A characterization of general position sets in graphs

Bijo S. Anand [a]   Ullas Chandran S. V. [b]   Manoj Changat [c]
Sandi Klavžar [d,e,f]   Elias John Thomas [g]

November 16, 2018

[a] Department of Mathematics, Sree Narayana College, Punalur-691305, Kerala, India; `bijos_anand@yahoo.com`

[b] Department of Mathematics, Mahatma Gandhi College, Kesavadasapuram, Thiruvananthapuram-695004, Kerala, India; `svuc.math@gmail.com`

[c] Department of Futures Studies, University of Kerala Thiruvananthapuram-695034, Kerala, India; `mchangat@gmail.com`

[d] Faculty of Mathematics and Physics, University of Ljubljana, Slovenia `sandi.klavzar@fmf.uni-lj.si`

[e] Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia

[f] Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

[g] Department of Mathematics, Mar Ivanios College, Thiruvananthapuram-695015, Kerala, India; `eliasjohnkalarickal@gmail.com`

Abstract

A vertex subset S of a graph G is a general position set of G if no vertex of S lies on a geodesic between two other vertices of S. The cardinality of a largest general position set of G is the general position number $\mathrm{gp}(G)$ of G. It is proved that $S \subseteq V(G)$ is a general position set if and only if the components of $G[S]$ are complete subgraphs, the vertices of which form an in-transitive, distance-constant partition of S. If $\mathrm{diam}(G) = 2$, then $\mathrm{gp}(G)$ is the maximum of the clique number of G and the maximum order of an induced complete multipartite subgraph of the complement of G. As a consequence, $\mathrm{gp}(G)$ of a cograph G can be determined in polynomial time. If G is bipartite, then $\mathrm{gp}(G) \leq \alpha(G)$ with equality if $\mathrm{diam}(G) \in \{2, 3\}$. A formula for the general position number of the complement of a bipartite graph is deduced and simplified for the complements of trees, of grids, and of hypercubes.

1

1 Introduction

Motivated by the century old Dudeney's no-three-in-line problem [4] (see [9, 12, 15] for recent developments on it) and by the general position subset selection problem [5, 14] from discrete geometry, the natural related problem was introduced to graph theory in [10] as follows. Let $G = (V(G), E(G))$ be a graph. Then we wish to find a largest set of vertices $S \subseteq V(G)$, called a *gp-set* of G, such that no vertex of S lies on a geodesic (in G) between two other vertices of S. The *general position number* (gp-*number* for short), $\mathrm{gp}(G)$, of G is the cardinality of a gp-set of G.

As it happens, the same concept has already been studied two years earlier in [18] under the name *geodetic irredundant sets*. The concept was formally defined in a different, more technical language, see the preliminaries below. In [18] graphs G with $\mathrm{gp}(G) \in \{2, n(G) - 1, n(G)\}$ were characterized and several additional results about the general position number deduced. The term general position problem was coined in [10], where different general upper and lower bounds on the gp-number are proved. In the same paper it is demonstrated that the in a block graph the set of its set of simplicial vertices forms a gp-set and that the problem is NP-complete in the class of all graphs. In the subsequent paper [11] the gp-number is determined for a large class of subgraphs of the infinite grid graph, for the infinite diagonal grid, and for Beneš networks.

In this paper we continue the investigation of general position sets in graphs. In the following section definitions and preliminary observations are listed. In Section 3 we prove a characterization of general position sets and demonstrate that some earlier results follow directly from the characterization. In the subsequent section we consider graphs of diameter 2. We prove that if G is such a graph, then $\mathrm{gp}(G)$ is the maximum of the clique number of G and the maximum order of an induced complete multipartite subgraph of the complement of G. In the case of cographs the latter invariant can be replaced by the independence number which in turn implies that $\mathrm{gp}(G)$ of a cograph G can be determined in polynomial time. Moreover, we determine a formula for $\mathrm{gp}(G)$ for graphs with at least one universal vertex. In Section 5 we consider bipartite graphs and their complements. If G is bipartite, then $\mathrm{gp}(G) \leq \alpha(G)$ with equality if $\mathrm{diam}(G) \in \{2, 3\}$. We prove a formula for the general position number of the complement of a bipartite graph and simplify it for the complements of trees, of grids, and of hypercubes. In particular, if T is a tree, then $\mathrm{gp}(\overline{T}) = \max\{\alpha(T), \Delta(T) + 1\}$.

2 Preliminaries

Graphs in this paper are finite, undirected, and simple. Let G be a connected graph and $u, v \in V(G)$. The *distance* $d_G(u, v)$ between u and v is the minimum number of edges on a u, v-path. The maximum distance between all pairs of vertices of G is the *diameter* $\mathrm{diam}(G)$ of G. An u, v-path of length $d_G(u, v)$ is called an u, v-*geodesic*. The *interval* $I_G[u, v]$ between vertices u and v of a graph G is the set of vertices x such that there exists a u, v-geodesic which contains x. For $S \subseteq V(G)$ we set $I_G[S] = \bigcup_{u,v \in S} I_G[u, v]$. To simplify the writing, we may omit the index G in the above notation provided that G is clear from the context.

A set of vertices $S \subseteq V(G)$ is a *general position set* of G if no three vertices of S lie on a common shortest path. A gp-set is thus a largest general position set. Call a vertex $v \in T \subseteq V(G)$ to be an *interior vertex* of T, if $v \in I[T - \{v\}]$. Now, T is a general position set if and only if T contains no interior vertices. In this way general position sets were introduced in [18] under the name geodetic irredundant sets.

The set S is *convex* in G if $I[S] = S$. The *convex hull* $H(S)$ of S is the smallest convex set that contains S, and S is a *hull set* of G if $H(S) = V(G)$. A smallest hull set is a *minimum hull set* of G, its cardinality is the *hull number* $h(G)$ of G. A hull set S in a graph G is a *minimal hull set* if no proper subset of S is a hull set of G. The maximum cardinality of a minimal hull set of G is the *upper hull number* $h^+(G)$ of G. It is clear that in any graph G, every minimum hull set is a minimal hull set of G and hence $h(G) \leq h^+(G)$. The following fact is obvious.

Observation 2.1 *Let G be a connected graph, S a minimal hull set, $u, v \in S$, and P a u, v-geodesic. If $w \in P$, where $w \neq u, v$, then $w \notin S$.*

It follows from Observation 2.1 that every minimal hull set of G is its general position set. Consequently,

$$2 \leq h(G) \leq h^+(G) \leq \mathrm{gp}(G) \leq n(G),$$

where $n(G) = |V(G)|$. The upper hull number of a graph can be arbitrary smaller than its gp-number. For instance, if $n \geq 2$, then $h^+(K_{n,n}) = 2$ and $\mathrm{gp}(K_{n,n}) = n$.

With respect to convexities we mention the following parallel concept to the general position number, where in the definition of the interior vertex we replace "I" with "H". More precisely, the *rank* of a graph G is the cardinality of a largest set S such that $v \notin H(S - \{v\})$ for every $v \in S$, see [7]. Actually, the graph rank can be studied for any convexity, cf. [19], the one defined here is the rank w.r.t. the geodesic convexity.

A vertex x of a graph G is *simplicial* if its open neighborhood $N(x)$ induces a complete subgraph. Note that this is equivalent to saying that $N(x)$ is convex in G.

If $S \subseteq V(G)$, then the subgraph of G induced by S is denoted by $G[S]$. The size of a largest complete subgraph of a graph G and the size of its largest independent set are denoted by $\omega(G)$ and $\alpha(G)$, respectively. A vertex of degree $n(G) - 1$ is a *universal vertex* of G. The complement of a graph G will be denoted with $\overline{G}$. The *join* $G + H$ of disjoint graphs G and H is obtained from the disjoint union of G and H by adding all edges gh, where $g \in V(G)$ and $h \in V(H)$. Finally, set $[n] = \{1, \ldots, n\}$ for $n \in \mathbb{N}$.

3　The characterization

In this section we characterize general position sets in graphs. For this sake the following concepts are needed.

Let G be a connected graph, $S \subseteq V(G)$, and $\mathcal{P} = \{S_1, \ldots, S_p\}$ a partition of S. Then $\mathcal{P}$ is *distance-constant* if for any $i, j \in [p]$, $i \neq j$, the distance $d(u, v)$, where $u \in S_i$ and $v \in S_j$ is independent of the selection of u and v. (We note that in [7, p. 331] the distance-constant partition is called "distance-regular", but we decided to rather avoid this naming because distance-regular graphs form a well-established term, cf. [2].) If $\mathcal{P}$ is a distance-constant partition, and $i, j \in [p]$, $i \neq j$, then let $d(S_i, S_j)$ be the distance between a vertex from S_i and a vertex from S_j. Finally, we say that a distance-constant partition $\mathcal{P}$ is *in-transitive* if $d(S_i, S_k) \neq d(S_i, S_j) + d(S_j, S_k)$ holds for arbitrary pairwise different $i, j, k \in [p]$. With these concepts in hand we can characterize general position sets as follows.

Theorem 3.1 *Let G be a connected graph. Then $S \subseteq V(G)$ is a general position set if and only if the components of $G[S]$ are complete subgraphs, the vertices of which form an in-transitive, distance-constant partition of S.*

Proof. Let G be a connected graph and let $S \subseteq V(G)$. Let $G_1, \ldots, G_p$ be the components of $G[S]$ and let $\mathcal{P}$ be the partition of S induced by the vertex sets of the components, that is, $\mathcal{P} = \{V(G_1), \ldots, V(G_p)\}$. To simplify the notation let $V_i = V(G_i)$ for $i \in [p]$, so that $\mathcal{P} = \{V_1, \ldots, V_p\}$.

Suppose first that $G_1, \ldots, G_p$ are complete subgraphs of G and that $\mathcal{P}$ forms an in-transitive, distance-constant partition of S. We claim that S is a general position set and assume by the way of contradiction that S contains three vertices u, v, w, such that v lies on a u, w-geodesic. Since $G_1, \ldots, G_p$ are complete subgraphs, u and w lie in different parts of $\mathcal{P}$, say $u \in V_i$ and $w \in V_j$, where $i, j \in [p]$, $i \neq j$. Since $\mathcal{P}$ is distance-constant, we infer that $v \notin V_i$ as well as $v \notin V_j$. Therefore, $v \in V_k$ for some $k \in [p]$, $k \neq i, j$. But then $d(V_i, V_j) = d(V_i, V_k) + d(V_k, V_j)$, a contradiction with the assumption that $\mathcal{P}$ is an in-transitive partition. This proves the asserted claim for S.

Conversely, let S be a general position set. If G_i is not complete for some $i \in [p]$, then G_i contains an induced P_3, say uvw. But this means that S is not a general position set. Hence G_i is a complete subgraph of G for every $i \in [p]$. Next, let $u, v \in V_i$ and $w \in V_j$ for $i, j \in [p]$, $i \neq j$. Since G_i and G_j are complete, $uv \in E(G)$ and hence $|d(u, w) - d(v, w)| \leq 1$. Moreover, neither v can be on a u, w-geodesic, nor u lies on a v, w-geodesic and consequently $d(u, w) = d(v, w)$. Since u, v are arbitrary vertices of G_i and w and arbitrary vertex of G_j, this means that $d(V_i, V_j) = d(u, w) = d(v, w)$ is well defined. Consequently, $\mathcal{P}$ is a distance-constant partition. Finally, $\mathcal{P}$ must also be an in-transitive partition, for otherwise we get an obvious contradiction with the assumption that S is a general position set. $\square$

Theorem 3.1 in particular implies some earlier results. First, it immediately implies [10, Lemma 3.5] asserting that the set of simplicial vertices of a graph is a general position set. Also, setting $d(e, f) = \min\{d(u, x), d(u, y), d(v, x), d(v, y)\}$ for edges $e = uv$ and $f = xy$ of a graph G, the following result was proved in [10, Proposition 4.4].

Corollary 3.2 *Let G be a graph with* $\mathrm{diam}(G) \geq 2$. *If $F \subseteq E(G)$ is such that $d(e, e') = \mathrm{diam}(G)$, $e, e' \in F$, $e \neq e'$, then* $\mathrm{gp}(G) \geq 2|F|$.

Proof. For $e \in F$ let x_e and y_e be the end-vertices of e. Then, having in mind that $\mathrm{diam}(G) \geq 2$, it is straightforward to see that $\{\{x_e, y_e\} : e \in F\}$ form an in-transitive, distance-constant partition. $\square$

4   Graphs of diameter 2

In this section we are going to use Theorem 3.1 in the case of graphs of diameter 2. For this sake we denote with $\eta(G)$ the maximum order of an induced complete multipartite subgraph of $\overline{G}$. Note that the complete graph K_n is a complete multipartite graph with $\eta(K_n) = 1$ and $\omega(K_n) = n$.

Theorem 4.1 *If* $\mathrm{diam}(G) = 2$, *then* $\mathrm{gp}(G) = \max\{\omega(G), \eta(G)\}$.

Proof. Since the vertices of an arbitrary complete subgraph of a graph G form a general position set of G, we have $\mathrm{gp}(G) \geq \omega(G)$. Suppose H is a complete multipartite subgraph of $\overline{G}$. Then in G the vertices of H induce a disjoint union of complete graphs. Since $\mathrm{diam}(G) = 2$, the vertices of these complete subgraphs clearly form an in-transitive, distance-constant partition. Hence by Theorem 3.1, $\mathrm{gp}(G) \geq \eta(G)$. Therefore, $\mathrm{gp}(G) \geq \max\{\omega(G), \eta(G)\}$.

Let now S be a general position set of G. Then by Theorem 3.1 the components of $G[S]$ are complete subgraphs, the vertices of which form an in-transitive, distance-constant partition of S. If there is only one such component, then $|S| \leq \omega(G)$, and if there are at least two components, then $|S| \leq \eta(G)$. Hence, $\mathrm{gp}(G) \leq \max\{\omega(G), \eta(G)\}$. $\qquad\square$

If P is the Petersen graph, then $\omega(P) = 2$ and $\eta(P) = 6$, hence by Theorem 4.1 we have $\mathrm{gp}(P) = 6 = \eta(P)$. Let further $G_{n,k}$, be the graph obtained from K_n and one more vertex that is adjacent to $k+1$ vertices of K_n, where $n \geq 3$ and $1 \leq k+1 < n$. Then $\omega(G) = n$ and $\eta(G) = n - k$, so that $\mathrm{gp}(G_{n,k}) = n = \omega(G)$. These examples show that the values from the maximum in Theorem 4.1 are independent.

Cographs form an important class of graphs that is still extensively investigated, [1, 17] is a selected couple of recent studies. Recall that G is a *cograph* if G contains no induced path on four vertices. These graphs were independently introduced several times and can be characterized in many different ways, see [3]. In particular, cographs are precisely the graphs that can be obtained from K_1 by means of the disjoint union and join of graphs. Note that this implies that every connected cograph of order at least 2 is the join of at least two smaller connected cographs. This implies that if G is a connected cograph, then $\mathrm{diam}(G) \leq 2$.

Theorem 4.2 *If G be a connected cograph, then $\mathrm{gp}(G) = \max\{\omega(G), \alpha(G)\}$.*

Proof. If $G = K_n$, then $\mathrm{gp}(K_n) = n = \max\{\omega(K_n), \alpha(K_n)\}$. Hence assume in the rest that G is a connected cograph with $\mathrm{diam}(G) = 2$.

We are going to show that $\eta(G) = \alpha(G)$ and proceed by induction on $n(G)$. The assertion is clear if $n(G) = 3$, that is, for $G = P_3$. Assume now that G is a connected cograph with $\mathrm{diam}(G) = 2$ and $n(G) \geq 4$. Then $G = G_1 + \cdots + G_k$, where $k \geq 2$ and G_i, $i \in [k]$, are connected cographs. Since for arbitrary graphs X and Y we have $\alpha(X + Y) = \max\{\alpha(X), \alpha(Y)\}$ and $\eta(X + Y) = \max\{\eta(X), \eta(Y)\}$, we get, by the induction assumption, that

$$\eta(G) = \max\{\eta(G_1), \ldots, \eta(G_k)\} = \max\{\alpha(G_1), \ldots, \alpha(G_k)\} = \alpha(G).$$

The result now follows from Theorem 4.1. $\qquad\square$

If G is a cograph, then $\alpha(G)$ and $\omega(G)$ can be determined in polynomial time, cf. [3, 13]. Hence Theorem 4.2 implies that the general position problem is polynomial on connected cographs. Since the general position function of a graph is clearly additive on its components, the general position problem is thus polynomial on all cographs.

Suppose that G is a non-complete graph that contains at least one universal vertex. Then $\mathrm{diam}(G) = 2$ and Theorem 4.1 applies. For this situation the theorem can be reformulated as follows.

Corollary 4.3 *Let G be a non-complete graph, $U \neq \emptyset$ the set of its universal vertices, and let $U' = V(G) - U$. Then*

$$\mathrm{gp}(G) = \max\{|U| + \omega(G[U']), \eta(G[U'])\}.$$

Proof. Since U contains universal vertices, every largest complete subgraph of G contains U. Hence $\omega(G) = |U| + \omega(G[U'])$. In $\overline{G}$ every vertex from U is isolated. Hence every induced complete multipartite subgraph of $\overline{G}$ with at least two parts contains only vertices from U'. It follows that $\eta(G) = \eta(G[U'])$. $\square$

Every graph G can be represented as the graph obtained from $K_{n(G)}$ by removing appropriate edges. To present examples how Corollary 4.3 can be applied, let us use the notation $K_n - E(H)$ to denote the graph obtained by considering H as a subgraph of K_n and then deleting the edges of H from K_n. Then we have the following formulas that can be easily deduced from Corollary 4.3.

- $\mathrm{gp}(K_n - E(K_k)) = \max\{k, n - k + 1\}$, where $2 \leq k < n$.

- $\mathrm{gp}(K_n - E(K_{1,k})) = \max\{k + 1, n - 1\}$, where $2 \leq k < n$.

- $\mathrm{gp}(K_n - E(P_k)) = \max\{3, n - k + \lceil \frac{n}{2} \rceil\}$, where $3 \leq k < n$.

- $\mathrm{gp}(K_n - E(K_{r,s})) = \max\{r + s, n - r\}$, where $2 \leq r \leq s$ and $r + s < n$.

- $\mathrm{gp}(K_n - E(W_k)) = \max\{3, n - k + \lfloor \frac{k-1}{2} \rfloor\}$, where $5 \leq k < n$.

- $\mathrm{gp}(K_n - E(C_k)) = \begin{cases} \max\{3, n - k + \lfloor \frac{k}{2} \rfloor\}, & 5 < k < n; \\ \max\{4, n - 2\}, & k = 4. \end{cases}$

5   Bipartite graphs and their complements

For bipartite graphs we have the following result.

Theorem 5.1 *If G is bipartite, then $\mathrm{gp}(G) \leq \alpha(G)$. Moreover, if $\mathrm{diam}(G) \in \{2, 3\}$, then $\mathrm{gp}(G) = \alpha(G)$.*

Proof. As observed in [18], if $G = P_n$ or $G = C_4$, then $\mathrm{gp}(G) = 2$. Thus $\mathrm{gp}(G) \leq \alpha(G)$ clearly holds in these cases. In the rest we may thus assume that G is neither a path nor C_4, so that $\mathrm{gp}(G) \geq 3$.

Let S be a gp-set of G and let $S_1, \ldots, S_k$ be the components of $G[S]$. As $\mathrm{gp}(G) \geq 3$ and G is bipartite, Theorem 3.1 implies that $k \geq 2$. Also, since G is bipartite, $|S_i| \in [2]$ for $i \in [k]$. We claim that actually $|S_i| = 1$ for every $i \in [k]$. Suppose on the contrary that, without loss of generality, $S_1 = \{u, v\}$. Let $w \in S_2$. Then,

since $uv \in E(G)$ and G is bipartite, $|d(u, w) - d(v, w)| = 1$, which in turn implies that either v lies on a u, w-geodesic or u lies on a v, w-geodesic. This contradiction proves the claim, that is, S is an independent set. We conclude that $\mathrm{gp}(G) \leq \alpha(G)$.

Assume now that $\mathrm{diam}(G) = 2$. The complement of an independent set I of a graph G induces the complete graph $K_{|I|}$ which is an instance of a complete multipartite graph. Hence $\alpha(G) \leq \eta(G)$ and Theorem 4.1 implies that $\alpha(G) \leq \mathrm{gp}(G)$ holds for a graph G of diameter 2.

Assume finally that $\mathrm{diam}(G) = 3$. Then we recall from [10, Corollary 4.3] that every independent set is a general position set. Hence $\alpha(G) \leq \mathrm{gp}(G)$ holds also in this case. $\qquad\square$

If G is bipartite, $\mathrm{gp}(G)$ can be arbitrary smaller than $\alpha(G)$. Consider first the paths P_n, $n \geq 2$, for which we have $\mathrm{gp}(P_n) = 2$ and $\alpha(P_n) = \lceil n/2 \rceil$. We also note that none of the two assertions of Theorem 5.1 need not hold if G is not bipartite. To see this, consider again the Petersen graph P. (Of course, $\mathrm{diam}(P) = 2$.) As already noticed, $\mathrm{gp}(P) = 6$, while $\alpha(P) = 4$. For a corresponding example of diameter 3 just add a pendant vertex to P.

We now turn our attention to complements of bipartite graphs for which some preparation is needed. If $G = (V(G), E(G))$ is a bipartite graph and $V(G) = A \cup B$ is its bipartition, then we will write G as triple $(A, B, E(G))$. If $G = (A, B, E(G))$ is a bipartite graph, then let M_G be the set of vertices of G that are of largest possible degree, more precisely,

$$M_G = \{u \in A : \deg(u) = |B|\} \cup \{u \in B : \deg(u) = |A|\}.$$

Let in addition $\psi(G)$ be the maximum order of an induced complete bipartite subgraph of G. Note that if G is a bipartite graph which is not a complete bipartite graph, then $\mathrm{diam}(\overline{G}) \leq 3$. Now we can formulate the following result.

Theorem 5.2 *If $G = (A, B, E(G))$ is a bipartite graph, then*

$$\mathrm{gp}(\overline{G}) = \begin{cases} n(G), & \mathrm{diam}(\overline{G}) \in \{1, \infty\}; \\ \max\{\alpha(G), \psi(G)\}, & \mathrm{diam}(\overline{G}) = 2; \\ \max\{\alpha(G), \psi(G \setminus (M_G \cap A)), \psi(G \setminus (M_G \cap B)), |M_G|\}, & \mathrm{diam}(\overline{G}) = 3. \end{cases}$$

Proof. Let $G = (A, B, E(G))$ be a bipartite graph. Then $\overline{G}$ is disconnected if and only if G is a complete bipartite graph. In this case we have $\mathrm{diam}(\overline{G}) = \infty$ and $\overline{G}$ is a disjoint union of $K_{|A|}$ and $K_{|B|}$. Therefore, $\mathrm{gp}(\overline{G}) = |A| + |B| = n(G)$. Further, $\mathrm{diam}(\overline{G}) = 1$ if and only if G is edge-less, hence again $\mathrm{gp}(\overline{G}) = n(G)$. If $\mathrm{diam}(\overline{G}) = 2$, then by Theorem 4.1 we have $\mathrm{gp}(\overline{G}) = \max\{\omega(\overline{G}), \eta(\overline{G})\}$. Since $\omega(\overline{G}) = \alpha(G)$ and $\eta(\overline{G}) = \psi(G)$, the assertion for the diameter 2 follows.

8

In the rest we may thus assume $|A| \geq 2$, $|B| \geq 2$, and $\text{diam}(\overline{G}) = 3$. Note that if $u \in M_G \cap A$, then u has no neighbor in B and if $u \in M_G \cap B$, then u has no neighbor in A. Consequently, in $\overline{G}$ two vertices are at distance 3 if and only if one lies in $M_G \cap A$ and the other in $M_G \cap B$. Since $\text{diam}(\overline{G}) = 3$ it follows that $M_G \cap A \neq \emptyset$ and $M_G \cap B \neq \emptyset$.

Consider a general position set T of $\overline{G}$ and set $T_A = T \cap A$, $T_B = T \cap B$. If T has at least one vertex in $M_G \cap A$, say x, and at least one vertex in $M_G \cap B$, say y, then every vertex from $(A \cup B) \setminus M_G$ lies on a x, y-geodesic. Therefore, $T \subseteq M_G$. This means that $|T| \leq |M_G|$. Suppose next that $T \cap (M_G \cap A) = \emptyset$. If there is an edge between a vertex from T_A and a vertex from T_B, then T must induce a clique and hence $|T| \leq \omega(\overline{G}) = \alpha(G)$. Otherwise, in view of Theorem 3.1, the vertices from T_A and from T_B are pairwise at distance 2. But then T induces a complete bipartite graph in $G \setminus (M_G \cap A)$ and therefore $|T| \leq \psi(G \setminus (M_G \cap A))$. Analogously, if $T \cap (M_G \cap B) = \emptyset$ then we get that $|T| \leq \alpha(G)$ or $|T| \leq \psi(G \setminus (M_G \cap B))$. In summary,

$$\text{gp}(\overline{G}) \leq \max\{\alpha(G), \psi(G \setminus (M_G \cap A)), \psi(G \setminus (M_G \cap B)), |M_G|\}.$$

On the other hand, we clearly have $\text{gp}(\overline{G}) \geq \omega(\overline{G}) = \alpha(G)$. Note next that each vertex from M_G is simplicial in $\overline{G}$ and consequently $\text{gp}(\overline{G}) \geq |M_G|$. Finally, an induced complete bipartite graph in $G \setminus (M_G \cap A)$ as well as in $G \setminus (M_G \cap A)$ corresponds to a disjoint union of cliques in $\overline{G}$ which form an in-transitive, distance constant partition (with constant 2). Hence we also have $\text{gp}(\overline{G}) \geq \psi(G \setminus (M_G \cap A))$ and $\text{gp}(\overline{G}) \geq \psi(G \setminus (M_G \cap B))$. $\qquad\square$

If $n \geq 5$, then $\text{diam}(\overline{P_n}) = 2$ and for $n \geq 7$ we have $\psi(P_n) = 3 < \lceil n/2 \rceil = \alpha(P_n)$. Let next G_n be a bipartite graph with the bipartition $A = \{x_1, \ldots, x_n, a_1, a_2\}$ and $B = \{y_1, \ldots, y_n, b_1, b_2\}$, where vertices $(A \cup B) \setminus \{a_1, a_2, b_1, b_2\}$ induce a complete bipartite graph $K_{n,n}$, and the remaining edges of G_n are a_1y_1, a_2y_2, b_1x_1, and b_2x_2. For $n \geq 3$ we have $\psi(G_n) = 2n > n + 2 = \alpha(G_n)$. As $\text{diam}(\overline{G_n}) = 2$, these two examples demonstrate that the values in Theorem 5.2 are independent in the case $\text{diam}(\overline{G}) = 2$.

Let $H(n, m, s, t)$, $n, m, s, t \geq 2$, be a bipartite graph with the bipartition $A = A_1 \cup A_2 \cup A_3$ and $B = B_1 \cup B_2 \cup B_3$, where $|A_1| = n$, $|B_1| = m$, $|A_2| = |B_3| = s$, and $|A_3| = |B_2| = t$. The vertices $(A_1 \cup A_2) \cup (B_1 \cup B_2)$ induce $K_{n+s,m+t}$, the vertices $A_2 \cup B_3$ induce $K_{s,s}$ and the vertices $A_3 \cup B_2$ induce $K_{t,t}$. These are all the edges of $H(n, m, s, t)$. Assume that $n \leq m$ and set $H = H(n, m, s, t)$. Then $M_H \cap A = A_2$ and $M_H \cap B = B_2$ and we have:

- $|M_H| = s + t$,

- $\alpha(H) = m + s + t$,

- $\psi(H \setminus A_2) = \max\{m + n + t, 2t\}$, and

- $\psi(H \setminus B_2) = \max\{m + n + s, 2s\}$.

It is now clear that the parameters n, s, and t can be selected such that exactly one of $\alpha(H)$, $\psi(H \setminus A_2)$, and $\psi(H \setminus B_2)$ is strictly larger than the other two (as well as bigger than $|M_H|$). Note finally that $\operatorname{diam}(\overline{H}) = 3$.

To see that $|M_G|$ can be strictly larger than the other three terms from Theorem 5.2 when $\operatorname{diam}(\overline{G}) = 3$, consider the edge deleted complete bipartite graph $K = K_{n,n} - e$. Then $\operatorname{diam}(\overline{K}) = 3$, and $|M_K| = 2n - 2$.

In the rest of the section we determine the general position number for some natural families of bipartite complements.

In [18, Theorem 2.5] and in [10, Corollary 3.7] it was independently observed that the gp-number of a tree T is the number of its leaves. (Actually, the set of leaves is the unique gp-set of T.) For the complements of trees we have:

Corollary 5.3 *It T is a tree, then* $\operatorname{gp}(\overline{T}) = \max\{\alpha(T), \Delta(T) + 1\}$.

Proof. Let $T = (A, B, E(T))$.

If $\operatorname{diam}(T) \leq 2$, then it is clear that T is a star. Hence $\operatorname{diam}(\overline{T}) = \infty$. By Theorem 5.2 we thus have $\operatorname{gp}(\overline{T}) = n(T) = \Delta(T) + 1$.

If $\operatorname{diam}(T) = 3$, then T it is straightforward to see that T is isomorphic to a double star. Therefore, $|M_T \cap A| = 1$ and $|M_T \cap B| = 1$. Thus $|M_T| = 2$, $\alpha(T) = |n(T)| - 2$, and $\psi(T \setminus (M_T \cap A)) = |A|$; $\psi(T \setminus (M_T \cap B)) = |B|$. Since $|n(T)| \geq 4$ and $|A| \geq 2$ and $|B| \geq 2$, we have $\operatorname{gp}(\overline{T}) = |n(T)| - 2 = \alpha(T)$.

Let finally $\operatorname{diam}(T) \geq 4$. Then from [16, Lemma 2.2] we deduce that $\operatorname{diam}(\overline{T}) = 2$. Since T has no cycles, we have that $\psi(T)$ is the order of a maximum induced star, that is, $\psi(T) = \Delta(T) + 1$. Thus $\operatorname{gp}(T) = \max\{\alpha(T), \Delta(T) + 1\}$. $\qquad\square$

Recall that the *Cartesian product* $G \square H$ of graphs G and H is the graph with $V(G \square H) = V(G) \times V(H)$, vertices (g, h) and (g', h') being adjacent if either $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$, cf. [6]. In [11, Theorem 3.1] it was proved that $\operatorname{gp}(P_\infty \square P_\infty) = 4$, where P_∞ is the two-way infinite path. If follows from this result that if $n, m \geq 3$, then $\operatorname{gp}(P_n \square P_m) = 4$ as well. For the complements of these grids we have:

Corollary 5.4 *If $n, m \geq 2$, then*

$$gp(\overline{P_n \square P_m}) = \begin{cases} 4, & n = m = 2; \\ \left\lceil \frac{n}{2} \right\rceil \left\lceil \frac{m}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor, & \text{otherwise} . \end{cases}$$

Proof. $\overline{P_2 \square P_2} = \overline{C_4}$, hence the assertion for $n = m = 2$. $\overline{P_2 \square P_3}$ is the graph obtained by adding edges $v_1 v_3$ and $v_4 v_6$ to the 6-cycle $v_1 v_2 \ldots v_6$. The assertion then follows immediately.

Suppose in the rest that $n, m \geq 3$ and set $G = P_n \square P_m$. Applying [16, Lemma 2.2] once more we get that $\operatorname{diam}(\overline{G}) = 2$. Hence by Theorem 5.2 we see that $\operatorname{gp}(G) = \max\{\alpha(G), \psi(G)\}$. Since the only induced complete bipartite subgraphs in $P_n \square P_m$ are isomorphic to $K_{2,2}$ or $K_{1,r}$, $r \in [4]$, we get $\operatorname{gp}(G) = \alpha(G)$. The assertion of the theorem now follows from the fact that $\alpha(P_n \square P_m) = \lceil \frac{n}{2} \rceil \lceil \frac{m}{2} \rceil + \lfloor \frac{n}{2} \rfloor \lfloor \frac{m}{2} \rfloor$, a result that can be deduced from [8, Theorem 4.2]. $\square$

Using parallel arguments as in the proof of Corollary 5.4 we also get the general position number of the complements of hypercubes. (The k-cube Q_k has the vertex set $\{0,1\}^k$, two vertices being adjacent if they differ in precisely one coordinate.)

Corollary 5.5 *If $k \geq 3$, then $\operatorname{gp}(\overline{Q_k}) = 2^{k-1}$.*

Acknowledgements

References

[1] L. E. Allem, F. Tura, Multiplicity of eigenvalues of cographs, Discrete Appl. Math. 247 (2018) 43–52.

[2] A. E. Brouwer, A. M. Cohen, A. Neumaier, Distance-Regular Graphs, Springer-Verlag, Berlin, 1989.

[3] D. G. Corneil, H. Lerchs, L. Stewart Burlingham, Complement reducible graphs, Discrete Appl. Math. 3 (1981) 163–174.

[4] H. E. Dudeney, Amusements in Mathematics, Nelson, Edinburgh, 1917.

[5] V. Froese, I. Kanj, A. Nichterlein, R. Niedermeier, Finding points in general position, Internat. J. Comput. Geom. Appl. 27 (2017) 277–296.

[6] R. Hammack, W. Imrich, S. Klavžar, Handbook of Product Graphs. Second Edition, CRC Press, Boca Raton, FL, 2011.

[7] M. M. Kanté, R. M. Sampaio, V. F. dos Santos, J. L. Szwarcfiter, On the geodetic rank of a graph, J. Comb. 8 (2017) 323–340.

[8] S. Klavžar, Some new bounds and exact results on the independence number of Cartesian product graphs, Ars Combin. 74 (2005) 173–186.

[9] C. Y. Ku, K. B. Wong, On no-three-in-line problem on m-dimensional torus, Graphs Combin. 34 (2018) 355–364.

[10] P. Manuel, S. Klavžar, A general position problem in graph theory, Bull. Aust. Math. Soc. 98 (2018) 177–187.

[11] P. Manuel, S. Klavžar, The graph theory general position problem on some interconnection networks Fund. Inform. 163 (2018) 339–350.

[12] A. Misiak, Z. Stępień, A. Szymaszkiewicz, L. Szymaszkiewicz, M. Zwierzchowski, A note on the no-three-in-line problem on a torus, Discrete Math. 339 (2016) 217–221.

[13] Y. Orlovich, J. Blazewicz, A. Dolgui, G. Finke, V. Gordon, On the complexity of the independent set problem in triangle graphs, Discrete Math. 311 (2011) 1670–1680.

[14] M. Payne, D. R. Wood, On the general position subset selection problem, SIAM J. Discrete Math. 27 (2013) 1727–1733.

[15] A. Por, D. R. Wood, No-Three-in-Line-in-3D, Algorithmica 47 (2007) 481–488.

[16] J. Senbagamalar, J. Baskar Babujee, I. Gutman, On Wiener index of graph complements, Trans. Comb. 3 (2014) 11–15.

[17] S. Tsujie, The chromatic symmetric functions of trivially perfect graphs and cographs, Graphs Combin. 34 (2018) 1037–1048.

[18] S. V. Ullas Chandran, G. Jaya Parthasarathy, The geodesic irredundant sets in graphs, Int. J. Math. Combin. 4 (2016) 135–143.

[19] M. L. J. van de Vel, Theory of Convex Structures, North-Holland Publishing Co., Amsterdam, 1993.