Hamming polynomials and their partial derivatives

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Abstract

Hamming graphs are Cartesian products of complete graphs and partial Hamming graphs are their isometric subgraphs. The Hamming polynomial $h(G)$ of a graph G is introduced as the Hamming subgraphs counting polynomial. K_k-derivates $\partial_k G$ ($k \geq 2$) of a partial Hamming graph are also introduced. It is proved that for a partial Hamming graph G, $\frac{\partial h(G)}{\partial x_k} = h(\partial_k G)$. A couple of combinatorial identities involving the coefficients of the Hamming polynomials of Hamming graphs are also proven.
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1. Introduction

In this paper we introduce Hamming polynomials as counting polynomials (in several variables) that count induced subgraphs of a given graph which are isomorphic to Hamming graphs. More precisely, we define the *Hamming polynomial*

$$h(G) = h(G; x_2, x_3, \ldots, x_\omega)$$

of a graph G as

$$\sum_{r_2, r_3, \ldots, r_\omega \geq 0} \alpha(G; r_2, r_3, \ldots, r_\omega) x_2^{r_2} x_3^{r_3} \cdots x_\omega^{r_\omega},$$

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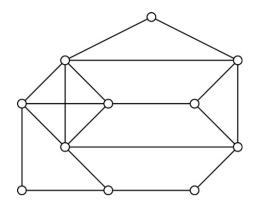
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Fig. 1. A partial Hamming graph.

where $\alpha(G; r_2, r_3, \ldots, r_\omega)$ is the number of induced subgraphs of G isomorphic to the Hamming graph $K_2^{r_2} \square K_3^{r_3} \square \cdots \square K_\omega^{r_\omega}$, and $\omega = \omega(G)$ is the clique number of G. Note that a variable x_1 is not present in the Hamming polynomial, the reason being that K_1 is the unit for the Cartesian product. For instance, $h(K_n; x_2, x_3, \ldots, x_n) = n + \sum_{r=2}^{n} \binom{n}{r} x_r$. For another example consider the graph from Fig. 1. Its Hamming polynomial is equal to $11 + 19x_2 + 5x_2^2 + 6x_3 + x_2x_3 + x_4$.

Special Hamming polynomials $\sum_{r_2 \geq 0} \alpha(G; r_2)x_2^{r_2}$ have been introduced in [5] and named *cube polynomials*. They have been further studied and applied in [2,6]. In particular, using the cube polynomials, one can obtain a very general form of the so-called Euler-type formulas for median graphs that were first presented in [14].

The natural environment for Hamming polynomials is formed by the isometric subgraphs of Hamming graphs, called partial Hamming graphs. These graphs have many intriguing structural properties and have been extensively studied, see [3,4,8,11,12,17,18].

We proceed as follows. In the rest of this section, we give definitions and concepts needed. In the next section we introduce the K_k-derivate $\partial_k G$ of a partial Hamming graph G. For this purpose, we in particular extend an idea of Chung, Graham, and Saks from [7]. Our main result then asserts that the partial derivatives of the Hamming polynomials and the K_k-derivates of partial Hamming graphs are connected as follows. For any partial Hamming graph G and for any $k, 2 \leq k \leq \omega$,

$$\frac{\partial h(G; x_2, \ldots, x_\omega)}{\partial x_k} = h(\partial_k G; x_2, \ldots, x_\omega).$$

In the last section we study the coefficients $\alpha(G; 0, \ldots, 0, d)$ where G is a Hamming graph, and obtain a couple of combinatorial identities. Along the way, we also observe that Hamming polynomials are multiplicative on the Cartesian product of graphs.

The *Cartesian product* $G \square H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H)$ where the vertex (a, x) is adjacent to the vertex (b, y) whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. The one-vertex graph K_1 is the unit for this operation. The Cartesian product is commutative and associative, hence the Cartesian product of several factors is well defined. The product of k copies of K_2 is called the *k-cube* while the graphs of the form

$$G = K_{n_1} \square K_{n_2} \square \cdots \square K_{n_r},$$

where $r \geq 1$ and $n_i \geq 1$, $i = 1, \ldots, r$, are known as *Hamming graphs*. It is clear what we mean by K_n^r for $r \geq 1$. We also define $K_n^0 = K_1$.

A subgraph H of a graph G is called *isometric* if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$, where $d_G(u, v)$ denotes the usual shortest path distance. H is *convex* if, together with $u, v \in V(H)$, all shortest u, v-paths from G belong to H. Isometric subgraphs of k-cubes are known as *partial cubes*, while *partial Hamming graphs* are isometric subgraphs of Hamming graphs.

2. K_k-derivates of partial Hamming graphs

In the study of partial Hamming graphs, the following relation was used. Edges $ab, xy \in E(G)$ are in relation $\sim$ if $d(x, a) = d(y, b) = d(x, b) - 1 = d(y, b) - 1$. It was first introduced by Djoković in the context of bipartite graphs [9] where it coincides with the Winkler's relation Θ [18]. Wilkeit [17] was the first to use the relation $\sim$ in the context of partial Hamming graphs.

The relation $\sim$ is reflexive and symmetric, but it is in general not transitive, cf. $K_{2,3}$. However, in partial Hamming graphs, $\sim$ is an equivalence relation on $E(G)$ and $\sim \subset \Theta$.

In this section we first extend the relation $\sim$ to a family of relations $\sim_k$ and then use them to define the K_k-derivates of partial Hamming graphs. Then we prove our main result — Theorem 2.3.

Let G be a partial Hamming graph and let k be an integer, $2 \leq k \leq \omega$. Let $\mathcal{K}_k(G)$ be the set of all complete subgraphs of G on k vertices. Then we introduce the relation $\sim_k$ defined on $\mathcal{K}_k(G)$ in the following way. Complete subgraphs $X, Y \in \mathcal{K}_k(G)$ on vertices $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$, respectively, are in relation $\sim_k$, if the notation of vertices can be chosen in such a way that there exists an integer p such that

$$d_G(x_i, y_j) = p + 1 \quad \text{for } i \neq j, \text{ and } d_G(x_i, y_i) = p. \tag{1}$$

Note that $\sim_2 = \sim$. We also say that X and Y are *parallel* when they are in relation $\sim_k$. Such a parallelism relation was studied by Chung et al. [7], where the definition is restricted to quasi-median graphs and to maximal complete subgraphs. (Recall that quasi-median graphs form a proper subclass of partial Hamming graphs, cf. [1,11].)

Lemma 2.1. *Let G be a partial Hamming graph and $2 \leq k \leq \omega$. Then $\sim_k$ is an equivalence relation on $\mathcal{K}_k(G)$.*

Proof. Clearly, $\sim_k$ is reflexive and symmetric, and different complete subgraphs $X, Y \in \mathcal{K}_k(G)$ that share a vertex cannot be in relation $\sim_k$. Let X, Y, and Z be pairwise disjoint complete subgraphs from $\mathcal{K}_k(G)$ such that $X \sim_k Y$ and $Y \sim_k Z$. Let x_i, y_i, and z_i, $1 \leq i \leq k$, be the vertices of X, Y, and Z, respectively, where the notation is selected in the spirit of (1). Then for any $i \neq j$, the edge $x_i x_j$ of X is in relation $\sim$ with the edge $y_i y_j$ of Y and $y_i y_j \sim z_i z_j$. Since $\sim$ is transitive, we derive $x_i x_j \sim z_i z_j$ and hence $X \sim_k Z$. We conclude that $\sim_k$ is transitive as well. $\square$

Let E be an equivalence class of the relation $\sim_k$, and let $\langle E \rangle$ be the subgraph of G induced by the vertices from E. Note that $\langle E \rangle$ might not be connected. For instance, in C_6 each equivalence class with respect to $\sim_2$ consists of two disjoint K_2's. We next show that $\langle E \rangle$ has a product structure.

Lemma 2.2. *Let G be a partial Hamming graph, $2 \leq k \leq \omega$, and E an equivalence class of $\sim_k$. Then there exists a graph U_E such that $\langle E \rangle = K_k \square U_E$.*

Proof. Let $\langle E \rangle$ be induced by complete subgraphs $X^{(1)}, X^{(2)}, \ldots, X^{(t)}$. We have already observed that these complete subgraphs are pairwise disjoint. Let $V(X^{(i)}) = \{x_1^{(i)}, x_2^{(i)}, \ldots, x_k^{(i)}\}$

where the notation is selected such that $d(x_1^{(1)}, x_1^{(i)}) < d(x_1^{(1)}, x_2^{(i)})$ for $i = 2, \ldots, t$. Since $x_1^{(1)}x_2^{(1)} \sim x_1^{(i)}x_2^{(i)}$, $x_1^{(1)}x_2^{(1)} \sim x_1^{(j)}x_2^{(j)}$ ($i, j \neq 1, i \neq j$), and because $\sim$ is transitive, we infer that $d(x_r^{(i)}, x_r^{(j)}) < d(x_r^{(i)}, x_s^{(j)})$ for any $i \neq j$ and $r \neq s$. Set now $U_E = \langle\{x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(t)}\}\rangle$. Let $i \neq j$. Then for $r \neq s$ we have $x_r^{(i)}x_s^{(j)} \notin E(G)$, while $x_r^{(i)}x_r^{(j)} \in E(G)$ if and only if $x_1^{(i)}x_1^{(j)} \in E(G)$. It follows that $\langle E \rangle = K_k \square U_E$.   □

Let G be a partial Hamming graph and $E_1, \ldots, E_r$ the equivalence classes of relation $\sim_k$. By Lemma 2.2, there exist graphs U_i, $1 \leq i \leq r$, such that $\langle E_i \rangle = K_k \square U_i$. Then we define the K_k-derivate $\partial_k G$ of G (with respect to k) as the disjoint union of the graphs U_i:

$$\partial_k G = \bigcup_{i=1}^{r} U_i.$$

This definition is a generalization of the concept ∂G introduced for median graphs G in [4], since for a median graph G, $\partial_k G$ is defined only for $k = 2$ where $\partial_2 G = \partial G$.

We are now ready for our main theorem.

Theorem 2.3. *Let G be a partial Hamming graph. Then for any k, $2 \leq k \leq \omega$,*

$$\frac{\partial h(G; x_2, \ldots, x_\omega)}{\partial x_k} = h(\partial_k G; x_2, \ldots, x_\omega).$$

Proof. Let H be a subgraph of G isomorphic to $K_2^{r_2} \square K_3^{r_3} \square \cdots \square K_\omega^{r_\omega}$, where $r_k \geq 1$ and $r_i \geq 0$ for $i \neq k$. Clearly, H contributes 1 to the coefficient $\alpha(G; r_2, r_3, \ldots, r_\omega)$. For any factor graph of H isomorphic to K_k, let us denote it with K, we can write

$$H = K \square (K_2^{r_2} \square \cdots \square K_k^{r_k-1} \square \cdots \square K_\omega^{r_\omega}).$$

Let E be the equivalence class of K with respect to $\sim_k$. Then K has a vertex in common with any other factor of H, hence $K_2^{r_2} \square \cdots \square K_k^{r_k-1} \square \cdots \square K_\omega^{r_\omega} \subseteq U_E$. Therefore K contributes 1 to the coefficient $\alpha(\partial_k G; r_2, r_3, \ldots, r_k-1, \ldots, r_\omega)$ and there are r_k copies of K_k each belonging to its own equivalence class.

Similarly, if we have a Hamming graph X that contributes to a coefficient $\alpha(\partial_k G; r_2, r_3, \ldots, r_k - 1, \ldots, r_\omega)$, then Lemma 2.2 implies that there exists a U_E such that $X \subseteq U_E$, where $U_E \square K_k$ is an induced subgraph of G. Hence $X \square K_k$ contributes 1 to the coefficient $\alpha(G; r_2, r_3, \ldots, r_\omega)$. We conclude that

$$\frac{\partial h(G; x_2, \ldots, x_\omega)}{\partial x_k} = \sum_{r_2, \ldots, r_\omega \geq 0} r_k \alpha(G; r_2, \ldots, r_k, \ldots, r_\omega) x_2^{r_2} \cdots x_k^{r_k-1} \cdots x_\omega^{r_\omega}$$
$$= h(\partial_k G; x_2, \ldots, x_\omega).   □$$

To illustrate Theorem 2.3 consider the partial Hamming graph G from Fig. 1. The equivalence classes of the relation $\sim_2$ are represented by the edge labels in Fig. 2.

The graph $\partial_2 G$ is shown in Fig. 3. Thus, $h(\partial_2 G) = 19 + 10x_2 + x_3$ which is indeed equal to $\frac{\partial h(G)}{\partial x_2}$.

The equivalence classes of the relation $\sim_3$ induce a $K_2 \square K_3$ and four triangles, that is, four copies of $K_1 \square K_3$. Hence, the graph $\partial_3 G$ is the disjoint union of an edge and four isolated vertices, hence $h(\partial_3 G) = 6 + x_2$. Finally, $\partial_4 G$ is just a vertex and therefore $h(\partial_4 G) = 1$.
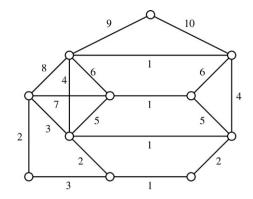
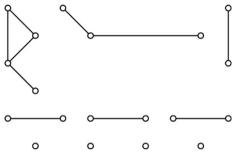Fig. 2. Equivalence classes of the relation $\sim_2$ on $E(G)$.

Fig. 3. The K_2-derivate of G.

In order to be able to introduce K_k-derivates of a higher order in the sense of [4], all components of every U_i should be partial Hamming graphs, because then also $\partial_k G$ would consist of partial Hamming graphs as components. We formulate as an open problem whether or not this concept can be formulated this way.

Problem 2.4. Let G be a partial Hamming graph, and E an equivalence class of the relation $\sim_k$. Is each connected component of $\langle E \rangle$ a partial Hamming graph?

Note that in the case of quasi-median graphs, this is true; more precisely $\langle E \rangle$ is a convex subgraph in G, hence a quasi-median graph. One can derive this fact from [1, Theorem 1].

3. Coefficients of Hamming polynomials in Hamming graphs

In this section we take a closer look at the coefficients $\alpha(G; 0, \ldots, 0, d)$, where G is a Hamming graph. This enables us to obtain a couple of combinatorial identities. Related identities have been previously proved for hypercubes and median graphs, cf. [15,16], as well as for quasi-median graphs [4]. We first recall the following well-known fact.

Lemma 3.1. Let $H = K_n$ be an (induced) subgraph of a Cartesian product graph G. Then H projects isomorphically onto a factor of G.

With this lemma the following result follows easily.

Proposition 3.2. *For any graphs G and H, $h(G \square H) = h(G)h(H)$.*

We next state the main result of this section. To prove it we apply a double counting of Hamming subgraphs and the binomial inversion, extending similar ideas for hypercubes from [10,13]. Recall that the binomial inversion asserts that

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k \quad (\forall n \geq 0)$$

if and only if

$$b_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k \quad (\forall n \geq 0).$$

Theorem 3.3. *Let G be the Hamming graph K_r^n and let $\alpha_d = \alpha(G; 0, \ldots, 0, d)$. Then*

$$\sum_{k=0}^{n} (-1)^k \alpha_k = (r-1)^n \quad \text{and} \quad -\sum_{k=0}^{n} (-1)^k k \alpha_k = n(r-1)^{n-1}.$$

Proof. Note first that the vertices of $G = K_r^n$ can be identified with the words of length n over the alphabet $\{1, 2, \ldots, r\}$, where two words are adjacent if and only if they differ in precisely one position. Hence, by Lemma 3.1, α_d can be considered as the number of words of length n over the alphabet $\{1, 2, \ldots, r, K\}$ containing d times the character K. Each character represents the part of a factor K_r from the product we take to form a K_r^d. It may be either the whole clique (character K) or just a vertex. In the latter case, we can choose among r vertices represented by $1, \ldots, r$.

We now compute α_d in two different ways.

First, we choose d positions out of n for characters K. The remaining $n - d$ positions can be filled with the integers in the range $[1, r]$. Therefore,

$$\alpha_d = \binom{n}{d} r^{n-d}.$$

On the other hand, we can first position the symbol 1. There can be any number of them from 0 to $n - d$. Next, we place d copies of the character K in the remaining l positions, $d \leq l \leq n$. Finally, we place symbols $2, \ldots, r$ in the remaining $l - d$ positions. So we get

$$\alpha_d = \sum_{l=d}^{n} \binom{n}{l} \binom{l}{d} (r-1)^{l-d}.$$

The above double counting thus yields the following identity

$$\binom{n}{d} r^{n-d} = \sum_{l=d}^{n} \binom{n}{l} \binom{l}{d} (r-1)^{l-d} = \sum_{l=0}^{n} \binom{n}{l} \binom{l}{d} (r-1)^{l-d}.$$

Applying the binomial inversion we get

$$\binom{n}{d} (r-1)^{n-d} = \sum_{l=0}^{n} (-1)^{n-l} \binom{n}{l} \binom{l}{d} r^{l-d}. \tag{2}$$

When $d = 0$, this implies that

$$(r - 1)^n = \sum_{l=0}^{n} (-1)^{n-l} \binom{n}{l} r^l = \sum_{k=0}^{n} (-1)^k \binom{n}{k} r^{n-k} = \sum_{k=0}^{n} (-1)^k \alpha_k$$

which proves the first identity of our theorem.

For the second identity consider Eq. (2) for $d = 1$:

$$n(r - 1)^{n-1} = \sum_{l=0}^{n} (-1)^{n-l} \binom{n}{l} l r^{l-1} = \sum_{k=0}^{n} (-1)^k (n - k) \frac{\alpha_k}{r}.$$

Now, using the first identity, we obtain

$$-\sum_{k=0}^{n} (-1)^k k \alpha_k = nr(r - 1)^{n-1} - n(r - 1)^n = n(r - 1)^{n-1}$$

which completes the proof. $\square$

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References

[1] H.-J. Bandelt, H.M. Mulder, E. Wilkeit, Quasi-median graphs and algebras, J. Graph Theory 18 (1994) 681–703.
[2] H.-J. Bandelt, L. Quintana-Murci, A. Salas, V. Macaumay, The fingerprint of phantom mutations in mitochondrial DNA data, American J. Human Genetics 71 (2002) 1150–1160.
[3] B. Brešar, Partial Hamming graphs and expansion procedures, Discrete Math. 237 (2001) 13–27.
[4] B. Brešar, S. Klavžar, R. Škrekovski, Quasi-median graphs, their generalizations, and tree-like equalities, European J. Combin. 24 (2003) 557–572.
[5] B. Brešar, S. Klavžar, R. Škrekovski, The cube polynomial and its derivatives: The case of median graphs, Electron. J. Combin. 10 (2003) (#R3) 11.
[6] B. Brešar, S. Klavžar, R. Škrekovski, On cube-free median graphs, Discrete Math. (in press).
[7] F.R.K. Chung, R.L. Graham, M.E. Saks, A dynamic location problem for graphs, Combinatorica 9 (1989) 111–131.
[8] V. Chepoi, Isometric subgraphs of Hamming graphs and d-convexity, Cybernetics 1 (1988) 6–11.
[9] D.Ž. Djoković, Distance-preserving subgraphs of hypercubes, J. Combin. Theory Ser. B 14 (1973) 263–267.
[10] M.L. Garbano, J.F. Malerba, M. Lewinter, Hypercubes and Pascal's triangle: A tale of two proofs, Math. Mag. 76 (2003) 216–217.
[11] W. Imrich, S. Klavžar, Product Graphs: Structure and Recognition, John Wiley & Sons, New York, 2000.
[12] S. Klavžar, On the canonical metric representation, average distance, and partial Hamming graphs, European J. Combin. 27 (2006) 68–73.
[13] S. Klavžar, Counting hypercubes in hypercubes, Discrete Math. (in press).
[14] S. Klavžar, H.M. Mulder, R. Škrekovski, An Euler-type formula for median graphs, Discrete Math. 187 (1998) 255–258.
[15] R. Škrekovski, Two relations for median graphs, Discrete Math. 226 (2001) 351–353.
[16] P.S. Soltan, V. Chepoi, Solution of the Weber problem for discrete median metric spaces, Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR 85 (1987) 52–76 (Russian).
[17] E. Wilkeit, Isometric embeddings in Hamming graphs, J. Combin. Theory Ser. B 50 (1990) 179–197.
[18] P.M. Winkler, Isometric embedding in products of complete graphs, Discrete Appl. Math. 7 (1984) 221–225.