

# Graphs $S(n, k)$ and a variant of the Tower of Hanoi problem \*

Sandi Klavžar and Uroš Milutinović  
Department of Mathematics, PF  
University of Maribor  
Koroška cesta 160  
62000 Maribor  
Slovenia

## Abstract

For any  $n \geq 1$  and any  $k \geq 1$ , a graph  $S(n, k)$  is introduced. Vertices of  $S(n, k)$  are  $n$ -tuples over  $\{1, 2, \dots, k\}$  and two  $n$ -tuples are adjacent if they are in a certain relation. These graphs are graphs of a particular variant of the Tower of Hanoi problem. Namely, the graphs  $S(n, 3)$  are isomorphic to the graphs of the Tower of Hanoi problem. It is proved that there are at most two shortest paths between any two vertices of  $S(n, k)$ . Together with a formula for the distance, this result is used to compute the distance between two vertices in  $O(n)$  time. It is also shown that for  $k \geq 3$ , the graphs  $S(n, k)$  are Hamiltonian.

## 1 Introduction

In Lipscomb [10, 11] a relation  $\sim$  is introduced on the set of infinite sequences with values from an arbitrary set. This relation is defined in order to obtain some universal topological spaces. A natural question arises whether the relation  $\sim$  restricted to the finite case yields any interesting structure. This is indeed used in Milutinović [13, 14] to obtain some more topological results on universal spaces. Direct connections with the Sierpiński gasket (triangular Sierpiński curve) are established in [12, 13, 14].

---

\*This work was supported in part by the Ministry of Science and Technology of Slovenia under the grants P1-0206-101 and P1-0214-101.

We use a slightly modified relation  $\sim$  to define a class of graphs  $S(n, k)$ . The set of vertices of  $S(n, k)$  is the Cartesian product of  $n$  sets  $\{1, 2, \dots, k\}$ , while the edges are defined according to the relation  $\sim$ . There are several classes of graphs defined on the Cartesian product of sets and/or using certain relations to define edges. Vertices of the most important graph products are Cartesian products of vertices of the factor graphs, see Feigenbaum and Schäffer [4], or Imrich and Izbicki [9] for the definitions. Among Cartesian products of graphs, Hamming graphs play a very special role, cf. Bandelt, Mulder and Wilkeit [1], or Wilkeit [15]. Note that hypercubes are binary Hamming graphs. The edges of Hamming graphs are defined with a particular relation, Hamming distance, between the pairs of the corresponding tuples. We may henceforth consider the graphs  $S(n, k)$  as being of “Hamming type”.

In our investigation of these graphs we came across the well-known Tower of Hanoi problem. Although the problem is more than 100 years old [2], only recently a correct treatment of regular states was given by Hinz [5]. In fact there were several approaches before based on the wrong assumption that the largest disk moves at most once. We will not go into details here. We only refer to the papers [5] and [6] of Hinz for the large bibliography on the topic, historical overview, correct treatment and an algorithmic aspect of the problem.

The rest of this paper is organized as follows. In the next section we define graphs  $S(n, k)$ . It is shown that the graphs  $S(n, k)$  are graphs of a variant of the Tower of Hanoi problem and that the graphs  $S(n, 3)$  are isomorphic to the graphs of the Tower of Hanoi problem. We also demonstrate that graphs  $S(n, k)$  are Hamiltonian for  $k \geq 3$ . In Section 3 the shortest path problem is studied. We first prove a formula for the distance between any pair of vertices. It is also proved that there are at most two shortest paths between any pair of vertices. These two results enable us to compute the distance between any two vertices of  $S(n, k)$  in  $O(n)$  time. Finally we explicitly construct all the shortest paths.

## 2 Graphs $S(n, k)$ and the Tower of Hanoi

All graphs considered in this paper are finite undirected graphs without loops and multiple edges. For a graph  $G$  let  $V(G)$  and  $E(G)$  denote its vertex set and edge set, respectively. As usual, the distance between vertices  $u$  and  $v$  of a graph  $G$  is the shortest path distance and will be denoted by  $d(u, v)$ .

For any  $k \geq 1$  and any  $n \geq 1$  we define a graph  $S(n, k)$  as follows. Its vertex set is

$$V(S(n, k)) = \{1, 2, \dots, k\}^n$$

and two different vertices  $I = (i_1, i_2, \dots, i_n)$  and  $J = (j_1, j_2, \dots, j_n)$  are adjacent if  $I \sim J$ , where

$$I \sim J \iff \exists h \in \{1, 2, \dots, n\}$$

such that

- i)  $\forall t, t < h \implies i_t = j_t$ ,
- ii)  $i_h \neq j_h$ ,
- iii)  $\forall t, t > h \implies i_t = j_h \ \& \ j_t = i_h$ .

We point out that  $h$  may equal  $n$ , in which case the condition iii) is formally true being empty. In the rest of the paper we will write  $i_1 i_2 \dots i_n$  instead of  $(i_1, i_2, \dots, i_n)$  for brevity.

We use the notation  $S(n, k)$  because our original motivation is related to Sierpiński.

For any  $n \geq 1$ ,  $S(n, 1)$  is isomorphic to the one vertex graph  $K_1$  and for any  $n \geq 1$ ,  $S(n, 2)$  is isomorphic to the path on  $2^n$  vertices  $P_{2^n}$ . Hence these paths play an analogous role among graphs  $S(n, k)$  as hypercubes among the Hamming graphs. Furthermore, for any  $k \geq 1$ ,  $S(1, k)$  is the complete graph on  $k$  vertices. More interesting graphs appear when  $k \geq 3$  and  $n \geq 2$ . For instance, the graph  $S(3, 4)$  is shown in Fig. 1.

The problem of the Tower of Hanoi (the problem of TH for short) is well-known, thus we will not repeat the definition here, see for instance Hinz [5, 7]. The problem with three pegs is well understood. However, if we have more than three pegs it is still an open problem to determine the minimum number of moves needed to transfer  $n$  disks from one peg to another, cf. Hinz [7].

Consider the following variant of the TH with  $n$  disks and  $k$  pegs. Regular and perfect states are the same as in the classical problem: a state is *regular* if no larger disk lies on a smaller one, and a regular state with all disks on a single peg is called *perfect*. Legal moves are defined as follows. Suppose we have a regular state in which the  $t \geq 0$  topmost disks on a peg  $i$  are the  $t$  smallest disks. Then if the  $(t + 1)$ -st smallest disk is on a peg  $j \neq i$  we are allowed to switch the  $t$  disks from the peg  $i$  with the disk on the peg  $j$  (see Fig. 2). Note that for  $t = 0$  this means that arbitrary moves of the

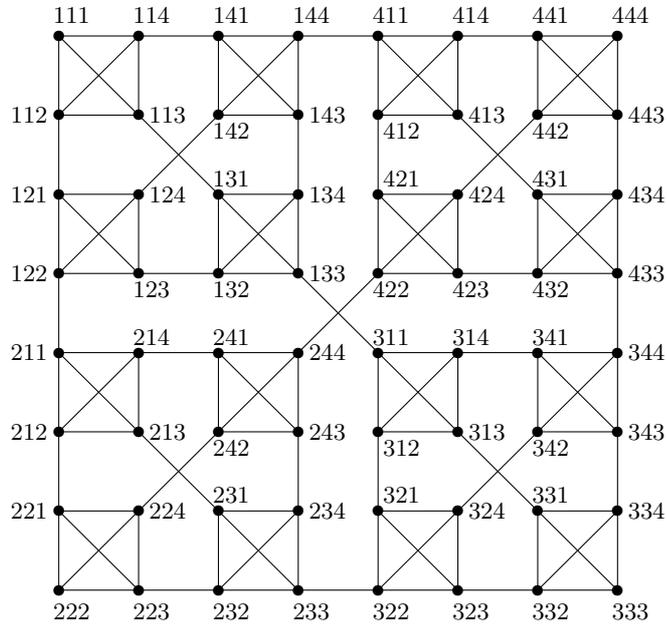


Figure 1: The graph  $S(3, 4)$

smallest disk are legal moves. Let us henceforth call this variant of the TH the *switching Tower of Hanoi* or *STH* for short.

Note that a switch preserves regular states and that the switching operation is reversible. Therefore we can define the (undirected) graph of STH as usual: its vertices are regular states and two vertices are adjacent if we can move from one state to another by a legal move. Then we have

**Theorem 1** *Let  $n \geq 1$  and  $k \geq 1$ . Then the graph of STH with  $n$  disks and  $k$  pegs is isomorphic to the graph  $S(n, k)$ .*

**Proof.** It is obvious that regular states of STH bijectively correspond to the sequences

$$i_1 i_2 \dots i_n \in \{1, 2, \dots, k\}^n,$$

according to the interpretation that  $i_j = h$  means that the  $j$ -th largest disk is on the peg  $h$ . Recalling the definition of  $\sim$  we then easily see that two

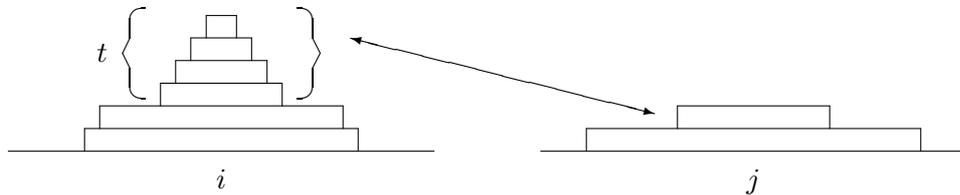


Figure 2: A legal move

vertices of the graph of STH are adjacent if and only if the corresponding sequences are in the relation  $\sim$ .  $\square$

Theorem 1 in particular implies that STH is also defined for two pegs, which is not the case with the classical problem. In addition, as  $S(n, 2)$  is a path, there is exactly one (shortest) path between any two regular states of STH with two pegs.

The interpretation of vertices as sequences used in the proof of Theorem 1 is just opposite to the one used in [3, 5, 7] for the interpretation of the TH (see the proof of Theorem 2). Since legal moves are quite different, the corresponding graphs of TH and STH would be expected to differ (even with the reinterpretation of the vertices of the TH graph by switching the order among the disks). This is in general indeed the case. However, to our surprise we get the same graphs in the case  $k = 3$ .

**Theorem 2** *For any  $n \geq 1$ , the graph  $S(n, 3)$  is isomorphic to the graph of the TH with  $n$  disks.*

**Proof.** Let  $TH_n$  be the graph of the TH with  $n$  disks and three pegs. Its vertices are sequences  $i_1 i_2 \dots i_n \in \{1, 2, 3\}^n$ , according to the interpretation that  $i_j = h$  means that the  $j$ -th smallest disk is on the peg  $h$ , cf. [5, 7].

By induction on  $n$  we construct isomorphisms  $f_n : S(n, 3) \longrightarrow TH_n$ . For  $n = 1$  both graphs are complete graphs on three vertices.

Let  $n \geq 2$  and consider a partition of  $V(S(n, 3))$  into sets  $V_1, V_2$  and  $V_3$ , where  $V_i$  consists of all vertices beginning with  $i$ ,  $i = 1, 2, 3$ . Then for any  $i$  and  $j$ ,  $i \neq j$  there is exactly one edge between  $V_i$  and  $V_j$ , i.e. the edge between the vertices  $i j j \dots j$  and  $j i i \dots i$ . We will call such an edge a *bridging edge*.

In a similar way consider a partition of  $V(TH_n)$  into sets  $W_1, W_2$  and  $W_3$ , where  $W_i$  consists of all vertices ending with  $i$ ,  $i = 1, 2, 3$ . Then for any  $i$  and  $j$ ,  $i \neq j$  there is exactly one bridging edge between  $W_i$  and  $W_j$ , i.e. the edge between the vertices  $k \cdots k k j$  and  $k \cdots k k i$ ,  $k \neq i, k \neq j$ .

Then we may isomorphically map  $V_i$  onto  $W_i$ , using  $f_{n-1}$  and an appropriate automorphism (induced by a permutation of the set  $\{1, 2, 3\}$ ) of  $TH_{n-1}$  for adjustment, in such a way that the ends of the bridging edges are mapped onto the corresponding ends of the bridging edges. Considering the three maps as one map from  $V(S(n, k))$  onto  $V(TH_n)$  yields the map  $f_n$ .  $\square$

It is interesting to observe that, given any regular state of STH, we can return to it in such a way that we visit every regular state exactly once. In other words:

**Proposition 3** *For any  $n \geq 1$  and any  $k \geq 3$  the graph  $S(n, k)$  is Hamiltonian.*

**Proof.** For  $n = 1$  the proposition is trivial since  $S(1, k)$  is a complete graph. Let  $n \geq 2$  and consider the sequence of paths  $P_1, P_2, \dots, P_k$ , where  $P_1$  is a path between the vertices  $1kk \cdots k$  and  $122 \cdots 2$ ,  $P_k$  between the vertices  $k(k-1)(k-1) \cdots (k-1)$  and  $k11 \cdots 1$ , and for  $i = 2, 3, \dots, k-1$ ,  $P_i$  is a path between the vertices  $i(i-1)(i-1) \cdots (i-1)$  and  $i(i+1)(i+1) \cdots (i+1)$ . We claim that the paths  $P_i$  can be constructed in such a way that they include all the vertices beginning with  $i$ ,  $i = 1, 2, \dots, k$ .

To prove the claim it is enough to see that for any  $i, j$  and  $g, j \neq g$ , there is a path between  $ijj \cdots j$  and  $igg \cdots g$  which goes through all vertices beginning with  $i$ . Obviously that reduces the inductive argument to the statement that  $jj \cdots j$  and  $gg \cdots g$ ,  $j \neq g$ , may be connected in  $S(n, k)$  by a path going through all vertices (for all  $n$ ). Without loss of generality assume  $j = 1$  and  $g = k$ . By the inductive hypothesis we may find a path from  $11 \cdots 1$  to  $12 \cdots 2$  through all vertices beginning with 1. Add the edge between  $12 \cdots 2$  and  $21 \cdots 1$  to the path. By the same argument we may find a path from  $21 \cdots 1$  to  $23 \cdots 3$  through all vertices beginning with 2. Continue this procedure until  $(k-1)k \cdots k$  is joined to  $k(k-1) \cdots (k-1)$  and a path from  $k(k-1) \cdots (k-1)$  to  $kk \cdots k$  through all vertices beginning with  $k$  is added at the end.

It follows that the paths  $P_1, P_2, \dots, P_k$  form a Hamiltonian cycle.  $\square$

### 3 Shortest paths in $S(n, k)$ -graphs

Define

$$\rho_{i,j} = \begin{cases} 1; & i \neq j, \\ 0; & i = j. \end{cases}$$

(The symbol has been chosen in this way since rho graphically resembles the Kronecker's delta symbol put upside down.) In addition, let

$$\mathcal{P}_{j_1, \dots, j_m}^i = \rho_{i, j_1} \rho_{i, j_2} \cdots \rho_{i, j_m},$$

where the right-hand side term is a binary number, rhos representing its digits. Also, let  $V_i$  be the set of vertices of  $S(n, k)$  consisting of all vertices beginning with  $i$ .

**Lemma 4** *Let  $I = ii \dots i$  and  $J = j_1 j_2 \dots j_n$  be vertices of  $S(n, k)$ . Then  $d(I, J) = \mathcal{P}_{j_1, \dots, j_n}^i$  and there is exactly one shortest path between  $I$  and  $J$ . In particular, for  $i \neq j$ ,  $d(ii \dots i, jj \dots j) = 2^n - 1$ .*

**Proof.** By induction on  $n$ . The statement is trivial for  $n = 1$ .

Let  $n \geq 2$ .

If  $i = j_1$  then by the inductive assumption, the shortest path inside  $V_i$  has the length  $\mathcal{P}_{j_2, \dots, j_n}^i = \mathcal{P}_{j_1, \dots, j_n}^i$ . Consider now a path  $Q$  between  $I$  and  $J$  which is not completely in  $V_i$ . Let  $g, g \neq i$ , be such that the vertex  $gi \dots i$  is the last vertex of  $Q$  not belonging to  $V_i$ . Then  $Q$  contains a subpath from  $ig \dots g$  to  $I$  in  $V_i$  which by induction has length at least  $2^{n-1} - 1 \geq \mathcal{P}_{j_2, \dots, j_n}^i$ . Therefore  $|Q| > \mathcal{P}_{j_2, \dots, j_n}^i$ .

Let  $i \neq j_1$ . Then by the inductive assumption, among all paths between  $I$  and  $J$  containing the edge between  $ij_1 \dots j_1$  and  $j_1 i \dots i$ , there is a unique shortest one. Its length is  $(2^{n-1} - 1) + 1 + \mathcal{P}_{j_2, \dots, j_n}^i = \mathcal{P}_{j_1, \dots, j_n}^i \leq 2^n - 1$ . Consider a path  $Q$  between  $I$  and  $J$  containing an edge between the vertices  $ig \dots g$  and  $gi \dots i$ , where  $g$  is chosen as above. Then  $|Q| \geq (2^{n-1} - 1) + 1 + (2^{n-1} - 1) + 1 = 2^n$ . Thus  $Q$  is not a shortest path.

Note finally that there is only one shortest path in both cases.  $\square$

**Theorem 5** *Let  $I = i_1 i_2 \dots i_n$  and  $J = j_1 j_2 \dots j_n$  be vertices of  $S(n, k)$  such that  $i_1 = j_1, \dots, i_{\ell-1} = j_{\ell-1}$  and  $i_\ell \neq j_\ell, \ell \geq 1$ . Then  $d(I, J) = 1$  for  $\ell = n$ , and otherwise,  $d(I, J)$  is equal to*

$$\min\{\mathcal{P}_{i_{\ell+1}, \dots, i_n}^{j_\ell} + 1 + \mathcal{P}_{j_{\ell+1}, \dots, j_n}^{i_\ell}, \mathcal{P}_{i_{\ell+1}, \dots, i_n}^h + 1 + 2^{n-\ell} + \mathcal{P}_{j_{\ell+1}, \dots, j_n}^h \mid h \neq i_\ell, j_\ell\}.$$

**Proof.** By induction on  $n$ . If  $n = 1$  then  $\ell = 1$  and  $d(I, J) = 1$  as claimed.

By a similar argument as in Lemma 4 we first note that it suffices to consider paths in the subgraph of  $S(n, k)$  induced by the vertices beginning with  $i_1 \dots i_{\ell-1}$ . Omitting  $i_1 \dots i_{\ell-1}$  from the vertices gives a natural isomorphism between the subgraph and  $S(n - \ell + 1, k)$ . If  $\ell > 1$  then the theorem holds by the induction hypothesis. Hence it remains to prove the statement for  $\ell = 1$ .

Let  $n \geq 2$ . For brevity let  $i_1 = i$  and  $j_1 = j$ . Consider a shortest path  $Q$  among those paths between  $I$  and  $J$  which have vertices only from  $V_i \cup V_j$ . Then, by Lemma 4,  $|Q| = \mathcal{P}_{i_2, \dots, i_n}^j + 1 + \mathcal{P}_{j_2, \dots, j_n}^i$ , because  $Q$  must contain the edge between the vertices  $ij \dots j$  and  $ji \dots i$ . Also by the lemma,  $Q$  is unique. We will call such a path the *direct* path between  $I$  and  $J$ .

Consider next a shortest path  $Q'$  among the paths between  $I$  and  $J$  with vertices only from  $V_i \cup V_j \cup V_h$ ,  $h \neq i, j$  where  $Q \cap V_h \neq \emptyset$ . Since  $Q'$  must contain the edges between  $ih \dots h$  and  $hi \dots i$ , and between  $hj \dots j$  and  $jh \dots h$ , Lemma 4 implies  $|Q'| = \mathcal{P}_{i_2, \dots, i_n}^h + 1 + (2^{n-1} - 1) + 1 + \mathcal{P}_{j_2, \dots, j_n}^h$ . Furthermore, Lemma 4 also implies uniqueness of  $Q'$  (for a fixed  $h$ ). We call such a path the  $V_h$ -*path* between  $I$  and  $J$ .

Clearly, for the direct path  $Q$  we have  $|Q| < 2^n$  and thus the distance between  $I$  and  $J$  is strictly less than  $2^n$ . But since any path containing also vertices from  $V_g$  and  $V_h$ , where  $i, j, h$  and  $g$  are pairwise different, has length at least  $2^{n-1} + 2^{n-1} + 1 = 2^n + 1$ , the theorem follows.  $\square$

From the computational point of view, Theorem 5 can be used to compute  $d(I, J)$  in  $O(nk)$  time. The next theorem will enable us to improve this complexity.

**Theorem 6** *There are at most two shortest paths between any two vertices of  $S(n, k)$ .*

**Proof.** Let  $I = i_1 i_2 \dots i_n$  and  $J = j_1 j_2 \dots j_n$  be vertices of  $S(n, k)$  and assume without loss of generality that  $i_1 \neq j_1$ . For brevity let  $i_1 = i$  and  $j_1 = j$ . Note that the proof of Theorem 5 implies that the length of the direct path between  $I$  and  $J$  is  $\mathcal{P}_{i_2, \dots, i_n}^j + 1 + \mathcal{P}_{j_2, \dots, j_n}^i$ , while the length of the  $V_h$ -path is  $\mathcal{P}_{i_2, \dots, i_n}^h + 1 + 2^{n-1} + \mathcal{P}_{j_2, \dots, j_n}^h$  for any  $h \neq i, j$ .

We distinguish several cases.

*Case 1:*  $i_2 = i, j_2 = j$ .

Any  $V_h$ -path is of length at least  $2^{n-2} + 1 + 2^{n-1} + 2^{n-2}$ , because  $\rho_{h, i_2} =$

$\rho_{h,j_2} = 1$ . Since the direct path is of length at most  $2^n - 1$ , it is the unique shortest path in this case.

*Case 2:  $i_2 = j$ .*

Any  $V_h$ -path is of length at least  $2^{n-2} + 2^{n-1} + 1$ , because  $\rho_{h,j_2} = 1$ . Since  $\rho_{j,i_2} = 0$ , the direct path is of length at most  $(2^{n-2} - 1) + 1 + (2^{n-1} - 1)$ . Hence it is again the unique shortest path between  $I$  and  $J$ .

The case  $j_2 = i$  is treated analogously.

*Case 3:  $i_2 = i, j_2 = h, h \neq i, j$ .*

Let  $g \neq i, j, h$ . Then  $\rho_{g,i_2} = \rho_{g,j_2} = 1$ . As in Case 1 it follows that the length of the  $V_g$ -path is at least  $2^n + 1$ . Thus a shortest path can only be the direct path or the  $V_h$ -path. (Consider for example vertices 113 and 233 of  $S(3, 4)$  in Figure 1 to see that both paths may be the shortest ones.)

The case  $j_2 = j$  and  $i_2 = h$  is treated analogously.

*Case 4:  $i_2 = j_2 = h, h \neq i, j$ .*

This case can be treated exactly as the previous one. (Consider for example vertices 133 and 231 of  $S(3, 4)$  in Figure 1 to see that the direct path and the  $V_h$ -path may have equal length. Also consider vertices 122 and 322 to see that the  $V_h$ -path may be shorter than the direct one.)

*Case 5:  $i_2 = g, j_2 = h, i, j, g$  and  $h$  are pairwise different.*

Let  $f \neq i, j, g, h$ . Then as in the previous two cases we get that the  $V_f$ -path cannot be a shortest path.

The length of the direct path is equal to

$$\mathcal{P}_{g,i_3,\dots,i_n}^j + 1 + \mathcal{P}_{h,j_3,\dots,j_n}^i = 2^{n-1} + \mathcal{P}_{i_3,\dots,i_n}^j + 1 + \mathcal{P}_{j_3,\dots,j_n}^i,$$

which is in turn equal to  $2^{n-1}$  plus the length of the direct path between  $ii_3 \dots i_n$  and  $jj_3 \dots j_n$  in  $S(n-1, k)$ .

The length of the  $V_h$ -path is equal to

$$\mathcal{P}_{g,i_3,\dots,i_n}^h + 1 + 2^{n-1} + \mathcal{P}_{h,j_3,\dots,j_n}^h = 2^{n-1} + \mathcal{P}_{i_3,\dots,i_n}^h + 1 + 2^{n-2} + \mathcal{P}_{j_3,\dots,j_n}^h,$$

which is equal to  $2^{n-1}$  plus the length of the  $V_h$ -path between  $ii_3 \dots i_n$  and  $jj_3 \dots j_n$  in  $S(n-1, k)$ . An analogous statement holds for the  $V_g$ -path.

This proves that shortest paths between  $igi_3 \dots i_n$  and  $jhj_3 \dots j_n$  in  $S(n, k)$  correspond to shortest paths between  $ii_3 \dots i_n$  and  $jj_3 \dots j_n$  in the graph  $S(n-1, k)$ . These paths can only be the direct path, the  $V_{i_3}$ -path, or the  $V_{j_3}$ -path.

Hence if  $\{i_3, j_3\} \neq \{g, h\}$ , at most one of the  $V_g$ -path and the  $V_h$ -path may be a required shortest path in  $S(n, k)$ .

If  $\{i_3, j_3\} = \{g, h\}$  we may use the initial argument once again. Finally, if  $\{i_t, j_t\} = \{g, h\}$  holds for  $t = 3, 4, \dots, n-1$ , shortest paths between  $ig_i \dots i_n$  and  $jh_j \dots j_n$  in  $S(n, k)$  correspond to shortest paths between  $ii_n$  and  $jj_n$  in  $S(2, k)$ . In  $S(2, k)$  the direct path between  $ii_n$  and  $jj_n$  is of length  $\rho_{i,j_n} + \rho_{j,i_n} + 1 \leq 3$ . The length of the  $V_h$ -path is  $\rho_{h,j_n} + \rho_{h,i_n} + 2^1 + 1 \geq 3$ . Clearly, if the  $V_h$ -path is a shortest path then its length must be equal 3, which is possible only if  $i_n = j_n = h$ . Analogously we see that the  $V_g$ -path may be a shortest path only if  $i_n = j_n = g$ .

We conclude that at most two of these three paths may be shortest paths.

□

The proof of Theorem 6 in particular shows that  $d(I, J)$ , where again without loss of generality  $i_1 \neq j_1$ , from Theorem 5 is obtained as minimum of

$$\mathcal{P}_{i_2, \dots, i_n}^{j_1} + 1 + \mathcal{P}_{j_2, \dots, j_n}^{i_1}$$

and

$$\min\{\mathcal{P}_{i_2, \dots, i_n}^h + 1 + 2^{n-1} + \mathcal{P}_{j_2, \dots, j_n}^h \mid h \in \{i_2, j_2\} \setminus \{i_1, j_1\}\},$$

if  $\{i_2, j_2\} \setminus \{i_1, j_1\}$  is nonempty, otherwise  $d(I, J) = \mathcal{P}_{i_2, \dots, i_n}^{j_1} + 1 + \mathcal{P}_{j_2, \dots, j_n}^{i_1}$ . This gives us the following

**Corollary 7** *The distance between any two vertices of  $S(n, k)$  can be computed in  $O(n)$  time.*

When we know the distance between two vertices we can also easily find all shortest paths, i.e. one or two of them. To see this it is enough to construct a shortest path between  $I = i_1 i_2 \dots i_n$  and  $J = i_1 j_1 \dots j_1$  of length  $\mathcal{P}_{i_2, \dots, i_n}^{j_1}$ . Indeed, using such paths together with the corresponding bridging edges gives the shortest paths between the original vertices.

Consider a sequence  $\sigma$  of vertices starting with  $I$  and ending with  $J$  where the next term is obtained from the previous one analogously to the way one represents addition of 1 in binary notation. The beginning of  $\sigma$  thus is

$$\begin{aligned} & i_1 \dots i_{n-3} i_{n-2} i_{n-1} i_n, \\ & i_1 \dots i_{n-3} i_{n-2} i_{n-1} j_1, \\ & i_1 \dots i_{n-3} i_{n-2} j_1 i_{n-1}, \\ & i_1 \dots i_{n-3} i_{n-2} j_1 j_1, \\ & i_1 \dots i_{n-3} j_1 i_{n-2} i_{n-2}, \dots \end{aligned}$$

In case when two consecutive terms of the above list are equal we of course omit the redundant one. That means that if  $i_\ell = j_\ell$  then there are  $2^{n-\ell}$  such redundant terms. Thus  $\sigma$  is indeed the shortest path between  $I$  and  $J$  of the desired length.

Note that the shortest path between  $ii \dots i$  and  $jj \dots j$ ,  $i \neq j$ , is obtained exactly by adding 1 in binary notation if we replace  $i$  by 0 and  $j$  by 1. Moreover, this describes the path  $S(n, 2)$ .

To conclude the paper we remark that in view of Theorem 2, Theorems 5 and 6 (in the case  $k = 3$ ) offer an alternative approach to the classical TH problem.

## Acknowledgement

We are grateful to Ciril Pezdir who turned our attention to the Tower of Hanoi problem and to Andreas M. Hinz for several useful remarks.

## References

- [1] *H.-J. Bandelt, H.M. Mulder, E. Wilkeit*: Quasi-median graphs and algebras, *J. Graph Theory* 18 (1994), 681–703.
- [2] *N. Claus (= E. Lucas)*: La Tour d’Hanoi, Jeu de calcul, *Science et Nature* 1(8) (1884), 127–128.
- [3] *M.C. Er*: An analysis of the generalized Towers of Hanoi problem, *BIT* 23 (1983), 295–302.
- [4] *J. Feigenbaum, A.A. Schäffer*: Finding the prime factors of strong direct product graphs in polynomial time, *Discrete Math.* 109 (1992), 77–102.
- [5] *A.M. Hinz*: The Tower of Hanoi, *Enseign. Math.* 35 (1989), 289–321.
- [6] *A.M. Hinz*: Shortest paths between regular states of the Tower of Hanoi, *Inform. Sci.* 63 (1992), 173–181.
- [7] *A.M. Hinz*: Pascal’s triangle and the Tower of Hanoi, *Amer. Math. Monthly* 99 (1992), 538–544.
- [8] *A.M. Hinz, A. Schief*: The average distance on the Sierpiński gasket, *Probab. Theory Related Fields* 87 (1990), 129–138.
- [9] *W. Imrich, H. Izbicki*: Associative products of graphs, *Monatsh. Math.* 80 (1975), 277–281.

- [10] *S. L. Lipscomb*: A universal one-dimensional metric space, In *TOPO 72 - General Topology and its Applications, Second Pittsburgh Internat. Conf.*, Volume 378 of *Lecture Notes in Math.* Springer-Verlag, New York, 1974, 248–257.
- [11] *S. L. Lipscomb*: On imbedding finite-dimensional metric spaces, *Trans. Amer. Math. Soc.* *211* (1975), 143–160.
- [12] *S. L. Lipscomb, J. C. Perry*: Lipscomb’s  $L(A)$  space fractalized in Hilbert’s  $l^2(A)$  space, *Proc. Amer. Math. Soc.* *115* (1992), 1157–1165.
- [13] *U. Milutinović*: Completeness of the Lipscomb space, *Glas. Mat. Ser. III* *27(47)* (1992), 343–364.
- [14] *U. Milutinović*: A universal separable metric space of Lipscomb’s type, *Preprint Series Univ. of Ljubljana* *32* (1994), 443.
- [15] *E. Wilkeit*: Isometric embeddings in Hamming graphs, *J. Combin. Theory Ser. B* *50* (1990), 179–197.