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Fast recognition algorithms for classes of partial cubes

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Abstract

Isometric subgraphs of hypercubes, or partial cubes as they are also called, are a rich class of graphs that include median graphs, subdivision graphs of complete graphs, and classes of graphs arising in mathematical chemistry and biology. In general, one can recognize whether a graph on n vertices and m edges is a partial cube in $O(mn)$ steps, faster recognition algorithms are only known for median graphs. This paper exhibits classes of partial cubes that are not median graphs but can be recognized in $O(m \log n)$ steps. On the way relevant decomposition theorems for partial cubes are derived, one of them correcting an error in a previous paper (Eur. J. Combin. 19 (1998) 677).

Keywords: Isometric subgraph; Hypercube; Algorithm; Complexity

1. Introduction

The structure of isometric subgraphs of hypercubes, or partial cubes, as they are also known, is well understood; see the classical papers of Djoković [9], Winkler [23], and Chepoi [5]. Interestingly, they were first introduced in computer science [10,11], where a nice characterizing property of partial cubes was used, namely, vertices of a partial cube can be labeled by words of a fixed length using two symbols in such a way

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that the shortest path distance between any two vertices coincides with the Hamming distance of their labels.

Here we are concerned with the fundamental problem of determining the recognition complexities for such classes of graphs. The fastest general recognition algorithm for partial cubes on n vertices and m edges has complexity $O(mn)$. The first such algorithm is due to Aurenhammer and Hagauer [1]. It is direct but difficult. For more transparent algorithms see [13,16]. Unfortunately only the trivial lower bound $\Omega(m)$ is known for the recognition of partial cubes.

Median graphs [16,19–21] are presumably the most important subclass of partial cubes, they include trees, hypercubes, and can also be characterized as the class of retracts of hypercubes. An $O(mn)$ recognition algorithm for median graphs is given in [18], while a simple algorithm of the same complexity can be found in [14]. In [12] the complexity was reduced to $O(m\sqrt{n})$. Currently the recognition complexity is $O((m \log n)^{1.41})$; see [16]. Because of a connection with triangle-free graphs there is strong evidence that median graphs cannot be recognized much faster; cf. [17].

In order to narrow the gap between $\Omega(m)$ and $O(mn)$ for the recognition complexity of partial cubes several classes of graphs have been introduced, notably semi-median graphs and almost-median graphs [14]. Although these graphs can be isometrically embedded into hypercubes in $O(m \log n)$ time once they have been recognized [16, Lemma 7.10], their recognition complexity is still $O(mn)$. Here, we show that prism-free almost-median graphs and many related classes can be recognized in $O(m \log n)$ time. These classes are the first examples of easily describable classes of partial cubes that are not median graphs but can be recognized faster than median graphs.

Expansion theorems and related expansion procedures are fundamental tools in designing recognition algorithms for partial cubes and their subclasses. The expansion theorem for median graphs is due to Mulder [20,21] and the expansion theorem for partial cubes to Chepoi [5]; cf. also [22]. In Section 3, we prove two expansion theorems, one for semi-median graphs and one for almost-median graphs. The first theorem corrects the corresponding expansion result from [14]. In Section 4 we then present an algorithm of complexity $O(m \log n)$ that recognizes partial cubes in which certain sets U_{ab} induce isometric trees; such graphs can also be described as prism-free, almost-median graphs. We conclude with generalizations of this result to larger classes of almost-median graphs.

2. Preliminaries

For a graph G, the *distance* $d_G(u,v)$, or briefly $d(u,v)$, between vertices u and v is defined as the number of edges on a shortest u,v-path. A subgraph H of G is called *isometric* if $d_H(u,v) = d_G(u,v)$ for all $u,v \in V(H)$. A subgraph H of G is *convex*, if for any u, $v \in V(H)$, all shortest u,v-paths belong to H.

The *Cartesian product* $G \Box H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ in which the vertex (a,x) is adjacent to the vertex (b,y) whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. The graphs $K_2 \Box C_n$, $n \geqslant 3$, are called *prisms*, and the graphs $K_2 \Box P_n$ *ladders*. The Cartesian product of k copies of K_2 is a *hypercube* or *k-cube* Q_k. Isometric subgraphs of hypercubes are called *partial cubes*. An important subclass of partial cubes are *median graphs*. These are the graphs G in which there exists a unique vertex x to every triple of vertices u, v, and w such that x lies simultaneously on a shortest u,v-path, a shortest u,w-path, and a shortest w,v-path.

For partial cubes, the following vertex sets play a crucial role. Let ab be an edge of a connected, bipartite graph $G=(V,E)$. Then

$$W_{ab}=\{w\in V \mid d_G(a,w)<d_G(b,w)\},$$

$$U_{ab}=\{w\in W_{ab} \mid w \text{ has a neighbor in } W_{ba}\},$$

$$F_{ab}=\{e\in E \mid e \text{ is an edge between } W_{ab} \text{ and } W_{ba}\}.$$

By abuse of language we shall use the same notation for the sets W_{ab}, U_{ab} and the subgraphs induced by them.

Clearly, W_{ab} and W_{ba} are disjoint. Moreover, as all graphs considered are bipartite, $V=W_{ab}\cup W_{ba}$. Djoković [9] proved that a graph G is a partial cube if and only if it is bipartite and if for any edge ab of G the subgraph W_{ab} is convex. It follows from results in [3] that median graphs are precisely the bipartite graphs in which all U_{ab}'s are convex. By this result, the following definitions make sense.

A bipartite graph is a *semi-median* graph if it is a partial cube in which for any edge ab the subgraph induced by U_{ab} is connected. Similarly, a bipartite graph is an *almost-median* graph if it is a partial cube such that for any edge ab the subgraph induced by U_{ab} is isometric.

Two edges $e=xy$ and $f=uv$ of G are in the Djoković–Winkler [9,23] relation Θ if

$$d_G(x,u)+d_G(y,v)\neq d_G(x,v)+d_G(y,u).$$

Clearly, Θ is reflexive and symmetric. Winkler [23] proved that a bipartite graph is a partial cube if and only if $\Theta=\Theta^*$, where Θ^* denotes the transitive closure of Θ. We will need the following basic property of Θ; see [13]:

Lemma 1. *Suppose P is a walk connecting the endpoints of an edge e. Then P contains an edge f with $e\Theta f$.*

Another relevant relation defined on the edge set of a graph is δ. We say an edge e is in relation δ to an edge f if e and f are opposite edges of a 4-cycle without diagonals in G or if $e=f$. Clearly δ is reflexive and symmetric. Moreover, it is contained in Θ. Thus its transitive closure δ^* is contained in Θ^*. In [14] it is shown that a bipartite graph is a semi-median graph if and only if $\Theta=\delta^*$, in analogy to Winkler's characterization of partial cubes.

Let G be a connected graph. A *proper cover* of G consists of two isometric subgraphs G_1 and G_2 with union G and nonempty intersection. In symbols, $G=G_1\cup G_2$ and $G_0=G_1\cap G_2\neq\emptyset$. G_0 is called the *intersection* of the cover. The *expansion* of G with respect to G_1, G_2 is the graph H obtained by the following procedures:

(i) Replacement of each vertex $v\in G_1\cap G_2$ by vertices v_1, v_2 and insertion of the edge v_1v_2.

(ii) Insertion of edges between v_1 and the neighbors of v in $G_1 \setminus G_2$ as well as between v_2 and the neighbors of v in $G_2 \setminus G_1$.
(iii) Insertion of the edges v_1u_1 and v_2u_2 whenever $v, u \in G_1 \cap G_2$ are adjacent in G.

(We refer to [4,5,14,16,20–22] for various types of expansions.) Suppose $e = e_1\delta e_2\delta \dots \delta e_k = f$ is a sequence of edges by virtue of which e and f are in relation δ^*. If, in addition, the endpoints of these edges induce two isometric paths, then the union of the squares that contain e_i, e_{i+1} for $i = 1, 2, \dots, k-1$ forms a ladder, that is, the Cartesian product of a path of length $k-1$ by an edge. In such a case we shall frequently use the expression that two edges e, f in relation δ^* are connected by a "ladder".

Note that there is a natural projection of H onto G. In this projection the edges introduced by (i) are all contracted into single vertices. We call this mapping a *contraction*; it is the inverse of the expansion.

Clearly the edges introduced by (i) form a Θ-class in H. We call it the new Θ-class of H. All the other edges of H are in relation Θ_H if and only if their preimages are in relation Θ_G; cf. the proof of Theorem 2.

If G_0 is connected, isometric, or convex in G, we speak of a *connected*, *isometric*, or *convex expansion*, respectively. If H can be obtained from the one-vertex graph K_1 by a sequence of expansions of a given type, then we say that H is obtainable by an *expansion procedure* of that type.

3. Decomposing semi-median and almost-median graphs

In this section we prove two expansion theorems; one for semi-median graphs and one for almost-median graphs.

In [14] it is asserted that a graph is a semi-median graph if and only if it can be obtained by a connected expansion procedure. However, Chepoi [6] showed that this condition is not sufficient. To see this, take the semi-median graph Q_3^- (that is, the 3-cube minus a vertex) and expand it with respect to its isometric 6-cycle and with respect to the subgraph induced by the vertices of Q_3^- minus a vertex of degree 2. This is a connected expansion but the obtained graph is not semi-median; cf. Fig. 1.

We can correct this result by imposing an additional condition on the expansion steps.

Theorem 2. *A graph H is semi-median if and only if it can be obtained by a connected expansion procedure with the following property*:

(P) *If two edges $e \in G_1$, $f \in G_2$ are in relation δ and if each one of them has a vertex outside $G_0 = G_1 \cap G_2$, then there exists an edge $g \in G_0$ with $e\delta^*_{G_1}g$ and $f\delta^*_{G_2}g$.*

Proof. Let H be a semi-median graph. We already know that it can be obtained by a connected expansion procedure. We have to prove that all expansions in the sequence satisfy property (P). Suppose, on the contrary, that one of the expansions in the sequence does not satisfy (P). Since every contraction of a semi-median graph is semi-median, we can assume without loss of generality that this happens in the last

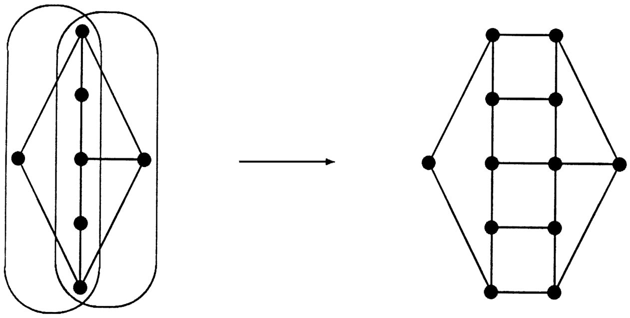

Fig. 1. Connected expansion yielding a non semi-median graph.

expansion step by which we obtain H from a graph G. Thus G is semi-median. Let G_1, G_2 be the corresponding proper cover of G. Since the proper cover does not satisfy property (P), there exist edges $e \in E(G_1)$ and $f \in E(G_2)$ with $e\delta f$, but no edge g in G_0 with $e\delta^*_{G_1} g$ and $f\delta^*_{G_2} g$. Then we still have $e\Theta_H f$, yet e and f clearly cannot be in relation δ^*_H. Hence H is not semi-median.

For the converse, let G be a semi-median graph, and G_1, G_2 be a proper cover that satisfies the conditions of the theorem. We need to prove that the graph H obtained by the expansion is semi-median, that is $\delta^*_H = \Theta_H$. Clearly, if xy and uv are edges of the new Θ-class, say F_{ab}, then $xy\delta^* uv$, because the expansion is connected. So let xy and uv be the edges of G that correspond to edges xy and uv of H. As we already noted, $xy\Theta_H uv$ holds precisely when the corresponding edges are in relation Θ_G. Indeed, if xy and uv are both edges of W_{ab} or of W_{ba} then the distances between endvertices are not changed by the expansion; otherwise, the distances between endvertices increase by one.

Suppose now that $xy\Theta_H uv$. Since G is semi-median, $xy\delta^*_G uv$ and G contains a ladder from xy to uv. In H we wish to use the same ladder to show the validity of $xy\delta^*_H uv$. This is possible unless it contains two consecutive edges $e \in G_1$ and $f \in G_2$. In this case we use property (P) to find a ladder from e to f in H. Hence we also obtain $xy\delta^*_H uv$.

On the other hand, if xy and uv are not in relation Θ, then they are not in relation δ^* in G, since G is semi-median. Thus xy and uv cannot be in relation δ^* in H.

We conclude that H is semi-median. □

Note that property (P) of the above theorem holds when G_0 is 2-isometric. (A subgraph H of a graph G is called 2-isometric if any two vertices $u, v \in H$ of distance 2 in G have distance 2 in H as well.) In particular this holds for median graphs as all G_0 are convex and thus 2-isometric. Moreover, this is also true for almost-median graphs which can be obtained by an isometric expansion procedure.

The problem with an expansion theorem for almost-median graphs is that an isometric expansion of an almost-median graph need not be almost-median. For instance, the graph G_{13} of Fig. 2 is obtained by an isometric expansion from an almost-median graph Q_3^-. (We expand the outer 6-cycle of Q_3^-.)

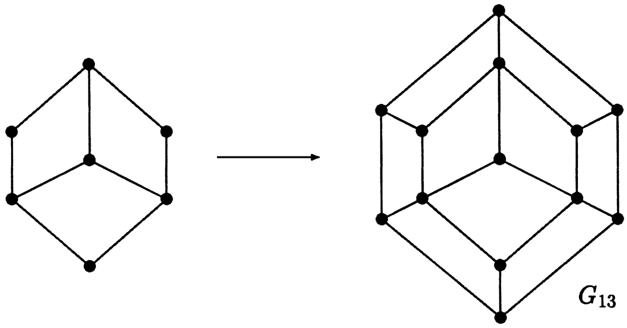

Fig. 2. An isometric expansion of Q_3^-.

To obtain an expansion theorem for this class we recall the following well-known property of median graphs: the covering sets in convex expansions of median graphs are also median graphs. This observation leads to Theorem 4.

First a lemma. For a fixed edge uv we will consider sets U_{uv} and F_{uv} with respect to different (sub)graphs H. We will hence use the notation $U_{uv}(H)$ and $F_{uv}(H)$ to emphasize that U_{uv} or F_{uv} are considered with respect to H.

Lemma 3. *Let ab be an edge of a partial cube G. If uv is an edge from W_{ab} then $U_{uv}(W_{ab}) = U_{uv}(G) \cap W_{ab}$.*

Proof. Note that $F_{uv} = F_{u'v'}$ for any $u'v' \in F_{uv}$ by the transitivity of Θ. Since W_{ab} is convex in G, Θ with respect to W_{ab} coincides with Θ with respect to G. Hence $F_{uv}(W_{ab}) \subseteq F_{uv}(G)$. Clearly this implies $F_{uv}(W_{ab}) = F_{uv}(G) \cap W_{ab}$, which yields the desired result. $\square$

Theorem 4. *A graph is almost-median if and only if it can be obtained by an isometric expansion procedure whose covering sets induce almost-median graphs.*

Proof. Let H be an almost-median graph. We need to prove that both W_{ab} and W_{ba} induce almost-median graphs (since then H can be obtained from a smaller almost-median graph G with covering sets isomorphic to W_{ab} and W_{ba}).

For the proof that W_{ab} is almost-median we first note that W_{ab} is a partial cube. So we need to prove that for any $uv \in W_{ab}$ the set $U_{uv}(W_{ab})$ is isometric. Because W_{ab} is convex, any shortest path between vertices x, y of $U_{uv}(W_{ab})$ is in W_{ab}. By Lemma 3, $U_{uv}(W_{ab}) = U_{uv}(G) \cap W_{ab}$. Hence, as $U_{uv}(G)$ is isometric in G, there is a shortest x, y-path that lies in $U_{uv}(W_{ab})$.

For the converse let H be a graph obtained by an isometric expansion procedure in which covering sets induce almost-median graphs. Let H be obtained by such an expansion from G with covering sets G_1 and G_2. Let uv be any edge of H and assume without loss of generality that uv is not in the new Θ-class obtained by the expansion. Let ab be a representative of this new class. Let x and y be vertices of U_{uv}. By Lemma 3 we have $U_{uv}(W_{ab}) = U_{uv}(G) \cap W_{ab}$ and $U_{uv}(W_{ba}) = U_{uv}(G) \cap W_{ba}$. Hence, if x and y lie either both in W_{ab} or both in W_{ba}, then U_{uv} already contains a shortest x, y-path of G since G is almost-median. Finally, let $x \in W_{ab}$ and $y \in W_{ba}$. The proof is completed by the observation that the length of any shortest x, y-path increases by one with respect to such paths in G. So we can use the isometry of U_{uv} in G again. □

4. Partial cubes whose U_{ab}'s are isometric trees

In this section, we present an algorithm of complexity $O(m \log n)$ that recognizes partial cubes in which the sets U_{ab} induce isometric trees.

Lemma 5. *A graph G is a partial cube in which the sets U_{ab} induce isometric trees if and only if G is a prism-free almost-median graph.*

Proof. Let G be a partial cube in which the sets U_{ab} induce isometric trees and assume that G contains $X = K_2 \Box C_{2r}$ as an induced subgraph. Since the U_{ab}'s are isometric (and thus connected), G is a semi-median graph. Thus $\delta^* = \Theta$. It follows that the edges of X corresponding to the factor K_2 belong to the same Θ-class. This in turn implies that the corresponding U_{ab}'s contain cycles.

The converse is clear. □

The class of prism-free almost median graphs includes all cube-free median graphs and arbitrary Cartesian products of two trees. As the second example shows they are not planar, in general, but they may be viewed as a kind of two-dimensional almost median graphs.

Partial cubes whose U_{ab}'s are isometric trees can also be characterized as almost-median graphs whose U_{ab}'s are trees. To recognize such graphs we first check whether they are bipartite with not too many edges. Then we determine δ^*, construct the U^*_{ab}'s, check whether they induce trees, no two edges of which are in the same δ^* class, and whether an F^*_{ab} induces an isomorphism between U_{ab} and U_{ba}. Finally, it suffices to show that $\Theta = \delta^*$, because then Θ is transitive.

We proceed as follows. Let $E^*_1, \ldots, E^*_k$ be the equivalence classes of δ^*, which we also call *color classes with respect to* δ^*. We first construct G by joining these classes one by one, checking certain properties which necessarily have to hold if $\Theta = \delta^*$. If they are not satisfied, G cannot be a semi-median graph and is rejected.

Algorithm A. *Input*: A connected, bipartite graph G with $m \leqslant n \log n$.

Output: TRUE if G is a prism-free almost-median graph, FALSE otherwise.

1. Determine all squares, compute δ^* and the U^*_{ab}'s. Let $E^*_1, \ldots, E^*_k$ be the equivalence classes of δ^*.
2. For each edge ab check if U^*_{ab} and U^*_{ba} induce distinct, isometric trees that are isomorphic. If not, then return FALSE and stop.

3. Let $G_0 = (V(G), \emptyset)$. For $i = 1, \ldots, n$ let $G_i = G_{i-1} \cup E_i^*$. If an edge of E_i^* has both endpoints in the same component of G_{i-1}, then return FALSE and stop.
4. Return TRUE.

Lemma 6. *Every prism-free almost-median graph is accepted by Algorithm A.*

Proof. Let G be a prism-free almost-median graph. Then G is a partial cube and $\delta^* = \Theta^* = \Theta$. Therefore $F_{uv}^* = F_{uv}$ and $U_{uv}^* = U_{uv}$, and so G clearly passes Step 2. Concerning Step 2 we infer that if there exists an edge e of E_i^* that has both endpoints in the same component of G_{i-1}, then e is in relation $\Theta = \delta^*$ to an edge of the component of G_{i-1} by Lemma 3.2 of [13]. Since this is not possible, a semi-median graph passes Step 3. □

Lemma 7. *Let G be a graph accepted by Algorithm A. Then $\delta^* = \Theta$.*

Proof. Suppose this is not the case, then there are two edges e, f that are either in relation Θ or in δ^*, but not in both. Let e, f be such a pair with minimum distance d in G and ℓ the smallest index such that both e and f are in one and the same component of G_ℓ, we denote it by $C_\ell^{e,f}$.

Claim. *Every shortest path from e to f is in $C_\ell^{e,f}$.*

Proof. Let P be a shortest path from e to f. By Lemma 1 no two edges of P are in relation Θ. Hence, by the minimality of d, we infer that they are also not in δ^*.

If P were not in G_ℓ, then it would contain edges in a class E_j^* with $j > \ell$. Let j' be the largest such index. As all edges of P are in different classes E_j^*, this implies that we add an edge to the graph $G_{j'-1}$ of which both endpoints are in one and the same component of $G_{j'-1}$, and so G is rejected by Step 3 of the algorithm. □

We continue with the proof of Lemma 7.

Case 1: $e\Theta f$. In this case there are shortest paths P, Q of length d in G_ℓ between e and f such that $C = e \cup P \cup f \cup Q$ is a cycle. Note that neither e nor f can be in relation δ^* with any of the edges of P or Q (because of the minimality of d).

Assume first that no two edges of the cycle have the same color. Then the algorithm rejects G in Step 3.

Thus, suppose $g\delta^* h$, where $g \in P$ and $h \in Q$. Clearly the distance between g and h is at most d. Let R be one side of the ladder between g and h. As any two edges of it are in different δ^*-classes, we derive by the same argument as in the proof of the claim that R is in G_ℓ. Also R is the shortest path, since any shortest path between the corresponding vertices must have the same colors as R by the very same argument. Let R' be the other side of the ladder, and let S be the part of C with the same endpoints as R and S' the one with the same endpoints as R'. Again without loss of generality, we may assume that the walk $W = R \cup S$ is not longer than the walk $R' \cup S'$ and that W contains e. Then the length of W is at most $2d$. Clearly any two edges of W have distance at most $d-1$. Thus, by Lemma 1 and the minimality, there is an edge on R that is in relation δ^* with e.

Now we consider two subcases.

Subcase 1.1: $|R| = d$. Then S, R' and S' also have length d. This implies that they all have the same δ^*-colors. Hence e must be in relation δ^* to an edge of S'. Since this edge can be on neither P nor Q, it must be f.

Subcase 1.2: $|R| < d$. Then $R' \cup g \cup S \cup h$ also has length at most $2d$. Hence there is an edge e' in R' with $e\delta^* e'$. Because of Step 3 we find an edge e'' on S' with $e'\delta^* e''$. By the transitivity of δ^* we have $e'' = f$.

We have thus shown that e and f are in relation δ^* if they are in relation Θ.

Case 2: $e\delta^* f$. By the claim there is a shortest path P of length d between e and f, no two edges of which have the same δ^*-color. Consider a ladder between e and f. As before, it must be in G_ℓ and its sides must be shortest paths in G. But then $e\Theta f$.

Theorem 8. *Algorithm A correctly recognizes prism-free almost-median graphs and can be implemented to run in* $O(m \log n)$ *time.*[2]

Proof. By Lemma 6 prism-free almost-median graphs are accepted by the algorithm. Suppose now that G is accepted. Then, by Lemma 7, $\Theta = \delta^*$ which implies that for any edge uv, $U^*_{uv} = U_{uv}$. Since the U^*_{uv}'s are checked for being isometric trees, G is a prism-free almost-median graph.

It remains to determine the algorithm's complexity.

With the algorithm of [7] we first construct all quadrangles of G. The complexity of the algorithm is $O(m\,a(G))$, where $a(G)$ denotes the *arboricity* of G, that is, the minimum number of disjoint spanning forests into which G can be decomposed. It considers every edge uv and all edges incident with an endpoint of uv of degree $\min\{d_G(u), d_G(v)\}$. Since

$$\sum_{uv \in E(G)} \min\{d_G(u), d_G(v)\} \leqslant 2a(G)m$$

and $a(G) \leqslant \log n$ for partial cubes, we can stop the algorithm if

$$\sum_{uv \in E(G)} \min\{d_G(u), d_G(v)\} > m \log n.$$

It follows that δ^* can be determined in $O(m \log n)$ time; cf. Proposition 7.6 (ii) of [16].

We observe next that

$$\sum_{uv \in E(G)} |U^{\delta^*}_{uv}| = \sum_{j=1}^{k} |E^*_j| = m.$$

Now to Step 2. By standard Make-Set, Union and Find-Set operations we can check in $O(m \log n)$ time whether for each edge ab, U^*_{ab} and U^*_{ba} induce distinct trees that are isomorphic. To check if the U^*_{ab}'s are isometric, we only need to check if the edges of a U^*_{ab} have pairwise different colors. Indeed, if a graph G is accepted, then $U^*_{ab} = U_{ab}$, and $\delta^* = \Theta$, so pairwise different colors of the edges of U_{ab} imply isometry by Lemma 1.

[2] As one of the referees asked, the factor $\log n$ can be removed. See the last paragraph of the paper.

In Step 2 we have to perform two Find-Set operations and possibly one Union for every one of the m edges in the graph. It is well known that these operations can be executed within time complexity $O(m \log n)$, cf. [8]. □

5. Concluding remarks

1. Algorithm A thus recognizes prism-free almost-median graphs. Let us briefly call them $AM(T)$ graphs, as they can be characterized as almost-median graphs whose U_{ab}'s are trees. One of the main properties of trees T that are isometrically embedded in a partial cube G is that removal of a Θ-class of G decomposes T into two isometrically embedded trees. We say the class of isometrically embedded trees is invariant under removal of Θ-classes. If we replace the class of trees in $AM(T)$ by a class S of partial cubes that is invariant under removal of Θ-classes, then we obtain a new class of almost-median graphs which we denote by $AM(S)$. This class can be obtained by isometric expansion procedures whose covering sets G_1, G_2 are almost-median and whose intersection $G_1 \cap G_2$ are in S. Let $S(m,n)$ denote the recognition complexity of $AM(S)$.

By the same arguments as in the proof of Theorem 8 their recognition complexity is the maximum of $S(m,n)$ and $O(m \log n)$. For example, S could be the set $\mathscr{C}$ of all cycles and all trees or the class $\mathscr{H}$ of hypercubes. Since their recognition complexities are linear, $AM(\mathscr{C})$ and $AM(\mathscr{H})$ have recognition complexities $O(m \log n)$. Both classes contain non-median graphs and the latter all acyclic cubical complexes; cf. [2,15].

2. It was remarked by one of the referees that the number of edges m of prism-free almost median graphs might be linear in n. A positive answer would support the conjecture that these graphs can be recognized in linear time, which is the second question.

The answer to the first question is easy. For, consider a contraction procedure of such a graph. Let n_i be the number of vertices that are contracted in the ith step. Then these vertices induce a tree, and hence at most $2n_i - 1$ edges collapse. Thus $m = O(n)$.

The answer to the second question is also positive but requires some arguments that will be included elsewhere.

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