(G_A) \leq \gamma_{tg}(G_B)$ and $\gamma'_{tg}(G_A) \leq \gamma'_{tg}(G_B)$.*

Proof. Two games will be played, Game A on the graph G_A and Game B on the graph G_B. The first of these will be the real game, while Game B will only be imagined by Dominator. In Game A, Staller will play optimally while in Game B, Dominator will play optimally.

We claim that in each stage of the games, the set of vertices that are totally dominated in Game B is a subset of the vertices that are totally dominated in Game A. This is clearly true at the start of the games. Suppose now that Staller has (optimally) selected vertex u in Game A. Then by the induction assumption, vertex u is a legal move in Game B because a new vertex v that was totally dominated by u in Game A, is not yet totally dominated in Game B. Then Dominator copies the move of Staller and plays vertex u in Game B. Dominator then replies with an optimal move in Game B. If this move is legal in Game A, Dominator plays it in Game A as well. Otherwise, if the game is not yet over, Dominator plays any other legal move in Game A. In both cases the claim assumption is preserved, which by induction also proves the claim.

We have thus proved that Game A finishes no later than Game B. Suppose thus that Game B lasted r moves. Because Dominator was playing optimally in Game B, it follows that $r \leq \gamma_{tg}(G_B)$. On the other hand, because Staller was playing optimally in Game A and Dominator has a strategy to finish the game in r moves, we infer that $\gamma_{tg}(G_A) \leq r$. Therefore,

$$\gamma_{tg}(G_A) \leq r \leq \gamma_{tg}(G_B),$$

and we are done if Dominator is the first to play. Note that in the above arguments we did not assume who starts first, hence in both cases Game A will finish no later than Game B. Hence the conclusion holds for γ'_{tg} as well. □

As a consequence of the Total Continuation Principle whenever x and y are legal moves for Dominator and $N(x) \subseteq N(y)$, then Dominator will play y instead of x. We remark that the proof of Lemma 3.1 could be modified to work on other (but not all) variants of possible domination games, but since we do not wish to initiate an inflation of such games, we stated the result for the total version only.

Lemma 3.1 leads to the following fundamental property:

Theorem 3.2 *For any graph G, we have $|\gamma_{tg}(G) - \gamma'_{tg}(G)| \le 1$.*

Proof. Consider Game 1 and let v be the first move of Dominator. Let $A = N(v)$ and consider the partially total dominated graph G_A. Set in addition $B = \emptyset$ and note that $G_B = G$. Note first that $\gamma_{tg}(G) = 1 + \gamma'_{tg}(G_A)$. By the Total Continuation Principle, $\gamma'_{tg}(G_A) \le \gamma'_{tg}(G_B) = \gamma'_{tg}(G)$. Therefore,

$$\gamma_{tg}(G) \le \gamma'_{tg}(G_A) + 1 \le \gamma'_{tg}(G) + 1\,.$$

By a parallel argument, $\gamma'_{tg}(G) \le \gamma_{tg}(G) + 1$. □

One can check directly that $\gamma_{tg}(P_4) = \gamma'_{tg}(P_4) = 3$, $\gamma_{tg}(P_5) = 3 = \gamma'_{tg}(P_5) - 1$, and $\gamma_{tg}(C_8) = 5 = \gamma'_{tg}(C_8) + 1$. For instance, in Game 2 on C_8, an optimal first move of Dominator is the vertex opposite to the vertex played by Staller in the first move of the game. The second move of Staller must be adjacent to one of the two vertices already played, and then Dominator can finish the game by picking the vertex that is opposite to the second move of Staller. So there are graphs G_1, G_2 and G_3 such that $\gamma_{tg}(G_1) = \gamma'_{tg}(G_1)$, $\gamma_{tg}(G_2) = \gamma'_{tg}(G_2) + 1$, and $\gamma_{tg}(G_3) = \gamma'_{tg}(G_3) - 1$. Infinite families of such examples can be obtained using the next result. It involves graphs G with $\gamma_{tg}(G) = \gamma'_{tg}(G) = 2$. Examples of such graphs are complete multipartite graphs $K_{n_1,\ldots,n_r}$ where $r \ge 2$, and all graphs that contain a universal vertex. With $\cup_i G_i$ we denote the disjoint union of the graphs G_i.

Proposition 3.3 *Let G_0 be a graph, r a positive integer, and let G_i, $1 \le i \le r$, be graphs with $\gamma_{tg}(G_i) = \gamma'_{tg}(G_i) = 2$. Then*

$$\gamma_{tg}(\cup_{i=0}^{r} G_i) = \gamma_{tg}(G_0) + 2r \quad \textit{and} \quad \gamma'_{tg}(\cup_{i=0}^{r} G_i) = \gamma'_{tg}(G_0) + 2r\,.$$

Proof. Set $G = \cup_{i=0}^{r} G_i$. Consider first Game 1 and the following strategy of Dominator. He starts with an optimal (with respect to the game restricted to G_0) move in G_0. In each later move he follows Staller in G_0 as long as possible and, if necessary, plays an optimal reply to a move of Staller in a G_i, $i \ge 1$, in which she played. In this way Dominator ensures that the game restricted to G_0 is Game 1 so that he can guarantee that at most $\gamma_{tg}(G_0)$ moves will be played on G_0. Since $\gamma_{tg}(G_i) = \gamma'_{tg}(G_i) = 2$, exactly two moves will be played on each G_i, $i \ge 1$. Consequently, $\gamma_{tg}(G) \le \gamma_{tg}(G_0) + 2r$. On the other hand, a similar strategy of Staller, namely that she follows a move of Dominator in G_0 if possible, guarantees her that at least $\gamma_{tg}(G_0)$ moves will be played on G_0, and consequently $\gamma_{tg}(G) \geq \gamma_{tg}(G_0) + 2r$. This establishes the first assertion.

The second assertion follows by analogous strategies by Staller and Dominator, respectively. □

To conclude the section we remark that recently a detailed study of the (usual) domination game played on the disjoint union of two graphs was done in [5].

4 Game total domination versus total domination

The game total domination number can be bounded by the total domination number as follows.

Theorem 4.1 *If G is a graph on at least two vertices, then*

$$\gamma_t(G) \leq \gamma_{tg}(G) \leq 2\gamma_t(G) - 1\,.$$

Moreover, given any integer $n \geq 2$ and any $0 \leq \ell \leq n-1$ there exists a connected graph H such that $\gamma_t(H) = n$ and $\gamma_{tg}(H) = n + \ell$.

Proof. At the end of Game 1 played on a graph G the vertices played form a TD-set of G, hence $\gamma_{tg}(G) \geq \gamma_t(G)$. Moreover, if D is a minimum TD-set of G, then the strategy of Dominator to consecutively play vertices from D if possible, and otherwise playing some other vertex, guarantees that the game ends in no more than $2\gamma_t(G) - 1$ moves. This proves the claimed bounds. For the rest of the proof fix a positive integer $n \geq 2$.

Let G_n be the graph obtained from the disjoint union of the path $P^{(0)} = P_n$ on n vertices and n copies of P_{n+1}, denoted $P^{(i)}$, $1 \leq i \leq n$, by joining the ith vertex of $P^{(0)}$ $(1 \leq i \leq n)$ to all vertices of $P^{(i)}$. Then it is straightforward to see that $\gamma_t(G_n) = n$. Hence by the already proved upper bound, $\gamma_{tg}(G_n) \leq 2n - 1$. On the other hand, Staller has a strategy to play at least $n-1$ vertices of the subgraphs $P^{(i)}$, $1 \leq i \leq n$, which then implies that $\gamma_{tg}(G_n) \geq 2n-1$. This settles the theorem's case $\ell = n-1$.

Let k be a positive integer and let ℓ be a non-negative integer such that $n-1 = k + \ell$. Let X_i, $1 \leq i \leq k$, be the complete graph K_2 with $V(X_i) = \{x_i, y_i\}$. Let Y_i, $1 \leq i \leq \ell$, be the graph with $V(Y_i) = \{u_i\} \cup \{a_{i,1}, \ldots, a_{i,\ell}, b_{i,1}, \ldots, b_{i,\ell}\}$, in which u_i is a universal vertex (that is, adjacent to all other vertices), the remaining edges being $a_{i,1}b_{i,1}, \ldots, a_{i,\ell}b_{i,\ell}$. (In other words, Y_i is obtained from ℓ disjoint triangles by identifying a vertex in each of them.) Let ${}_kG_\ell$ be the graph constructed from the disjoint union of $X_1, \ldots, X_k$, $Y_1, \ldots, Y_\ell$, and a vertex w by adding the edges $wx_1, \ldots, wx_k$ and $wu_1, \ldots, wu_\ell$.

Consider the following strategy of Dominator when the total domination game is played on ${}_kG_\ell$. Dominator will play first at vertex w. Note that this move prevents any y_i to be played in the rest of the game. Therefore, regardless of how the game proceeds all of the vertices in the set $\{x_1, \ldots, x_k\}$ will be chosen. Whenever Staller plays a vertex in some Y_i, Dominator responds by playing u_i. This is always possible as it is clearly not optimal for Staller to select u_i as the first vertex to be played in Y_i. Using this strategy of Dominator not more than two vertices will be played in each Y_i. Consequently,

$$\gamma_{tg}({}_kG_\ell) \leq 1 + k + 2\ell\,.$$

Consider next Staller's strategy. Suppose first that Dominator plays w as his first move. In the course of the game Dominator will play vertices u_i in order to finish the game on Y_i. The strategy of Staller is to first play, say, $a_{1,1}$, and then reply to a move u_i of Dominator by playing on some Y_j on which Dominator did not play yet. There she selects any legal move different from u_j. Note that in this way Staller always has a legal move as long as at least one u_i has not yet been played. Finally, if u_j is the last such vertex, Dominator must play u_j to finish the game on Y_j. In this way Staller forces the game to last at least $1 + k + 2\ell$ moves.

Suppose next that w is not the first move of Dominator. Assume that Dominator first played u_i. Then Staller replies with a move in Y_i, say $a_{i,1}$. As long as Dominator plays vertices u_j, Staller accordingly replies in Y_j. In this way in all subgraphs Y_j that were played, two vertices were selected. Suppose that at some stage of the game, Dominator plays w. In that case, Staller follows the same strategy on the remaining subgraphs Y_j as she was playing in the case when Dominator started the game on w. This will ensure that the game will last at least $1 + k + 2\ell$ moves. Assume finally that Dominator will never play the vertex w. Then, as soon as he plays in some X_i, Staller replies with a move in the same X_i. In this way (at least) 2ℓ vertices will be played on the subgraphs Y_i and (at least) $k+1$ on the X_i's. (Note that here we essentially need the assumption that $k \geq 1$.) Hence also in any case at least $1 + k + 2\ell$ moves will be played, therefore

$$\gamma_{tg}({}_kG_\ell) \geq 1 + k + 2\ell = n + \ell\,.$$

Since this holds for any $0 \leq \ell \leq n - 2$ and because it is clear that

$$\gamma_t({}_kG_\ell) = k + \ell + 1 = n\,,$$

we are done. □

The game domination number and the game total domination number are related as follows:

Corollary 4.2 *If G is a graph on at least two vertices, then $\gamma_g(G) \leq 2\gamma_{tg}(G) - 1$.*

Proof. It was proved in [1] that $\gamma_g(G) \le 2\gamma(G) - 1$, hence

$$\begin{aligned}\gamma_g(G) &\le 2\gamma(G) - 1 \quad ([1, \text{Theorem 1}])\\ &\le 2\gamma_t(G) - 1\\ &\le 2\gamma_{tg}(G) - 1 \quad (\text{Theorem 4.1})\,.\end{aligned}$$

□

To see that Corollary 4.2 is close to being optimal consider the following examples. For any $n \ge 2$, let G_n be the graph obtained from the complete graph on n vertices by attaching n leaves to each of its vertices. We claim that

$$\gamma_{tg}(G_n) = n + 1 \quad \text{and} \quad \gamma_g(G_n) = 2n - 1\,.$$

Consider first the total domination game and let Dominator start the game in a vertex u of the n-clique. (By the following it is clear that playing a leaf in the first move is not optimal for Dominator.) An optimal reply of Staller is to play a leaf attached to u, because otherwise she could play no leaf in due course. After the first two moves both players must alternatively play all the $n-1$ remaining vertices of the n-clique. Therefore, $\gamma_{tg}(G_n) = n + 1$.

Consider next the usual domination game. From the Continuation Principle it follows that an optimal first move for Dominator is a vertex of the n-clique. Since each vertex of it has n leafs attached, Staller will be able to play a leaf as long as Dominator has not played all the vertices from the clique. So $\gamma_g(G_n) \ge 2n - 1$. On the other hand, $\gamma_g(G_n) \le 2\gamma(G_n) - 1 = 2n - 1$.

5 Game total domination versus domination

The game total domination number can be bounded by the domination number as follows.

Theorem 5.1 *If G is a graph such that $\gamma(G) \ge 2$, then $\gamma(G) \le \gamma_{tg}(G) \le 3\gamma(G)-2$. Moreover, the bounds are sharp.*

Proof. The lower bound follows from the inequality chain $\gamma(G) \le \gamma_t(G) \le \gamma_{tg}(G)$. Note first that $\gamma(K_{2,n}) = \gamma_{tg}(K_{2,n}) = 2$, $n \ge 3$. Let u be a vertex of degree 2 in $K_{2,n}$ and append to it a path of length 2. Call this graph H_n. Then $\gamma(H_n) = 3$ and hence also $\gamma_{tg}(H_n) \ge 3$. Suppose now that Dominator plays u. Then the only legal move for Staller is one of the three neighbors of u in H_n. But then Dominator can finish the game in the next move, thus $\gamma_{tg}(H_n) \le 3$. Consequently $\gamma(H_n) = \gamma_{tg}(H_n) = 3$. To get all other possible values that attain the lower bound apply Proposition 3.3.

To prove the upper bound, let D be an arbitrary $\gamma(G)$-set. Dominator's strategy is to select vertices in D sequentially whenever such a move is legal. Once Dominator has played all allowable vertices in D, at most $2|D| - 1 = 2\gamma(G) - 1$ moves have been made. At this point of the game all vertices in $N(D)$ are totally dominated.

Case 1. No vertex in D is currently totally dominated.
In this case D is an independent set and both Dominator and Staller only played vertices from D. That is, exactly $|D| = \gamma(G)$ moves have been made. The only remaining legal moves are those that totally dominate vertices in D, implying that at most $|D|$ additional moves are required to complete the game. Hence the total number of moves played is at most $2|D| = 2\gamma(G) \leq 3\gamma(G) - 2$.

Case 2. At least one vertex in D is currently totally dominated.
Now the only legal moves remaining in the game are those that totally dominate vertices in D if any are not yet totally dominated. This implies that at most $|D| - 1$ additional moves are required to complete the game. Hence the total number of moves is at most $(2|D| - 1) + (|D| - 1) = 3\gamma(G) - 2$.

This proves the upper bound. In order to show its sharpness, let B_k, $k \geq 2$, be the graph constructed as follows. For $i = 1, 2, \ldots, k^2$ let Q_i be a complete graph of order k with the vertex set $\{y_1^{(i)}, y_2^{(i)}, \ldots, y_k^{(i)}\}$. Then take the disjoint union of these cliques, add vertices $x_1, x_2, \ldots, x_k$, and for $i = 1, 2, \ldots, k$, join x_i to the k^2 vertices $y_i^{(1)}, y_i^{(2)}, \ldots, y_i^{(k^2)}$. Finally, add a pendant edge to each vertex x_i and call the resulting leaf w_i. See Fig. 1 for B_k. For further reference let $X = \{x_1, x_2, \ldots, x_k\}$, and for $i = 1, 2, \ldots, k$ let $Y_i = \{y_i^{(j)} :\ j = 1, 2, \ldots, k^2\}$.

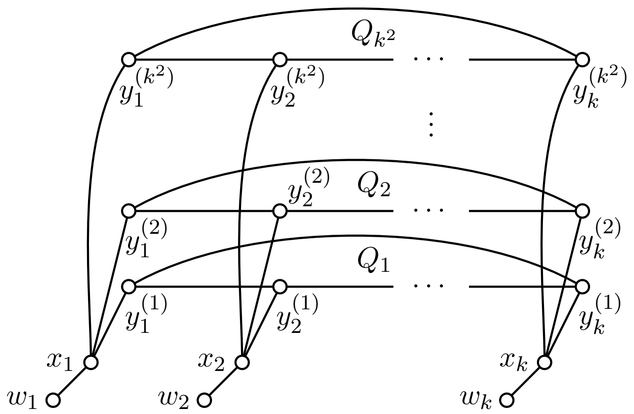

Figure 1: The graph B_k

Note first that $\gamma(B_k) = k$ and in fact, X is a unique minimum dominating set of B_k. Since we have already proved that $\gamma_{tg}(B_k) \leq 3k - 2$, it remains to show that Staller has a strategy that will ensure the game to last $3k - 2$ moves. Her strategy is to play vertices from Y_1 as long as possible and to never play a vertex from X.

Since as long as not all vertices from X are played, Staller can play inside Y_1, at least $k-1$ moves are played on vertices from Y_1. Note next that during the game all vertices from X must be played in order to totally dominate vertices w_i. In order to totally dominate the vertices from $X' = X \setminus \{x_1\}$, an additional $k-1$ moves are needed since X' is a 2-packing. Hence at least $(k-1)+k+(k-1) = 3k-2$ moves are needed to complete the game. □

6 Paths and Cycles

In this section we study the game total domination number, $\gamma_{tg}(G)$, and the Staller-start game total domination number, $\gamma'_{tg}(G)$, when G is a path P_n or a cycle C_n on n vertices. In a partially total dominated cycle or path, a *run* is a maximal sequence of consecutive totally dominated vertices that contain at least two vertices.

6.1 Cycles

We focus our attention first on cycles. For small n, the values of $\gamma_{tg}(C_n)$ and $\gamma'_{tg}(C_n)$ can be checked by computer and are shown in Table 1.

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\gamma_{tg}(C_n)$	2	2	3	4	5	5	6	6	7	8	9	9	10	10	11	12	13	13
$\gamma'_{tg}(C_n)$	2	2	3	4	4	4	6	6	7	8	8	8	10	10	11	12	12	12

Table 1. $\gamma_{tg}(C_n)$ and $\gamma'_{tg}(C_n)$ for small cycles C_n.

We first establish lower bounds on $\gamma_{tg}(C_n)$ and $\gamma'_{tg}(C_n)$.

Lemma 6.1 *For $n \ge 5$, the following holds.*
(a) $\gamma_{tg}(C_n) \ge 2n/3 - 1$.
(b) $\gamma'_{tg}(C_n) \ge 2(n-1)/3 - 1$.

Proof. We present the following strategy for Staller which will yield the desired lower bounds. Staller's strategy is as follows.

(a) Suppose we are in Game 1 and that Dominator has selected vertex v. Staller then picks an arbitrary neighbor of v as her first move, and in so doing creates a run (on four vertices). We note that Staller's first move totally dominates two new vertices. On each of Staller's subsequent moves, she arbitrarily chooses a run and plays a vertex x on one end of this run. The neighbor of x not on this run is the only new vertex totally dominated by x. Thus on each of Staller's moves different from her initial move she totally dominates exactly one new vertex. We note that since we are playing on a cycle, each move of Dominator totally dominates at most two new vertices. Thus after $2k-1$ moves, where $k \geq 2$, we note that at most $3k$ vertices have been totally dominated, $2k$ vertices by the k moves of Dominator and $2+(k-2)=k$ vertices by the $k-1$ moves of Staller, while after $2k$ moves, at most $3k+1$ vertices have been totally dominated.

For notational simplicity, let $M = \gamma_{tg}(C_n)$. On the one hand, suppose that $M = 2k-1$ for some $k \geq 2$. Then all n vertices on the cycles must be totally dominated, implying that $n \leq 3k = 3(M+1)/2$, or, equivalently, $M \geq 2n/3 - 1$. On the other hand, suppose that $M = 2k$ for some $k \geq 2$. Then this implies that $n \leq 3k+1 = 3M/2+1$, or, equivalently, $M \geq 2(n-1)/3$. In both cases, $\gamma_{tg}(C_n) = M \geq 2n/3 - 1$. This proves Part (a).

(b) Suppose next we are in Game 2. Staller selects an arbitrary vertex v. If Dominator plays a move that creates a run, then Staller's strategy is as before: she arbitrarily chooses a run and plays a vertex on one end of this run on each of her subsequent moves, thereby totally dominating exactly one new vertex on each of her moves except for her first move (which totally dominates two vertices). If however Dominator's first move does not create a run, then Staller on her second move picks an arbitrary neighbor of a played vertex, and in so doing creates a run. On all her subsequent moves, she follows her earlier strategy and arbitrarily chooses a run and plays a vertex on one end of this run. Thus after $2k-1$ moves, where $k \geq 2$, we note that at most $3k+1$ vertices have been totally dominated, $2k$ vertices by the k moves of Dominator and $2+2+(k-3) = k+1$ vertices by the $k-1$ moves of Staller, while after $2k$ moves, at most $3k+2$ vertices have been totally dominated.

For notational simplicity, let $M' = \gamma'_{tg}(C_n)$. On the one hand, suppose that $M' = 2k-1$ for some $k \geq 2$. Then all n vertices on the cycles must be totally dominated, implying that $n \leq 3k+1 = 3(M'+1)/2+1$, or, equivalently, $M' \geq (2n-5)/3$. On the other hand, suppose that $M' = 2k$ for some $k \geq 2$. Then this implies that $n \leq 3k+2 = 3M'/2+2$, or, equivalently, $M' \geq 2(n-2)/3$. In both cases, $\gamma'_{tg}(C_n) = M' \geq (2n-5)/3$. This proves Part (b) and completes the proof of the lemma. □

We next establish upper bounds on $\gamma_{tg}(C_n)$ and $\gamma'_{tg}(C_n)$. For this purpose, we say that a vertex in a partially total dominated graph is *unplayable* if it is not a legal move.

Lemma 6.2 *For $n \geq 5$, the following holds.*
(a) $\gamma'_{tg}(C_n) \leq 2n/3$.
(b) $\gamma_{tg}(C_n) \leq (2n+1)/3$.

Proof. We present the following strategy for Dominator which will yield the desired upper bounds. At each point in the game, let U denote the set of unplayable vertices and let $|U| = u$. Further, let m be the number of moves made so far. Dominator's strategy is as follows.

(a) Suppose first that we are in Game 2 and that Staller has selected vertex v. Let e be an edge incident with v on the cycle and consider the path P obtained from the cycle by deleting the edge e. Dominator now plays the vertex at distance 4 from v on the resulting path P. We note then that the vertex on P at distance 2 from v is now unplayable and gets added to the set U.

More generally, suppose it is Staller's move and she plays a vertex v_1. Let v_2 denote a new vertex totally dominated by v_1, and so v_2 is a neighbor of v_1. Let v' be the other neighbor of v_1 on the cycle (possibly v' is also a new vertex totally dominated by v_1) and let $e = v_1v'$. We now consider the path, P, that starts at v_1 and is obtained from the cycle by deleting the edge e. Let the path P be given by $v_1, v_2, v_3, \ldots, v_n = v'$. Since the vertex v_2 is a new vertex totally dominated by v_1 when she played this vertex, we note that the vertex v_3 has not been played. If the game is not over, then Dominator strategy is as follows. If v_5 has not been played at this point, then Dominator plays v_5; otherwise, Dominator plays any legal move. We remark that if v_5 had not been played when Staller plays v_1, then in this case neither neighbor of v_4 had been played. Hence when Dominator plays the vertex v_5 the vertex v_4 becomes a new vertex totally dominated (and so v_5 is indeed a legal move). Further, we note that when Dominator plays the vertex v_5 the vertex v_3 now becomes unplayable (since both neighbors of v_3 are totally dominated) and gets added to the set U. We also remark that if v_5 had already been played when Staller plays v_1, then v_3 immediately becomes unplayable and gets added to the set U. Thus after Dominator has made his response to Staller's move v_1, then these two moves result in at least two new vertices becoming totally dominated and at least one new vertex becoming unplayable and added to the set U.

In summary, Dominator's strategy is to create at least one new unplayable vertex after every two consecutive moves of the game. Indeed Dominator's strategy guarantees that every two consecutive moves result in at least one new vertex becoming unplayable. For notational simplicity, let $m' = \gamma'_{tg}(C_n)$. Dominator's strategy implies that if m' is even, then $u \ge m'/2$. If m' is odd, then the last move of the game is played by Staller and this move results in at least one new vertex becoming unplayable, implying that $u \ge (m'+1)/2$. In both cases, $u \ge m'/2$. Since the moves of the game are only played using vertices not in U, this in turn implies that $m' = n - u \le n - m'/2$, or, equivalently, $\gamma'_{tg}(C_n) = m' \le 2n/3$. This proves Part (a).

(b) Suppose we are in Game 1. Dominator follows exactly the same strategy as in Part (a) except that he plays an arbitrary vertex on his first move. For notational simplicity, let $m = \gamma_{tg}(C_n)$. If m is odd, then $u \ge (m-1)/2$. If m is even, then the last move of the game is played by Staller and this move results in at least one new vertex becoming unplayable, implying that $u \ge m/2$. In both cases, $u \ge (m-1)/2$. Therefore, $m = n - u \le n - (m-1)/2$, or, equivalently, $\gamma_{tg}(C_n) = m \le (2n+1)/3$. This proves Part (b) and completes the proof of the lemma. □

As an immediate consequence of Lemma 6.1 and 6.2, we have the following results.

Corollary 6.3 *For $n \geq 3$, we have that $\gamma'_{tg}(C_n) \leq \lfloor \frac{2n}{3} \rfloor$. Further,*

$$\gamma_{tg}(C_n) \leq \begin{cases} \lceil \frac{2n}{3} \rceil & \text{when } n \equiv 1,4 \pmod 6 \\ \lfloor \frac{2n}{3} \rfloor & \text{otherwise.} \end{cases}$$

Corollary 6.4 *For $n \geq 3$ and $n \equiv 2,5 \pmod 6$, we have that*

$$\gamma_{tg}(C_n) = \left\lfloor \frac{2n}{3} \right\rfloor.$$

We believe that the upper bounds in Corollary 6.3 are almost tight. More precisely, we conjecture the following.

Conjecture 6.5 *For $n \geq 3$, we have*

$$\gamma_{tg}(C_n) = \begin{cases} \lceil \frac{2n}{3} \rceil & \text{when } n \equiv 1 \pmod 6 \\ \lfloor \frac{2n}{3} \rfloor & \text{otherwise} \end{cases}$$

and

$$\gamma'_{tg}(C_n) = \begin{cases} \lfloor \frac{2n}{3} \rfloor - 1 & \text{when } n \equiv 2 \pmod 6 \\ \lfloor \frac{2n}{3} \rfloor & \text{otherwise.} \end{cases}$$

6.2 Paths

For small n, the values of $\gamma_{tg}(P_n)$ and $\gamma'_{tg}(P_n)$ can be checked by computer and are shown in Table 2.

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\gamma_{tg}(P_n)$	2	3	3	4	5	6	6	7	7	8	9	10	10	11	11	12	13	14
$\gamma'_{tg}(P_n)$	2	3	4	4	5	6	6	7	8	8	9	10	10	11	12	12	13	14

Table 2. $\gamma_{tg}(P_n)$ and $\gamma'_{tg}(P_n)$ for small paths P_n.

We remark that an identical proof to that of Lemma 6.1 shows that the lower bounds in the statement of the lemma also hold in the case of a path. Further the strategy used to establish the upper bounds in the statement of Lemma 6.2 can be modified slightly to yield analogous upper bounds in the case of a path, showing that the game total domination number and the Staller-start game total domination number of a path P_n differs from that for a cycle C_n by at most 2. We conjecture the following.

Conjecture 6.6 *For $n \geq 3$, we have $\gamma'_{tg}(P_n) = \lceil \frac{2n}{3} \rceil$ and*

$$\gamma_{tg}(P_n) = \begin{cases} \lfloor \frac{2n}{3} \rfloor & \text{when } n \equiv 5 \pmod 6 \\ \lceil \frac{2n}{3} \rceil & \text{otherwise.} \end{cases}$$

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