On Isomorphism Classes of Generalized Fibonacci Cubes

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The generalized Fibonacci cube $Q_d(f)$ is the subgraph of the d-cube Q_d induced on the set of all strings of length d that do not contain f as a substring. It is proved that if $Q_d(f) \cong Q_d(f')$ then $|f| = |f'|$. The key tool to prove this result is a result of Guibas and Odlyzko about the autocorrelation polynomial associated to a binary string. An example of a family of such strings f, f', where $|f| = |f'| \geq \frac{2}{3}(d+1)$ are found. Strings f and f' with $|f| = |f'| = d-1$ for which $Q_d(f) \cong Q_d(f')$ are characterized.

1. Introduction

An element of $\{0,1\}^d$ is called a *binary string* (henceforth just called a *string*) of length d, with the usual concatenation notation. For example, $0^{d-1}1$ is the string of length d consisting of $d-1$ 0 bits followed by a single 1 bit. We will denote by $e_i = 0^{i-1}10^{d-i}$ the i^{th} unit string in $\{0,1\}^d$.

Let $d \geq 1$ be a fixed integer. The *d-cube* Q_d is the graph whose vertices are the binary strings of length d, with an edge connecting vertices v_1 and v_2 if the underlying strings differ in exactly one position. Given a graph G, the set of vertices of G is denoted by $V(G)$. We use $d_G(u,v)$ to denote the length of a shortest path connecting u and v in G. Lastly, we will write $G \cong H$ to signify that the graphs G and H are isomorphic.

For a given string f and integer d, the *generalized Fibonacci cube* $Q_d(f)$ is the subgraph of Q_d induced by the set of all strings of length d that do not contain f as a consecutive substring. Indeed, this

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generalizes the notion of the d-dimensional *Fibonacci cube* $\Gamma_d = Q_d(11)$, which is the graph obtained from the d-cube Q_d by removing all vertices that contain the substring 11. Set $n_d(f) = |V(Q_d(f))|$.

Fibonacci cubes were introduced by Hsu [3] as a model for interconnection networks. Like the hypercube graphs, Fibonacci cubes have several properties that make them ideal as a network topology, yet their size grows significantly slower than that of the hypercubes. More precisely, while the hypercube of dimension n has 2^n vertices, the order of Γ_n is asymptotically φ^{n+2}, where φ is the golden ratio. Fibonacci cubes have been extensively investigated; see, for example, the recent survey by Klavžar [7] and even more recent papers of Klavžar and Mollard [8] and Vesel [15]. In the first of these papers, different asymptotic properties of Fibonacci cubes are established, while in the latter a linear recognition algorithm is designed for recognizing Fibonacci cubes, improving the previous best recognition algorithm of Taranenko and Vesel [13].

Later, Ilić, Klavžar, and Rho [4] introduced the idea of generalized Fibonacci cubes (as defined above). Under the same name, the graphs $Q_d(1^s)$ were studied by Liu, Hsu, and Chung [10] and Zagaglia Salvi [14]. The analysis of the properties of generalized Fibonacci cubes led to the study of several problems related to the combinatorics of words. To study their isometric embeddability into hypercubes, good and bad words were introduced by Klavžar and Shpectorov [9], where it was proved that about eight percent of all words are good. Isometric embeddability and hamiltonicity of generalized Fibonacci cubes motivated the ideas of the index and parity of a binary word, as defined by Ilić, Klavžar, and Rho [5, 6]. Infinite families of bad strings were found [5, 16]. In [1] it was proved that $Q_d(f)$ is 2-connected for any f with $|f| \geq 3$.

In this paper we consider the following fundamental question about the generalized Fibonacci cubes: for which binary strings f and f' and positive integers d are the generalized Fibonacci cubes $Q_d(f)$ and $Q_d(f')$ isomorphic? In the next section, we prove that if $Q_d(f)$ and $Q_d(f')$ have the same order, then the equality $|f| = |f'|$ holds. In addition, a family of strings f, f', with $|f| = |f'| \geq \frac{2}{3}(d+1)$ is found for which $Q_d(f) \cong Q_d(f')$ holds. In the last section, we prove that if $|f| = d-1$, then $Q_d(f) \cong Q_d(f')$ if and only if f and f' have the same block structure. Several conjectures are posed along the way.

In the rest of the section we introduce additional terminology and notation needed throughout the paper. The complement of a bit x is denoted by $\overline{x}$. It is easy to see that if f' is the binary complement of f, or if f' is the reverse of f (the *reverse* of $f = f_1 \dots f_d$ is $f_d \dots f_1$), then $Q_d(f) \cong Q_d(f')$ for any dimension d. Hence we say that a pair of binary strings f, f' is *trivial* if f' can be obtained from f by binary complementation, reversal, or composition of these mappings. We are therefore only interested in the behavior of the other pairs, which we call the *non-trivial pairs*. A *block* of a binary sting f is a maximal (with respect to inclusion) substring of f consisting of consecutive equal bits. Let $f = 0^{r_1}1^{s_1}\cdots 0^{r_k}1^{s_k}$, where $r_1, s_k \geq 0$, $r_2, \dots, r_k, s_1, \dots, s_{k-1} \geq 1$ and $f' = 0^{r'_1}1^{s'_1}\cdots 0^{r'_\ell}1^{s'_\ell}$, where $r'_1, s'_\ell \geq 0$, $r'_2, \dots, r'_\ell, s'_1, \dots, s'_{\ell-1} \geq 1$ be binary strings. Then f and f' have the same block structure if the following three conditions are satisfied: $k = \ell$, $r_1 = 0$ if and only if $r'_1 = 0$, and $s_k = 0$ if and only if $s'_\ell = 0$.

2. The Length of Forbidden Words

In this section we first prove that a necessary condition for $Q_d(f)$ being isomorphic to $Q_d(f')$ is that $|f| = |f'|$. Then we pose the question whether there is some relation between $|f|$ $(= |f'|)$ and d provided that $Q_d(f) \cong Q_d(f')$. To this end we prove that there exist non-trivial pairs f, f' such that $Q_d(f) \cong Q_d(f')$ and $|f| \geq \frac{2}{3}(d+1)$. We also conjecture that for any non-trivial pair f, f' such that $Q_d(f) \cong Q_d(f')$, we must have $|f| \geq \frac{2}{3}(d+1)$.

The *autocorrelation polynomial* $p_f(z)$ associated to a binary string $f = f_1 \dots f_k \in \{0,1\}^k$ is defined as

$$p_f(z) = \sum_{i=0}^{k-1} c_i z^i \,,$$

where $c_i = 1$ if the length $k-i$ suffix of f is equal to the length $k-i$ prefix of f, i.e., if $f_{i+1}\dots f_k = f_1\dots f_{k-i}$, and $c_i = 0$ otherwise. The autocorrelation polynomial allows the immediate computation of the ordinary generating function counting binary words that avoid f by length:

$$S_f(z) = \frac{p_f(z)}{z^k + (1-2z)p_f(z)};$$

see Flajolet and Sedgewick [11, Proposition I.4]. Note that the autocorrelation polynomial of $f \in \{0,1\}^k$ is of degree at most $k-1$ and has degree $k-1$ if and only if the last bit of f is equal to the first bit of f. Observe also that

$$p_{0^k}(z) = \sum_{i=0}^{k-1} z^i \quad \text{and} \quad p_{0^{k-1}1}(z) = 1\,. \tag{1}$$

We note in passing that more generally, $p_f(z) = 1 + z + \cdots + z^{k-1}$ if and only if $f = 0^k$ or $f = 1^k$ and that $p_f(z) = 1$ if and only if every non-trivial suffix of f is different from the prefix of f of the same length. Such words were named *prime* in [6].

Set $n_d(f) = Q_d(f)$. If $k = |f|$, then in a word of length d that is not in $Q_d(f)$, the string f can begin in positions $1,\dots,d-k+1$, the other $d-k$ bits are arbitrary. Hence we note in passing that $n_d(f) \geq 2^d - 2^{d-k}(d-k+1)$. Theorem 2.2 establishes that only strings which have the same length can generate isomorphic generalized Fibonacci cubes, i.e., $|f| = |f'|$ is a necessary condition for $Q_d(f) \cong Q_d(f')$. Its proof relies on the following result due to Guibas and Odlyzko.

Lemma 2.1. ([2, Section 7]) *Let s and s' be two binary strings with $|s| = |s'|$. If $p_s(2) \geq p_{s'}(2)$, then $n_d(s) \geq n_d(s')$ for all d.*

We can now prove Theorem 2.2.

Theorem 2.2. *If $|f|, |f'| \leq d$ and $n_d(f) = n_d(f')$, then $|f| = |f'|$.*

Proof. We prove the contrapositive. Assume that $|f| < |f'|$. We will show that $n_d(f) < n_d(f')$, from which the theorem follows.

If $|f| = d$, then it is clear that $n_d(f) < n_d(f')$, so we now assume that $|f| < d$. It follows from Lemma 2.1 that if f is a binary string of length $k < d$, then

$$n_d(0^{k-1}1) \leq n_d(f) \leq n_d(0^k)\,. \tag{2}$$

In addition, since 0^k is a strict substring of 0^k1 we infer that

$$n_d(0^k) < n_d(0^k1)\,. \tag{3}$$

Combining (2) and (3) we conclude that

$$n_d(f) \leq n_d(0^k) < n_d(0^k1) \leq n_d(f')\,,$$

where the last inequality is due to the assumption that $|f'| > k$. □

Hence, non-trivial pairs f, f' such that $Q_d(f) \cong Q_d(f')$ are of the same length. We next ask what is the relation of this length with the dimension of the corresponding generalized Fibonacci cubes. The following result could be the extremal case.

Theorem 2.3. *If $k \geq 2$, then $Q_d(0^k1^k) \cong Q_d(0^{k+1}1^{k-1})$ for any $d \leq 3k-1$.*

Proof. Let $k \geq 2$ be a fixed integer and set $f = 0^k1^k$, $f' = 0^{k+1}1^{k-1}$, $G = Q_{3k-1}(f)$, and $G' = Q_{3k-1}(f')$. Let $X = V(Q_{3k-1}) \setminus V(G)$ and $X' = V(Q_{3k-1}) \setminus V(G')$. For any $0 \leq i \leq k-1$ let

$$X_i = \{ufv:\ |u| = i, |v| = k-1-i\}\,.$$

Then, by definition,

$$X = \bigcup_{i=0}^{k-1} X_i.$$

Let $w = w_1 \dots w_{3k-1}$ be an arbitrary vertex whose underlying string is in X_i. It then follows that $w_{k+1} \dots w_{2k-1} = 0^i 1^{k-i-1}$, and so $X_j \cap X_\ell = \emptyset$ holds for all $j \neq \ell$. Since $|X_i| = 2^{k-1}$, we have $|X| = k2^{k-1}$. With a parallel argument we infer that also $|X'| = k2^{k-1}$. This implies that $|V(G)| = |V(G')|$.

Consider now the mapping $\alpha : V(Q_{3k-1}) \to V(Q_{3k-1})$ defined by

$$\alpha(w_1 \dots w_{3k-1}) = w_1 \dots w_k \overline{w_{2k}} w_{k+1} \dots w_{2k-1} w_{2k+1} \dots w_{3k-1}\,.$$

In particular, α fixes the first k and the last $k-1$ coordinates. Since transposition of coordinates, complementation of a coordinate, and any composition of such mappings are all automorphisms of a hypercube, α is an automorphism of Q_{3k-1}. Consider now G and G' as subgraphs of Q_{3k-1} and the restriction $\alpha|_G$ of α to G. Since $|V(G)| = |V(G')|$, it remains to prove that $\alpha|_G : V(G) \to V(G')$. For any $w \in X$, $w \in X_i$ for some $0 \leq i \leq k-1$ and hence $w = ufv$ where $|u| = i, |v| = k-1-i$. Note that $w_{2k} = f_{2k-i} = 1$ and hence $\overline{w_{2k}} = 0$ as $k+1 \leq 2k-i \leq 2k$ and $w_1 \dots w_k \overline{w_{2k}} w_{k+1} \dots w_{2k-1} = f_1 \cdots f_k 0 f_{k+1} \dots f_{2k-1} = 0^{k+1}1^{k-1}$ if $i = 0$ and $w_{i+1} \dots w_k \overline{w_{2k}} w_{k+1} \dots w_{2k-1} w_{2k+1} \dots w_{2k+i} = f_1 \dots f_{k-i} 0 f_{k+1-i} \dots f_{2k-1-i} f_{2k+1-i} \dots f_{2k} = 0^{k+1}1^{k-1}$ if $i \geq 1$. Therefore $\alpha(w) = uf'v$. Thus $\alpha|_G : V(G) \to V(G')$.

We have thus proved the result for $d = 3k-1$. Note that all of the above arguments also work for $2k+1 \leq d \leq 3k-2$ and when $d \leq 2k$, the assertion is trivial. $\square$

Motivated by the last theorem we pose:

Conjecture 2.4. *If f and f' are binary strings such that $Q_d(f) \cong Q_d(f')$, then $Q_{d-1}(f) \cong Q_{d-1}(f')$.*

Conjecture 2.5. *Let f, f' be a non-trivial pair such that $Q_d(f) \cong Q_d(f')$. Then $|f| \geq \frac{2}{3}(d+1)$.*

We have verified these conjectures for small strings using the Sage package [12]. More precisely, we tested both conjectures for all $d \leq 11$ and all non-trivial pairs f, f'.

3. The Number of Blocks in Forbidden Words

In this section we characterize the binary strings f, f' of length $d-1$ for which $Q_d(f) \cong Q_d(f')$. It turns out that they are precisely the strings with the same block structure.

Let $\nu(f)$ denote one less than the number of blocks of $f = f_1 f_2 \dots f_{|f|}$. For example $\nu(0110) = 2$. When a bit is different from the previous bit we call its index an *index of bit change* and denote it by i_j. Therefore $f_{i_j - 1} \neq f_{i_j}$ for $i_1 < i_2 < \cdots < i_{\nu(f)}$.

Theorem 3.1. *Let $d \geq 2$ and let f, f' be binary strings of length $d-1$. Then $\nu(f) = \nu(f')$ if and only if $Q_d(f) \cong Q_d(f')$.*

Proof. Set $A = V(Q_d) \setminus V(Q_d(f))$. Then $A = \{\overline{f_1} f, f_1 f, f f_{|f|}, f \overline{f_{|f|}}\}$. Similarly set $A' = V(Q_d) \setminus V(Q_d(f')) = \{\overline{f'_1} f', f'_1 f', f' f'_{|f|}, f' \overline{f'_{|f'|}}\}$. Let $2 \leq i_1 < \cdots < i_{\nu(f)} \leq d-1$ be the indices of bit change of f and let $2 \leq i'_1 < \cdots < i'_{\nu(f')} \leq d-1$ be the indices of bit change of f'. Since $(\overline{f_1} f)_\tau = (f_1 f)_\tau = f_{\tau-1}$ for $2 \leq \tau \leq d-1$ and $(f f_{|f|})_\tau = (f \overline{f_{|f|}})_\tau = f_\tau$ for $2 \leq \tau \leq d-1$, the strings $f_1 f$ and $f f_{|f|}$ are different precisely for $\tau = i_j$, $1 \leq j \leq \nu(f)$. It follows that $d_{Q_d}(f_1 f, f f_{|f|}) = \nu(f)$ and by a parallel argument $d_{Q_d}(f'_1 f', f' f'_{|f'|}) = \nu(f')$.

Assume first that $\nu(f)=\nu(f')$. We may without loss of generality assume that $f_1=f_1'=0$. Then $f_{|f|}=f'_{|f|}$. Let ϕ be any permutation of $\{1,\dots,d\}$ such that $\phi(i_j)=i'_j$, $\phi(1)=1$, $\phi(d)=d$. Note that there could be multiple candidates for ϕ; any such candidate will suffice. Set $\psi(x_1\dots x_d)=y_1\dots y_d$, where $y_1=x_1$, $y_d=x_d$, and for all $2\le\tau\le d-1$

$$y_\tau=\begin{cases} x_{\phi^{-1}(\tau)}; & \text{if } f_{\phi^{-1}(\tau)}=f'_\tau \text{ or } \tau=d\\ \overline{x_{\phi^{-1}(\tau)}}; & \text{otherwise.}\end{cases}$$

We show that ψ sends the vertices of A to A'. Consider the case where $x_1x_2\dots x_d=f_1f_2\cdots f_{d-1}x_d$. Assume $\tau<d$. If $f_{\phi^{-1}(\tau)}=f'_\tau$, then $y_\tau=f_{\phi^{-1}(\tau)}=f'_\tau$. If $f_{\phi^{-1}(\tau)}\neq f'_\tau$, then $y_\tau=\overline{f_{\phi^{-1}(\tau)}}=f'_\tau$. Therefore in any case $y_1\dots y_d=f'_1f'_2\cdots f'_{d-1}x_d$. Now consider the case where $x_1x_2\cdots x_d=x_1f_1f_2\cdots f_{d-1}$, i.e., $x_\tau=f_{\tau-1}$ when $\tau\ge2$. Assume $\tau>1$. Note that $f_{\phi^{-1}(\tau)-1}=f_{\phi^{-1}(\tau)}$ if and only if $\phi^{-1}(\tau)\neq i_j$ if and only if $\tau\neq i'_j$ if and only if $f'_{\tau-1}=f'_\tau$. Firstly assume $f_{\phi^{-1}(\tau)}=f'_\tau$. Then $y_\tau=x_{\phi^{-1}(\tau)}=f_{\phi^{-1}(\tau)-1}$. If $\phi^{-1}(\tau)\neq i_j$, then $y_\tau=f_{\phi^{-1}(\tau)-1}=f_{\phi^{-1}(\tau)}=f'_\tau=f'_{\tau-1}$. If $\phi^{-1}(\tau)=i_j$, then $y_\tau=f_{\phi^{-1}(\tau)-1}=\overline{f_{\phi^{-1}(\tau)}}=\overline{f'_\tau}=f'_{\tau-1}$. Secondly assume $f_{\phi^{-1}(\tau)}\neq f'_\tau$. Then $y_\tau=\overline{f_{\phi^{-1}(\tau)-1}}$. If $\phi^{-1}(\tau)\neq i_j$, then $y_\tau=\overline{f_{\phi^{-1}(\tau)-1}}=\overline{f_{\phi^{-1}(\tau)}}=f'_\tau=f'_{\tau-1}$. If $\phi^{-1}(\tau)=i_j$, then $y_\tau=\overline{f_{\phi^{-1}(\tau)-1}}=f_{\phi^{-1}(\tau)}=\overline{f'_\tau}=f'_{\tau-1}$. Therefore in any case $y_1\dots y_d=x_1f'_1f'_2\cdots f'_{d-1}$.

It is well-known that transposition of coordinates, complementation of a coordinate, and compositions of such mappings are automorphisms of a hypercube. It follows therefore that ψ is an automorphism of Q_d. As shown above, ψ sends the vertices of A to A'. Thus ψ is an isomorphism from $Q_d(f)$ to $Q_d(f')$.

To prove the converse assume that $Q_d(f)\cong Q_d(f')$. We may without loss of generality assume that $\nu(f)\le\nu(f')$.

Assume $\nu(f)=0$. Then $f_1f=ff_{|f|}$ and hence $|A|=3$, while $|A'|=4$ if $\nu(f')\neq0$. Therefore $\nu(f')=0$.

Assume $\nu(f)=1$. Then note that the vertices from A induce a path on four vertices and hence $|E(Q_d(f))|=d2^{d-1}-(4d-3)$. If $\nu(f')>1$, then the vertices from A' induce two disjoint copies of K_2 and hence $|E(Q_d(f'))|=d2^{d-1}-(4d-2)\neq|E(Q_d(f))|$. We conclude that $\nu(f')=1$.

For the rest of the proof we can thus assume that $\nu(f)\ge2$. The subgraph of Q_d induced on A consists of two edges $\{\overline{f_1}f,f_1f\}$ and $\{ff_{|f|},f\overline{f_{|f|}}\}$ where $d(f_1f,ff_{|f|})=\nu(f)$ and $d(\overline{f_1}f,f\overline{f_{|f|}})=\nu(f)+2$. Denote $\overline{f_1}f$, f_1f, $ff_{|f|}$, $f\overline{f_{|f|}}$ by a, b, c, d, respectively. Consider the shortest b,c-path constructed by changing from left to right the bits in which b and c differ:

$$b=f_1f\to f_1f+e_{i_1}\to f_1f+e_{i_1}+e_{i_2}\to\cdots\to f_1f+e_{i_1}+\cdots+e_{i_{\nu(f)}}=ff_{|f|}=c,$$

where addition is taken modulo 2.

Denote the j-th internal vertex $f_1f+e_{i_1}+\cdots+e_{i_j}$ of this path by x_j for $1\le j\le\nu(f)-1$. Similarly, denote $\overline{f'_1}f'$, f'_1f', $f'f'_{|f|}$, $f'\overline{f'_{|f|}}$ by a', b', c', d', respectively. Set $k=\nu(f)-1$, $\ell=\nu(f')-1$, and recall that $k\ge1$. Let $\psi:Q_d(f)\to Q_d(f')$ be an isomorphism and let $x'_j=\psi(x_j)$ for $1\le j\le k$.

Assume $k=1$. Then x_1 is of degree $d-2$ and hence $\deg_{Q_d(f')}(x'_1)=d-2$. This means that x'_1 is adjacent to two vertices among a',b',c',d'. As Q_d is bipartite, x'_1 is adjacent to one of a',b' and one of c',d'. If x'_1 is adjacent to b' and c', then $\ell+1=d_{Q_d}(b',c')\le2$ and hence $\ell=1$. If x'_1 is adjacent to a' and c', then $\ell+2=d_{Q_d}(a',c')\le2$, a contradiction. Similarly we get contradictions if x'_1 is adjacent to d'.

Assume $k\ge2$. Then the vertices x_1 and x_k of $Q_d(f)$ are of degree $d-1$. Considering that $x_1\to\cdots\to x_k$ is a path in $Q_d(f)$, we see that $d_{Q_d(f)}(x_1,x_k)=k-1$. Therefore the vertices x'_1 and x'_k of $Q_d(f')$ are of degree $d-1$ and $d_{Q_d(f')}(x'_1,x'_k)=k-1$. We distinguish three cases.

Case 1: x'_1 and x'_k are adjacent to a common vertex among a',b',c',d'.
Now, $k-1=d_{Q_d}(x'_1,x'_k)\le2$ and hence $k\le3$. Also, k is odd as Q_d is bipartite, and thus $k=3$.

Considering that $x_1x_2x_3$ is a shortest path in Q_d, it follows that x_1 and x_3 have distance two and hence have a common neighbor which is different from x_2 in Q_d. Call it u. Then, u is not a, b, c, or d because of its distances from x_1 and x_3. Therefore $u \in Q_d(f)$. Set $u' = \psi(u)$. Then $u' \in Q_d(f')$ and hence $d_{Q_d}(x'_1, u') \leq d_{Q_d(f')}(x'_1, u') = d_{Q_d(f)}(x_1, u) = 1$, which means that $d_{Q_d}(x'_1, u') = 1$. Similarly, $d_{Q_d}(x'_3, u') = 1$. Hence in Q_d, x'_1 and x'_3 have three common neighbors: u', x'_2, and one of a', b', c', d'. This is a contradiction, because hypercubes are $K_{2,3}$-free.

From now on we regard that x'_1 and x'_k are not adjacent to a common vertex among a', b', c', d'.

Case 2: x'_1 and x'_k are either adjacent to a' and b' or adjacent to c' and d'.
We may without loss of generality assume the first. Then $k - 1 = d_{Q_d}(x'_1, x'_k) \leq 3$ and hence $k \leq 4$. Also, k is even as Q_d is bipartite. Therefore $k = 2$ or $k = 4$. We distinguish two subcases.

Case 2a: $k = 2$.
It is well-known that in Q_d a given edge lies in $d-1$ cycles of length 4. Among the 4-cycles containing the edge x_1x_2, one contains b and another contains c. This means that there are $d - 3$ cycles of length 4 containing the edge x_1x_2 in $Q_d(f)$. Among the 4-cycles containing the edge $x'_1x'_2$, only one contains a' and b' together, and no other contains a', b', c', or d'. This means that there are $d - 2$ cycles of length 4 containing the edge $x'_1x'_2$ in $Q_d(f')$, a contradiction.

Case 2b: $k = 4$.
It is known that for two given vertices at distance three, there are exactly three internally vertex-disjoint shortest paths in Q_d, and therefore there are such paths between x_1 and x_4. Let $R = x_1uvx_4$ be any one of them which is different from $x_1x_2x_3x_4$. Considering the distances of u, v from x_1, x_k, we obtain that $u, v \in Q_d(f)$ and hence R is a path in $Q_d(f)$. Therefore $\psi(R)$ is a path in $Q_d(f')$. By the assumption that $k = 4$, there is also an x'_1, x'_4-path through a' and b', implying that there are (at least) four internally disjoint shortest paths between x'_1 and x'_4 in Q_d, which is a contradiction.

Case 3: x'_1 is adjacent to one of a' and b' while x'_k is adjacent to one of c' and d', or vice versa.

Firstly, assume that x'_1 is adjacent to b' while x'_k is adjacent to c'. Then

$$\begin{aligned}
\ell + 1 &= d_{Q_d}(b', c') \\
&\leq d_{Q_d}(b', x'_1) + d_{Q_d}(x'_1, x'_k) + d_{Q_d}(x'_k, c') \\
&= 2 + d_{Q_d}(x'_1, x'_k) \\
&\leq 2 + d_{Q_d(f')}(x'_1, x'_k) \\
&= 2 + d_{Q_d(f)}(x_1, x_k) \\
&= k + 1\,.
\end{aligned}$$

Hence, under this assumption $\ell = k$.

Alternatively, assume that x'_1 is adjacent to a' while x'_k is adjacent to c'. Then,

$$\begin{aligned}
\ell + 2 &= d_{Q_d}(a', c') \\
&\leq d_{Q_d}(a', x'_1) + d_{Q_d}(x'_1, x'_k) + d_{Q_d}(x'_k, c') \\
&\leq 2 + d_{Q_d(f')}(x'_1, x'_k) \\
&= k + 1\,,
\end{aligned}$$

a contradiction. The other cases similarly lead to contradictions. □

If $|f| = |f'| \leq d - 2$, then $\nu(f) = \nu(f')$ in general no longer implies that $Q_d(f) \cong Q_d(f')$. For instance, it can be checked that $Q_6(0110) \not\cong Q_6(0100)$ despite the fact that $\nu(0110) = \nu(0100)$. On the other hand we pose the following conjecture.

Conjecture 3.2. *Let f, f' be a non-trivial pair such that $Q_d(f) \cong Q_d(f')$. Then $\nu(f) = \nu(f')$.*

This conjecture has also been verified for all dimensions up to 11 (and all non-trivial pairs f, f').

We conclude the paper with a brief description how the computer checking was performed on the case of Conjecture 2.4. The other conjectures were tested similarly. Let *genFib(d,f)* be a Python method returning $Q_d(f)$ and let *allStr(k)* be a method that returns all binary strings of length k modulo binary complements and inverses. Then a function *isom_classes* that returns a dictionary whose entries are classes of isomorphic strings is defined as follows:

```
def isom_classes(d):
    D = {}

    for k in range(3,d):
        for f in allStr(k):
            G = genFib(d,f)
            s = G.canonical_label().graph6_string()
            if s not in D:
                D[s] = [f]
            else:
                D[s]+=[f]
    return D
```

The idea of the above method is that to every generalized Fibonacci cube we compute a string associated to it such that isomorphic generalized Fibonacci cubes are assigned the same string. This string can be thus considered as an isomorphism certificate. In the method this is done in the assignment `s = G.canonical_label().graph6_string()`. Now, using the method, Conjecture 2.4 was tested as follows.

```
def testConj1(d):
     D = isom_classes(d)
     for key in D:
         for f1,f2 in Combinations(D[key],2):
             G1 = genFib(d-1,f1)
             G2 = genFib(d-1,f2)
             if not G1.is_isomorphic(G2):
                 return False
     return True
sage: all(testConj1(i) for i in xrange(2,12))
True
```

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