The Δ^2-Conjecture for $L(2,1)$-Labelings is True for Direct and Strong Products of Graphs

Sandi Klavzar and Simon Spacapan

***Abstract*—A variation of the channel-assignment problem is naturally modeled by $L(2,1)$-labelings of graphs. An $L(2,1)$-labeling of a graph G is an assignment of labels from $\{0,1,\ldots,\lambda\}$ to the vertices of G such that vertices at distance two get different labels and adjacent vertices get labels that are at least two apart and the λ-number $\lambda(G)$ of G is the minimum value λ such that G admits an $L(2,1)$-labeling. The Δ^2-conjecture asserts that for any graph G its λ-number is at most the square of its largest degree. In this paper it is shown that the conjecture holds for graphs that are direct or strong products of nontrivial graphs. Explicit labelings of such graphs are also constructed.**

***Index Terms*—$L(2,1)$-labeling, channel assignment, graph direct product, graph strong product.**

I. Introduction

An $L(2,1)$-labeling of a graph G is an assignment of labels from $\{0,1,\ldots,\lambda\}$ to the vertices of G such that vertices at distance two get different labels and adjacent vertices get labels that are at least two apart. The λ-number $\lambda(G)$ of G is the minimum value λ such that G admits an $L(2,1)$-labeling. For instance, in the complete graph on n vertices K_n every pair of different vertices must receive labels that differ by at least two, hence $\lambda(K_n)=2n-2$ for $n\geq 1$.

The $L(2,1)$-labeling concept grew up from the problem of assigning frequencies to radio transmitters at various nodes in a territory. To avoid interferences, transmitters that are close must receive frequencies that are sufficiently apart. Since frequencies are quantized in practice, there is no loss of generality in assuming that they admit integer values. The problem with the objective of minimizing the span of frequencies was first proposed in 1980 by Hale [9]. Later Griggs and Yeh [8] formulated the above-defined $L(2,1)$-labelings. Soon after the topic (and it generalization to the $L(h,k)$-labelings) became an extensive area of research, see the survey [2] with 114 references and recent papers [4], [5], [7], [15], [18].

One of the central problems in the area is the Δ^2-conjecture of Griggs and Yeh from [8] which asserts that for any graph G, $\lambda(G)\leq\Delta^2$, where Δ is the largest degree of G. The authors originally proved that $\lambda(G)\leq\Delta^2+2\Delta$. The bound has been improved to $\lambda(G)\leq\Delta^2+\Delta$ by Chang and Kuo [3] and further to $\lambda(G)\leq\Delta^2+\Delta-1$ by Král andŠkrekovski [17].

For the proof of the $\Delta^2+\Delta$ bound Chang and Kuo [3] introduced an algorithm to be described in Section II. Recently Shao and Yeh [20] proved that this approach can be used to establish the Δ^2-conjecture for graphs that are nontrivial Cartesian or lexicographic products of graphs. These two graph products, together with the direct product and the strong product, form the four standard graph products [10]. In this paper, we prove that the Δ^2-conjecture holds for direct products and strong products as well. We note that $L(2,1)$-labelings of direct and strong products have been studied before in [12], [13], [16] —mostly direct products and strong products of cycles have been considered.

In Section II, we describe the above-mentioned algorithm and introduce the two graph products of interest. Using the algorithm we then prove in Section III the Δ^2-conjecture for direct products and strong products. In this way the results of [20] are rounded up—the conjecture holds (with some minor exceptions) for all the four standard graph products. In the last section we also give two explicit labelings, one for the direct and one for the strong product.

II. Preliminaries

In this section, we introduce the three central concepts (besides the $L(2,1)$-labelings) of this paper: Algorithm A, the direct product of graphs, and the strong product of graphs.

Let G be a graph, then a vertex subset X is a *2-stable set* if for any vertices $x,y\in X$ we have $d(x,y)>2$, where $d(x,y)$ denotes the usual shortest path distance in the graph G. Chang and Kuo [3] proposed the following labeling procedure for a graph G, let us call it **Algorithm A**. In the beginning of the algorithm, no vertex of G is labeled.

- Set $X_{-1}=\emptyset$ and $i=-1$.
- Repeat
  $i\leftarrow i+1$.
  Let Y_i be the set of all vertices of G that are not yet labeled and are at distance at least two from any vertex of X_{i-1}.
  Select a maximal 2-stable subset of Y_i, denote it X_i.
  Label the vertices of X_i with i
  until all vertices of G are labeled.

Let k be the largest label obtained by Algorithm A and let x be a vertex with label k. Define the following sets.

- $I_1=\{i\mid 0\leq i\leq k-1$ and $d(x,y)=1$ for some $y\in X_i\}$.
- $I_2=\{i\mid 0\leq i\leq k-1$ and $d(x,y)\leq 2$ for some $y\in X_i\}$.

Manuscript received June 21, 2005. This work was supported in part by the Ministry of Science of Slovenia under Grant P1-0297. This paper was recommended by Associate Editor S. Tsukiyama.

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Digital Object Identifier 10.1109/TCSII.2005.861884

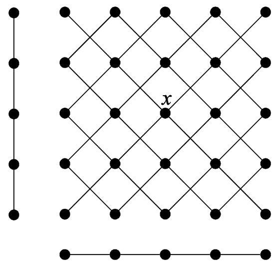

Fig. 1. Direct product $P_5 \times P_5$.

- $I_3 = \{i \mid 0 \leq i \leq k-1$ and $d(x,y) \geq 3$ for all $y \in X_i\}$.

Then, Chang and Kuo showed that

$$\lambda(G) \leq k \leq |I_1| + |I_2| \tag{1}$$

which in turn immediately implies the bound $\lambda(G) \leq \Delta^2 + \Delta$.

For graphs G and H, the *direct product* $G \times H$ of G and H is defined as follows: $V(G \times H) = V(G) \times V(H)$ and $E(G \times H) = \{(a,x),(b,y) : \{a,b\} \in E(G)$ and $\{x,y\} \in E(H)\}$. See Fig. 1 where the direct product of the path on five vertices with itself is shown. The direct product is commutative and associative in a natural way, hence we may also consider higher powers of the product. The direct product of graphs has several appealing properties, for instance, low diameter, high independence number and high odd girth, cf. [10], [11]. The direct product offers several applications in engineering, computer science and related disciplines. For example, the diagonal mesh studied by Tang and Padubirdi [21] with respect to multiprocessor network is representable as the direct product of two odd cycles while Ramirez and Melhem [19] present a fault-tolerant computational array whose underlying graph is isomorphic to a connected component of $P_{2i+1} \times P_{2i+1}$.

The strong product is closely related to the direct one and is defined as follows. Let G and H be graphs, then their *strong product* $G \boxtimes H$ is the graph with vertex set $V(G) \times V(H)$ and $(a,x)(b,y) \in E(G \boxtimes H)$ whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$, or $ab \in E(G)$ and $xy \in E(H)$. Note that the edge set of the strong product is the union of the edge set of the direct product and the Cartesian product (of the same factor graphs), cf. Fig. 2 with the strong product of the path on five vertices with itself. Among applications of the strong product let us mention it central role in the Shannon capacity of a graph, cf. [1], [10] and in the strong isometric dimension of a graph [6].

Let G_1 and G_2 be arbitrary graphs. Let G be the direct product $G = G_1 \times G_2$ or the strong product $G = G_1 \boxtimes G_2$. Then, for the rest of this paper, we adopt the following convention. Let Δ_1, Δ_2, and Δ be the largest degrees in G_1, G_2, and G, respectively. Note that

$$\Delta = \Delta_1 \Delta_2$$

in the case $G = G_1 \times G_2$ and

$$\Delta = \Delta_1 \Delta_2 + \Delta_1 + \Delta_2$$

when $G = G_1 \boxtimes G_2$.

Finally, in the studies of the $L(2,1)$-labeling problem we may clearly restrict to connected graphs, hence all factor graphs of products will be connected in this paper. Recall, however, that the direct product of two connected bipartite graphs consists of two connected components, cf. Fig. 1.

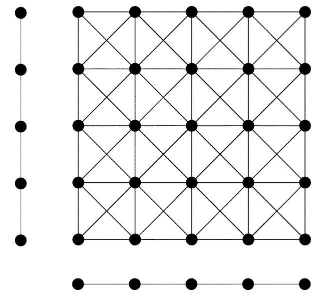

Fig. 2. Strong product $P_5 \boxtimes P_5$.

III. Δ^2-Conjecture for Direct and Strong Product

In this section, we prove that the Δ^2-conjecture holds for any graphs that are nontrivial direct or strong products. We first apply inequality (1) to prove the following result.

Theorem 1: Let G_1 and G_2 be nontrivial graphs. Then

$$\begin{aligned}\lambda(G_1 \times G_2) \leq \Delta^2 - \max\big\{&(\Delta_1 - 1)^2(\Delta_2 - 1)\\ &-\Delta_1 - \Delta_2 + 1, (\Delta_2 - 1)^2(\Delta_1 - 1) - \Delta_1 - \Delta_2 + 1\big\}.\end{aligned}$$

Proof: Let k be the largest label obtained by the Algorithm A (after running this algorithm on the graph $G_1 \times G_2$) and $x = (u,v)$ a vertex of $G_1 \times G_2$ with label k. Let $u_1, \ldots, u_{d_1}$ be the neighbors of u in G_1 and $v_1, \ldots, v_{d_2}$ be the neighbors of v in G_2. Let $\deg_{G_1}(u_i) = \alpha_i$ for $i = 1, \ldots, d_1$ and $\deg_{G_2}(v_i) = \beta_i$ for $i = 1, \ldots, d_2$. Then, the number of vertices on distance 2 from x is less or equal to

$$\sum_{i=1}^{d_1} \sum_{j=1}^{d_2} \alpha_i \beta_j - d_1 d_2 - \sum_{i=1}^{d_1} (\alpha_i - 1)(d_2 - 1).$$

To see this note that $\alpha_i \beta_j$ is equal to the degree of (u_i, v_j), hence the sum $\sum_{i=1}^{d_1} \sum_{j=1}^{d_2} \alpha_i \beta_j$ counts all neighbors of the neighbors of x (counted with their multiplicities). The number $d_1 d_2$ is subtracted since we have $d_1 d_2$ times counted x and $\sum_{i=1}^{d_1} (\alpha_i - 1)(d_2 - 1)$ is subtracted since for any $i \in \{1, 2, \ldots, d_1\}$ and any $j_1, j_2 \in \{1, 2, \ldots, d_2\}$ the vertices (u_i, v_{j_1}) and (u_i, v_{j_2}) have $\alpha_i - 1$ common neighbors (different from x), namely (w, v), where w is a neighbor of u_i. Thus, we have

$$\begin{aligned}|I_1| + |I_2| \leq 2 \deg_{G_1 \times G_2}(x) + \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} \alpha_i \beta_j - d_1 d_2\\ - \sum_{i=1}^{d_1} (\alpha_i - 1)(d_2 - 1)\end{aligned}$$

and, therefore

$$|I_1| + |I_2| \leq d_1 d_2 + \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} \alpha_i \beta_j - \sum_{i=1}^{d_1} (\alpha_i - 1)(d_2 - 1). \tag{2}$$

Analogously, we get

$$|I_1| + |I_2| \leq d_1 d_2 + \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} \alpha_i \beta_j - \sum_{i=1}^{d_2} (\beta_i - 1)(d_1 - 1). \tag{3}$$

Now define

$$f(d_1,d_2)=d_1d_2+\sum_{i=1}^{d_1}\sum_{j=1}^{d_2}\alpha_i\beta_j-\sum_{i=1}^{d_1}(\alpha_i-1)(d_2-1).$$

We shall see that for any fixed $d_2\geq 1$, f is an increasing function (as a function of d_1). Suppose $\widetilde{d}_1<d_1$, then

$$\begin{aligned}
f(\widetilde{d}_1,d_2)&=\widetilde{d}_1d_2+\sum_{i=1}^{\widetilde{d}_1}\sum_{j=1}^{d_2}\alpha_i\beta_j-\sum_{i=1}^{\widetilde{d}_1}(\alpha_i-1)(d_2-1)\\
&=d_1d_2+\sum_{i=1}^{d_1}\sum_{j=1}^{d_2}\alpha_i\beta_j-\sum_{i=1}^{d_1}(\alpha_i-1)(d_2-1)\\
&\quad-(d_1-\widetilde{d}_1)d_2\\
&\quad+\sum_{i=\widetilde{d}_1+1}^{d_1}\left((\alpha_i-1)(d_2-1)-\sum_{j=1}^{d_2}\alpha_i\beta_j\right)\\
&=f(d_1,d_2)+\sum_{i=\widetilde{d}_1+1}^{d_1}\alpha_i\left(d_2-1-\sum_{j=1}^{d_2}\beta_j\right)\\
&\quad+(d_1-\widetilde{d}_1)(1-2d_2)\\
&\leq f(d_1,d_2)
\end{aligned}$$

where the last inequality follows since $d_2\geq 1$ and $\beta_j\geq 1$ for $j=1,\ldots,d_2$.

Hence f is increasing for d_1. With a similar calculation we prove that f is an increasing function for d_2, hence the expressions (2) will be maximal if $d_1=\Delta_1$ and $d_2=\Delta_2$.

Analogously we prove that (3) is maximal if $d_1=\Delta_1$ and $d_2=\Delta_2$. Therefore, consider the expressions

$$\Delta_1\Delta_2+\sum_{i=1}^{\Delta_1}\sum_{j=1}^{\Delta_2}\alpha_i\beta_j-\sum_{i=1}^{\Delta_1}(\alpha_i-1)(\Delta_2-1) \tag{4}$$

and

$$\Delta_1\Delta_2+\sum_{i=1}^{\Delta_1}\sum_{j=1}^{\Delta_2}\alpha_i\beta_j-\sum_{i=1}^{\Delta_2}(\beta_i-1)(\Delta_1-1). \tag{5}$$

Since

$$\begin{aligned}
&\Delta_1\Delta_2+\sum_{i=1}^{\Delta_1}\sum_{j=1}^{\Delta_2}\alpha_i\beta_j-\sum_{i=1}^{\Delta_1}(\alpha_i-1)(\Delta_2-1)\\
&=\Delta_1\Delta_2+\sum_{i=1}^{\Delta_1}\sum_{j=1}^{\Delta_2}\Delta_1\Delta_2-\sum_{i=1}^{\Delta_1}(\Delta_1-1)(\Delta_2-1)\\
&\quad-\sum_{i=1}^{\Delta_1}\left(\sum_{j=1}^{\Delta_2}(\Delta_1\Delta_2-\alpha_i\beta_j)-(\Delta_1-\alpha_i)(\Delta_2-1)\right)
\end{aligned}$$

and since

$$\begin{aligned}
&\sum_{j=1}^{\Delta_2}(\Delta_1\Delta_2-\alpha_i\beta_j)-(\Delta_1-\alpha_i)(\Delta_2-1)\\
&\quad\geq\sum_{j=1}^{\Delta_2}(\Delta_1\Delta_2-\alpha_i\Delta_2)-(\Delta_1-\alpha_i)(\Delta_2-1)\\
&\quad=(\Delta_1-\alpha_i)\left(\Delta_2^2-\Delta_2+1\right)\geq 0
\end{aligned}$$

we find that (4) will be maximal if $\alpha_i=\Delta_1$ and $\beta_j=\Delta_2$ for all i, j. Similarly (5) will be maximal if $\alpha_i=\Delta_1$ and $\beta_j=\Delta_2$ for all i, j. Thus, we have

$$\begin{aligned}
|I_1|+|I_2|&\leq\Delta_1\Delta_2+\sum_{i=1}^{\Delta_1}\sum_{j=1}^{\Delta_2}\Delta_1\Delta_2-\sum_{i=1}^{\Delta_1}(\Delta_1-1)(\Delta_2-1)\\
&=\Delta^2-(\Delta_1-1)^2(\Delta_2-1)+\Delta_1+\Delta_2-1
\end{aligned}$$

and

$$\begin{aligned}
|I_1|+|I_2|&\leq\Delta_1\Delta_2+\sum_{i=1}^{\Delta_1}\sum_{j=1}^{\Delta_2}\Delta_1\Delta_2-\sum_{i=1}^{\Delta_2}(\Delta_2-1)(\Delta_1-1)\\
&=\Delta^2-(\Delta_2-1)^2(\Delta_1-1)+\Delta_1+\Delta_2-1
\end{aligned}$$

hence the result follows. ■

Theorem 1 implies the Δ^2-conjecture holds for all the direct product graphs with factors on at least three vertices.

Corollary 2: Let G_1 and G_2 be graphs with $\Delta_1\geq 2$ and $\Delta_2\geq 2$. Then $\lambda(G_1\times G_2)\leq\Delta^2$.

Proof: Suppose first that $\Delta_1=2$ and $\Delta_2=2$. Then, $\Delta^2=(\Delta_1\Delta_2)^2=16$. Observe that $|I_1|+|I_2|$ will be maximal if x is a vertex with maximal degree and all the neighbors of x have maximal degree. Hence, the graph $G_1\times G_2$ is locally isomorphic to $P_5\times P_5$, cf. Fig. 1. Clearly $|I_1|+|I_2|\leq 16$, hence $|I_1|+|I_2|\leq\Delta^2$.

Suppose next that $\Delta_1\geq 2$ and $\Delta_2\geq 2$, and not both of them are equal 2. Then

$$\begin{aligned}
\max\{&(\Delta_1-1)^2(\Delta_2-1)-\Delta_1-\Delta_2+1,\\
&(\Delta_2-1)^2(\Delta_1-1)-\Delta_1-\Delta_2+1\}\geq 0
\end{aligned}$$

hence Theorem 1 implies $|I_1|+|I_2|\leq\Delta^2$. ■

Corollary 2 assumes that both G_1 and G_2 have at least three vertices. Removing this assumption one would in particular prove that the Δ^2-conjecture holds for any bipartite graph. Indeed, if G is an arbitrary bipartite graph, then $K_2\times G$ consists of two connected components both isomorphic to G, see [14].

By arguments similar to those used in the proof of Theorem 1 one can also prove the Δ^2-conjecture for the strong product of graphs. The obtained bound is

$$\lambda(G_1\boxtimes G_2)\leq\Delta^2+\Delta_1+\Delta_2-5\Delta_1\Delta_2 \tag{6}$$

which clearly implies that

$$\lambda(G_1\boxtimes G_2)\leq\Delta^2-3.$$

We skip the proof of (6) since a better upper bound will be given in Section IV.

IV. Explicit Labelings

In the previous section we have shown that the Δ^2-conjecture holds for the direct and the strong products of graphs. The approach was based on inequality (1) that in turn follows from Algorithm A. Note that labelings obtained by Algorithm A are not uniquely defined and are computationally difficult to construct. From the practical point of view, we would like to have explicit labelings as well. In this section we give such explicit labelings. In the strong product case the proposed labeling in particular implies the Δ^2-conjecture. Moreover, the bound obtained here is better than (6).

We begin with the direct product as follows.

Proposition 3: Let G_1 and G_2 be nontrivial graphs with $|V(G_1)| = m$ and $|V(G_2)| = n$. Then $\lambda(G_1 \times G_2) \le mn - 1$.

Proof: Let $V(G_1) = \{x_0, \ldots, x_{m-1}\}$ and $V(G_2) = \{y_0, \ldots, y_{n-1}\}$. Consider the following labeling of the vertex set of $G_1 \times G_2$:

$$\ell(x_i, y_j) = \begin{cases} in + j, & \text{if } i \text{ is even} \\ (i+1)n - (j+1), & \text{if } i \text{ is odd.} \end{cases}$$

It is straightforward to verify that ℓ is an $L(2,1)$-labeling of $G_1 \times G_2$ and that the span of the labels is $mn - 1$. ∎

The explicit labeling of Proposition 3 implies the Δ^2-conjecture as soon as the degrees in factors are not very small. Moreover, it can also yield exact λ-numbers. We formulate this in the next corollaries.

Corollary 4: Let G_1 and G_2 be nontrivial graphs with $\Delta_1 \ge \sqrt{|V(G_1)|}$ and $\Delta_2 \ge \sqrt{|V(G_2)|}$. Then $\lambda(G_1 \times G_2) \le \Delta^2 - 1$.

Proof: By Proposition 3, $\lambda(G_1 \times G_2) \le |G_1||G_2| - 1 \le \Delta_1^2 \Delta_2^2 - 1 = \Delta^2 - 1$. ∎

Corollary 5: For any $n, m \ge 3$, $\lambda(K_n \times K_m) = nm - 1$.

Proof: By Proposition 3, $\lambda(K_n \times K_m) \le nm - 1$. Note that the diameter of $K_n \times K_m$ is 2. Hence, in any $L(2,1)$-labeling of $K_n \times K_m$ no two distinct vertices of $K_n \times K_m$ get the same label, thus $\lambda(K_n \times K_m) \ge nm - 1$. ∎

Let $\ell_1 : V(G_1) \to \mathbb{N}_0$ and $\ell_2 : V(G_2) \to \mathbb{N}_0$ be $L(2,1)$-labelings of G_1 and G_2, respectively. Let

$$X_i = \{u \in V(G_1) \mid \ell_1(u) = i\}$$

and

$$Y_i = \{u \in V(G_2) \mid \ell_2(u) = i\}$$

and let

$$\rho_1 = \max_{u \in V(G_1)} \{\ell_1(u)\} \text{ and } \rho_2 = \max_{u \in V(G_2)} \{\ell_2(u)\}.$$

Then, we have the following.

Theorem 6: Let G_1 and G_2 be nontrivial graphs and ℓ_1, ℓ_2, X_i, Y_i, ρ_1, ρ_2 as above. For $x \in X_k$ and $y \in Y_h$ let

$$\ell(x, y) = h(\rho_1 + 1) + k.$$

Then, ℓ is an $L(2,1)$-labeling of $G_1 \boxtimes G_2$ and

$$\lambda(G_1 \boxtimes G_2) \le (\rho_1 + 1)(\rho_2 + 1) - 1.$$

Proof: Suppose that $(u_1, u_2) \ne (v_1, v_2)$ and

$$d_{G_1 \boxtimes G_2}((u_1, u_2), (v_1, v_2)) = 2.$$

Then, $d_{G_1}(u_1, v_1) \le 2$ and $d_{G_2}(u_2, v_2) \le 2$, hence $\ell_1(u_1) \ne \ell_1(v_1)$ or $\ell_2(u_2) \ne \ell_2(v_2)$ and thus $v_1 \notin X_{\ell_1(u_1)}$ or $v_2 \notin X_{\ell_2(u_2)}$. It follows from the definition of ℓ that $\ell(u_1, u_2) \ne \ell(v_1, v_2)$. Suppose that

$$d_{G_1 \boxtimes G_2}((u_1, u_2), (v_1, v_2)) = 1.$$

Then, $d_{G_1}(u_1, v_1) \le 1$ and $d_{G_2}(u_2, v_2) \le 1$, hence $|\ell_1(u_1) - \ell_1(v_1)| \ge 2$ and $|\ell_2(u_2) - \ell_2(v_2)| \ge 2$. Let $\ell_1(u_1) = k_1$, $\ell_1(v_1) = k_2$, $\ell_2(u_2) = h_1$ and $\ell_2(v_2) = h_2$. Then, $\ell(u_1, u_2) = h_1(\rho_1 + 1) + k_1$ and $\ell(v_1, v_2) = h_2(\rho_1 + 1) + k_2$. Since $|h_1 - h_2| \ge 2$ and $0 \le k_1, k_2 \le \rho_1$ we find that $|\ell(u_1, u_2) - \ell(v_1, v_2)| \ge 2$. ∎

For the next corollary, we use the $\lambda(G) \le \Delta^2 + \Delta - 1$ bound of Král andŠkrekovski [17] that holds for any graph with the largest degree at least two.

Corollary 7: For any graphs G_1 and G_2 with $\Delta_1 \ge 2$ and $\Delta_2 \ge 2$

$$\lambda(G_1 \boxtimes G_2) \le \Delta^2 - \Delta_1^2\Delta_2 - \Delta_1\Delta_2^2 - \Delta_1^2 - \Delta_2^2 - \Delta_1\Delta_2 - 1.$$

Proof: By the bound of Král andŠkrekovski there exist labelings ℓ_1 and ℓ_2 of G_1 and G_2, such that $\rho_1 \le \Delta_1^2 + \Delta_1 - 1$ and $\rho_2 \le \Delta_2^2 + \Delta_2 - 1$. Therefore

$$\begin{aligned} \lambda(G_1 \boxtimes G_2) &\le \left(\Delta_1^2 + \Delta_1 + 1 - 1\right)\left(\Delta_2^2 + \Delta_2 + 1 - 1\right) - 1 \\ &= \Delta^2 - \Delta_1^2\Delta_2 - \Delta_1\Delta_2^2 \\ &\quad - \Delta_1^2 - \Delta_2^2 - \Delta_1\Delta_2 - 1. \end{aligned}$$

∎

Note that Corollary 7 immediately implies that for any graphs G_1 and G_2 with $\Delta_1 \ge 2$ and $\Delta_2 \ge 2$, $\lambda(G_1 \boxtimes G_2) \le \Delta^2 - 6$.

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