



Characterizing r -perfect codes in direct products of two and three cycles

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Abstract

An r -perfect code of a graph $G = (V, E)$ is a set $C \subseteq V$ such that the r -balls centered at vertices of C form a partition of V . It is proved that the direct product of C_m and C_n ($r \geq 1, m, n \geq 2r + 1$) contains an r -perfect code if and only if m and n are each a multiple of $(r + 1)^2 + r^2$ and that the direct product of $C_m, C_n,$ and C_ℓ ($r \geq 1, m, n, \ell \geq 2r + 1$) contains an r -perfect code if and only if $m, n,$ and ℓ are each a multiple of $r^3 + (r + 1)^3$. The corresponding r -codes are essentially unique. Also, r -perfect codes in $C_{2r} \times C_n$ ($r \geq 2, n \geq 2r$) are characterized.

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1. Introduction

The problem of efficient resource placement in a computer network or in a communication network can be naturally formulated as a search for a (perfect) r -code in the corresponding underlying graph [13].

The problem has been considered in several network topologies, for instance, in hypercubes [2] and in 3D torus [4]. In [21], the authors studied a new network, the diagonal mesh network, and compared it with the well-known toroidal mesh networks as models for parallel computations. Diagonal mesh networks correspond to the direct product of cycles, while toroidal mesh networks can be represented as the Cartesian product of cycles. As it turns out, cf. [13,21], the diagonal mesh networks surpass the toroidal mesh networks in many respects. Jha [12,13] studied partitions of the direct product of cycles into r -perfect codes and proved the following:

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Theorem 1.

- (i) [13] If $r \geq 1$ and m and n are each a multiple of $(r+1)^2 + r^2$ then (each connected component of) $C_m \times C_n$ can be partitioned into r -perfect codes.
- (ii) [12] If $r \geq 1$ and m, n , and ℓ are each a multiple of $(r+1)^3 + r^3$ then (each connected component of) $C_m \times C_n \times C_\ell$ can be partitioned into r -perfect codes.

(Similar constructions as the one for proving Theorem 1 have been used elsewhere, for instance, in [5] in the case of perfect codes in the Lee metric and in [6] for tilings of integer lattices with spheres defined by the Manhattan metric.)

Here we complement Theorem 1 by showing that $C_m \times C_n$ ($r \geq 1, m, n \geq 2r+1$) contains an r -perfect code if and only if m and n are each a multiple of $(r+1)^2 + r^2$ and that $G = C_m \times C_n \times C_\ell$ ($r \geq 1, m, n, \ell \geq 2r+1$) contains an r -perfect code if and only if m, n , and ℓ are each a multiple of $r^3 + (r+1)^3$. Moreover, in these cases codes are essentially unique. In addition, we also characterize r -perfect codes for $C_{2r} \times C_n$, where $r \geq 2$ and $n \geq 2r$. (For a related problem of determining the r -domination numbers of direct products of two paths see [16,17].)

The direct product $G \times H$ of graphs G and H is the graph defined on the Cartesian product of the vertex sets of the factors, with two vertices (u_1, u_2) and (v_1, v_2) adjacent if and only if $u_1v_1 \in E(G)$ and $u_2v_2 \in E(H)$. This product of graphs is commutative and associative in a natural way. Moreover, the direct product of two graphs is connected if and only if both factors are connected and at least one of them is not bipartite [22], cf. also [9]. If both factors are connected and bipartite, then their direct product consists of two connected components. In the special case when $G = C_{2m} \times C_{2n}$ it is in addition well known that the connected components of G are isomorphic. The direct product of graphs is one of the four standard graph products [9] and is known under many different names, for instance as the cardinal product, the Kronecker product and the categorical product. It is the product in the category of graphs [7] and has been considered from several points of view, cf. McKenzie [20] and Imrich [8].

For a graph $G = (V, E)$, the distance $d_G(u, v)$, or briefly $d(u, v)$, between vertices u and v , is defined

as the number of edges on a shortest u, v -path. For a vertex $v \in V$ let $B_r(v) = \{u \in V \mid d(u, v) \leq r\}$ be the r -ball centered at v . In particular, $N[v] = B_1(v)$ and $N(v) = N[v] \setminus \{v\}$. A set $C \subseteq V$ is an r -code in G if $B_r(u) \cap B_r(v) = \emptyset$ for any two distinct vertices $u, v \in C$. In addition, an r -code C is called an r -perfect code if $\{B_r(u) \mid u \in C\}$ forms a partition of V . For the results on codes in graphs up to 1991 see the monograph [18], while for some recent results see [3, 15]. Perfect codes in graphs arising from interconnection networks were studied in [19]. Since $C_m \times C_n$ is a 4-regular graph and $C_m \times C_n \times C_\ell$ is 8-regular, it is worth to add that the 1-perfect code problem remains NP-complete on k -regular graphs (for any fixed $k \geq 3$) [18, Theorem 7.2.2].

Throughout the paper we will set $V(C_n) = \{0, \dots, n-1\}$. Whenever applicable, the vertices of a cycle will be calculated modulo the number of its vertices. An explicit formula for the distance function in the direct product was first given by Kim in [14], but for our purposes the following approach from [1] is more useful.

Lemma 2. Let (a, x) and (b, y) be vertices of the direct product $X = G \times H$. Then $d_X((a, x), (b, y))$ is the smallest d such that there is an a, b -walk of length d in G and an x, y -walk of length d in H . In particular, if such walks do not exist, then (a, x) and (b, y) are in different connected components of X .

2. Products of two cycles

In this section we complement Theorem 1(i). For this sake, some preparation is needed.

Lemma 3. Let $r \geq 1$ and $n \geq m \geq 2r+1$ and let P be an r -perfect code of a connected component of $C_m \times C_n$. Assume that $(i, j) \in P$. Let $s = 2r+1$ and set

$$R_1 = \{(i+s, j+1), (i-1, j+s), \\ (i-s, j-1), (i+1, j-s)\},$$

$$R_2 = \{(i+1, j+s), (i-s, j+1), \\ (i-1, j-s), (i+s, j-1)\}.$$

If $P \cap R_k \neq \emptyset$, then $R_k \subseteq P$ ($1 \leq k \leq 2$).

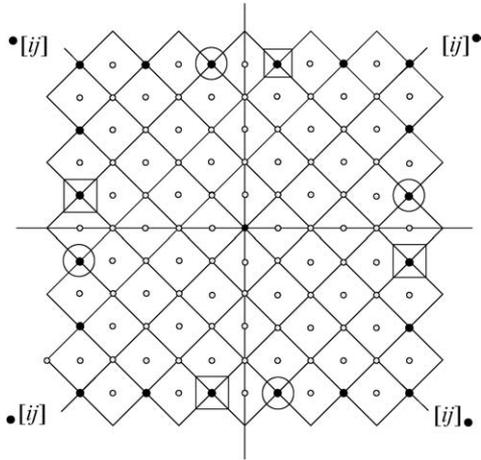


Fig. 1. Vertices from $[ij]^\bullet$, $\bullet[ij]$, $\bullet[ij]$, $[ij]_\bullet$ and (i, j) are marked black; R_1 is denoted by circles and R_2 by squares.

Proof. (The sets R_1 and R_2 are schematically shown on Fig. 1 for the case $r = 2$.) By symmetry it suffices to prove the lemma for R_1 . Using symmetry again we may assume that $(i + s, j + 1) \in P$. Let

$$[ij]^\bullet = \{(i + s, j + 1 + 2k), (i + s - 2k, j + s); 0 \leq k \leq (s - 1)/2\},$$

$$\bullet[ij] = \{(i - 1 - 2k, j + s), (i - s, j + s - 2k); 0 \leq k \leq (s - 1)/2\},$$

$$\bullet[ij] = \{(i - s, j - 1 - 2k), (i - s + 2k, j - s); 0 \leq k \leq (s - 1)/2\},$$

$$[ij]_\bullet = \{(i + 1 + 2k, j - s), (i + s, j - s + 2k); 0 \leq k \leq (s - 1)/2\},$$

cf. Fig. 1. Since $(i, j) \in P$, P does not contain any vertex of $B_{2r}(i, j)$, hence the vertices $(i \pm (r + 1), j \pm (r + 1))$ are not in P . As these vertices do not belong to $B_r(i, j)$, P must contain exactly one vertex of each of the sets $[ij]^\bullet$, $\bullet[ij]$, $\bullet[ij]$ and $[ij]_\bullet$. Since $(i + s, j + 1) \in [ij]^\bullet$, P does not contain any other vertex of $[ij]^\bullet$. Consider now the vertex $(i + r - 1, j + r + 1)$. This vertex can only lie in $B_r(i - 1, j + s)$, where $(i - 1, j + s) \in \bullet[ij]$. Hence $(i - 1, j + s) \in P$. If we repeat analogous arguments for the vertices $(i - r - 1, j + r - 1)$ and $(i - r + 1, j - r - 1)$, we get $(i - s, j - 1) \in P$ and $(i + 1, j - s) \in P$. We conclude that $R_1 \subseteq P$. \square

Lemma 4. Under the assumptions of Lemma 3, either $R_1 \subseteq P$ and $R_2 \cap P = \emptyset$, or $R_2 \subseteq P$ and $R_1 \cap P = \emptyset$.

Proof. Suppose that $R_1 \cap P \neq \emptyset$ and $R_2 \cap P \neq \emptyset$. Then by Lemma 3, $R_1 \subseteq P$ and $R_2 \subseteq P$. But since $R_1 \cup R_2$ contains vertices at distance two, this is not possible.

Suppose now that $R_2 \cap P = \emptyset$. Then $(i + s, j - 1) \notin P$ and the vertex $(i + r + 1, j + r - 1)$ must lie in one of the r -balls $B_r(i + s, j + 1 + 2k)$, $0 \leq k \leq (s - 3)/2$. If we consider the vertex $(i + r - 1, j + r + 1)$ as in the proof of Lemma 3, we get that $(i - 1, j + s) \in P$. By the same lemma, $R_1 \subseteq P$. The second assertion can be shown analogously. \square

Lemma 5. The assumptions of Lemma 3 together with $R_1 \subseteq P$ imply that $\{(i + 2, j - 2s), (i + 2s, j + 2), (i - 2, j + 2s), (i - 2s, j - 2)\} \subseteq P$.

Proof. Since $(i + 1, j - s) \in R_1 \subseteq P$, Lemma 4 implies that P contains exactly one of the following sets

$$\begin{aligned} &\{(i + 1 + s, j - s + 1), (i, j), (i + 1 - s, j - s - 1), \\ &\quad (i + 2, j - 2s)\}, \\ &\{(i + 2, j), (i + 1 - s, j - s + 1), (i, j - 2s), \\ &\quad (i + 1 + s, j - s - 1)\}. \end{aligned}$$

On the other hand, since $(i, j) \in P$, Lemma 3 implies that P contains the first set. Hence $(i + 2, j - 2s) \in P$.

For the other three vertices of R_1 we similarly get that

$$\begin{aligned} &\{(i + 2s, j + 2), (i - 2, j + 2s), (i - 2s, j - 2)\} \\ &\subseteq P. \quad \square \end{aligned}$$

Let P be an r -perfect code of a connected component of $G = C_m \times C_n$ and $(i, j) \in P$. By Lemma 4, either $R_1 \subseteq P$ or $R_2 \subseteq P$. Suppose $R_1 \subseteq P$. Consider the sets $P_0 = \{(i - t, j + ts) \mid t \in \mathbb{N}\}$ and $Q_0 = \{(i + qs, j + q) \mid q \in \mathbb{N}\}$. Let ℓ_1 be the smallest integer such that $(i + \ell_1 s, j + \ell_1) \in P_0$ and ℓ_2 the smallest integer such that $(i - \ell_2, j + \ell_2 s) \in Q_0$. Define $P_k = \{(i - t + ks, j + ts + k) \mid t \in \mathbb{N}\}$ for $k = 1, \dots, \ell_1 - 1$ and $Q_k = \{(i - k + qs, j + q + ks) \mid t \in \mathbb{N}\}$ for $k = 1, \dots, \ell_2 - 1$. By Lemma 5, $P_k \subseteq P$ and $Q_k \subseteq P$. Then

$$P = \bigcup_{k=0}^{\ell_1-1} P_k = \bigcup_{k=0}^{\ell_2-1} Q_k.$$

Since $|P_k| = pm$ and $|Q_k| = qn$ we infer that $|P| = \ell_1 pm = \ell_2 qn$. On the other hand, $|B_r(u)| = (r+1)^2 + r^2$, cf. [13, Lemma 2.1], hence $|P| = mn / ((r+1)^2 + r^2)$. We conclude that m and n are each a multiple of $(r+1)^2 + r^2$. Note that the above set P is uniquely determined by its vertices (i, j) and $(i+s, j+1)$. We denote this set by $P_{ij}(i+s, j+1)$.

In the case when $R_2 \subseteq P$ we argue analogously that m and n are each a multiple of $(r+1)^2 + r^2$. In this case P is uniquely determined by the vertices (i, j) and $(i+1, j+s)$. We denote this set by $P_{ij}(i+1, j+s)$.

Suppose that $P = P_{ij}(i+s, j+1)$ or $P = P_{ij}(i+1, j+s)$. Then it follows from Lemma 2 that $d(x, y) \geq 2r+1$ for any vertices $x, y \in C_m \times C_n$. We do not give details as this also follows directly from Theorem 1. The following result then follows immediately.

Theorem 6. *Let $r \geq 1$, $n \geq m \geq 2r+1$, and $G = C_m \times C_n$. Then a connected component G contains an r -perfect code P if and only if m and n are each a multiple of $(r+1)^2 + r^2$. Moreover, if $(i, j) \in P$ then either $P = P_{ij}(i+s, j+1)$ or $P = P_{ij}(i+1, j+s)$.*

Combining Theorems 6 and 1(i) we also have:

Corollary 7. *Let $r \geq 1$, $n \geq m \geq 2r+1$, and $G = C_m \times C_n$. Then (each connected component of) G can be partitioned into r -perfect codes if and only if m and n are each a multiple of $(r+1)^2 + r^2$.*

Corollary 8. *Let $r \geq 1$ and $n \geq 2r+1$. Then $C_{2r+1} \times C_n$ contains no r -perfect code.*

Proof. By Theorem 6, $2r+1$ must be a multiple of $(r+1)^2 + r^2$ if $C_{2r+1} \times C_n$ would contain an r -perfect code. \square

The conditions $r \geq 1$ and $n \geq m \geq 2r+1$ in the above results assure that for any vertex (i, j) of $G = C_m \times C_n$ we have $|B_r(i, j)| = (r+1)^2 + r^2$. This is no longer true if either n or m or both numbers are smaller than $2r+1$ since then we have vertices in $B_r(i, j)$ which can be reached by paths of length at most r by moving in both directions around the ‘torus’. The first such case is when $r \geq 2$ and $n \geq 2r$. In this case we have:

Proposition 9. *Let $r \geq 2$ and $n \geq 2r$. Then $C_{2r} \times C_n$ contains an r -perfect code precisely in the following two cases:*

- (i) $n = 2r$,
- (ii) $n > 2r$ and $n = \ell(2r+1)$, for some $\ell \in \mathbb{N}$.

Proof. Let $n = 2r$. Then $C_{2r} \times C_{2r}$ consists of two isomorphic components H_1 and H_2 . Select any vertices $u_1 \in H_1$ and $u_2 \in H_2$. Then, since $|B_r(u_i)| = 2r^2 = |V(H_i)|$, $C_{2r} \times C_{2r}$ contains an r -perfect code.

Suppose $n > 2r$ and let P be an r -perfect code of $C_{2r} \times C_n$. We first claim that n is a multiple of $2r+1$. Let $|P| = k$ and let B be a arbitrary r -ball of G . Then $|B| = r(2r+1)$ and hence $krn = kr(2r+1)$. Since $n \in \mathbb{N}$ it follows that k is even, and hence it follows that n is a multiple of $2r+1$.

It remains to show that $X = C_{2r} \times C_n$ contains an r -perfect code whenever $n = \ell(2r+1)$ for some $\ell \in \mathbb{N}$. Set

$$P = \bigcup_{t=0}^{\ell-1} \{(r-1, t(2r+1)), (r, t(2r+1))\}.$$

Lemma 2 implies that

$$d_X((r-1, p(2r+1)), (r, q(2r+1))) = \ell(2r+1)$$

whenever $p = q$ and at least $2r+1$ whenever $p \neq q$. Since $|B_r(x)| = r(2r+1)$ for any vertex x of X , we have

$$\sum_{x \in P} |B_r(x)| = 2\ell r(2r+1) = |X|. \quad \square$$

We note that in the case $X = C_{2r} \times C_{\ell(2r+1)}$ it is also easy to obtain a partition of $V(X)$ into r -perfect codes.

3. Products of three cycles

Similar arguments as for the product of two cycles also work for the product of three cycles. However the situation is a little bit more involved and the proof is rather lengthy so we only sketch it, the details can be found in [10].

To explain the ideas as briefly as possible, we introduce the following notation. Let $x = (i, j, k)$ be an arbitrary vertex of $C_m \times C_n \times C_\ell$, let $r \geq 1$ and $s =$

$2r + 1$. For $a_1, a_2, a_3 \in \{\oplus, \ominus, +, -\}$ let $x^{a_1 a_2 a_3}$ be the vertex $(i + i', j + j', k + k')$, where $i' = s, -s, 1, -1$ if $a_1 = \oplus, \ominus, +, -$, respectively, and j', k' are defined analogously. For instance,

$$x^{\oplus\oplus\oplus} = (i + s, j + 1, k + 1)$$

and

$$x^{-\ominus+} = (i - 1, j - s, k + 1).$$

In addition, let

$$X^{\oplus**} = \{x^{\oplus\oplus\oplus}, x^{\oplus\oplus-}, x^{\oplus\ominus-}, x^{\oplus--}\},$$

and define the sets $X^{\ominus**}, X^{*+\oplus}, X^{*\ominus+}, X^{**\oplus}$, and $X^{**\ominus}$ analogously. For instance,

$$X^{*+\oplus} = \{x^{+\oplus+}, x^{+\oplus-}, x^{-\ominus+}, x^{-\ominus-}\}.$$

With this notation we first have the following two lemmata.

Lemma 10. Let P be an r -perfect code of $C_m \times C_n \times C_\ell$ ($m, n, \ell \geq 2r + 1$), and let $x = (i, j, k) \in P$. Then

$$\begin{aligned} |X^{\oplus**} \cap P| &= |X^{\ominus**} \cap P| = |X^{*+\oplus} \cap P| \\ &= |X^{*\ominus+} \cap P| = |X^{**\oplus} \cap P| \\ &= |X^{**\ominus} \cap P| = 1. \end{aligned}$$

Lemma 11. Let P be an r -perfect code of $C_m \times C_n \times C_\ell$ ($m, n, \ell \geq 2r + 1$), let $x = (i, j, k) \in P$, and let:

$$\begin{aligned} R_1 &= \{x^{\oplus\oplus\oplus}, x^{-\oplus+}\}, & R_2 &= \{x^{\oplus\oplus\oplus}, x^{-\oplus-}\}, \\ R_3 &= \{x^{\oplus\oplus-}, x^{-\oplus+}\}, & R_4 &= \{x^{\oplus\oplus-}, x^{-\oplus-}\}, \\ R_5 &= \{x^{\oplus\ominus-}, x^{+\oplus+}\}, & R_6 &= \{x^{\oplus\ominus-}, x^{+\oplus-}\}, \\ R_7 &= \{x^{\oplus--}, x^{+\oplus+}\}, & R_8 &= \{x^{\oplus--}, x^{+\oplus-}\}. \end{aligned}$$

Then there is exactly one $i \in \{1, \dots, 8\}$ such that $R_i \subseteq P$. In addition, P is uniquely determined by $R_i \cup \{x\}$.

The elements and the corresponding r -balls (for $r = 2$) of the code P are shown in Fig. 2. The third component of the center of each r -ball is written in the upper right corner of the corresponding ball.

The above two lemmata reflect the local structure of an r -perfect code. In order to determine the lengths of the cycles in mind, we next consider the code on a larger scale and obtain the following lemma.

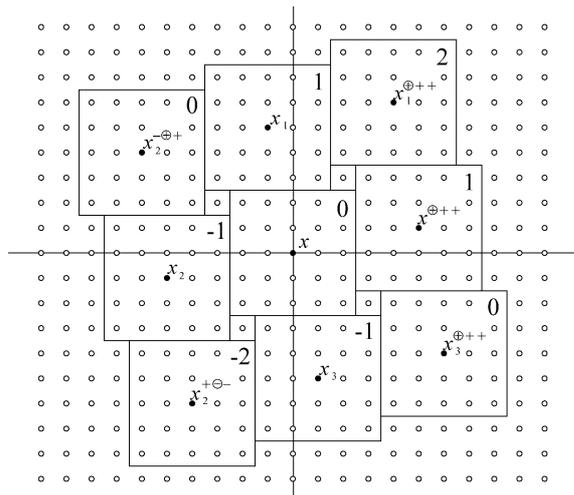


Fig. 2. 2-balls of code P .

Lemma 12. Let P be an r -perfect code of $C_m \times C_n \times C_\ell$ ($m, n, \ell \geq 2r + 1$) and $R_1 \cup \{(i, j, k)\} \subseteq P$. Then for some $t \in \mathbb{N}$ we have

$$\begin{aligned} (i - 1, j + s, k + 1) \\ = (i - 1 + 2t((r + 1)^3 + r^3), j + s, k + 1). \end{aligned}$$

Finally, we also have the following lemma.

Lemma 13. Let P be an r -perfect code of $C_m \times C_n \times C_\ell$ ($m, n, \ell \geq 2r + 1$) and $R_1 \cup \{(i, j, k)\} \subseteq P$. Let t_0 be the smallest integer such that

$$\begin{aligned} (i - 1, j + s, k + 1) \\ = (i - 1 + 2t_0((r + 1)^3 + r^3), j + s, k + 1). \end{aligned}$$

Then $2t_0((r + 1)^3 + r^3) \leq 2m$.

Lemmas 12 and 13 are formulated and proved for the case when $R_1 \cup \{(i, j, k)\} \subseteq P$. By symmetry both results also hold for all cases $R_i \cup \{(i, j, k)\} \subseteq P$ ($i = 2, \dots, 8$). Therefore, combining these two lemmas we get:

Corollary 14. Let $r \geq 1$, $m, n, \ell \geq 2r + 1$, and let P be an r -perfect code of $C_m \times C_n \times C_\ell$. Then m is a multiple of $r^3 + (r + 1)^3$.

By the commutativity of the direct product the result of Corollary 14 can also be applied to the other

two cycles of the product. We thus have as the main result of this section:

Theorem 15. *Let $r \geq 1$, $m, n, \ell \geq 2r + 1$, and $G = C_m \times C_n \times C_\ell$. Then each connected component of G contains an r -perfect code if and only if m , n , and ℓ are each a multiple of $r^3 + (r + 1)^3$.*

Finally, combining Theorem 15 with Theorem 1(ii) we can also state:

Corollary 16. *Let $r \geq 1$, $m, n, \ell \geq 2r + 1$, and $G = C_m \times C_n \times C_\ell$. Then each connected component of G can be partitioned into r -perfect codes if and only if m , n , and ℓ are each a multiple of $r^3 + (r + 1)^3$.*

4. Concluding remarks

The direct product of graphs has found several applications in computer science, engineering and related areas. In particular, direct products of cycles have been proposed as a network model for parallel computations. The applicability of these graphs is in particular imposed by their rich cycle structure, cf. [11]. Recent developments from [1] also enable us to tackle location type problems in direct products of graph more efficiently than before.

The problem of efficient resource placement in a computer/communication network can be modeled as a search for (perfect) codes in the corresponding underlying graphs. The major contribution of our paper is an explicit characterization of perfect codes in direct products of two and three cycles. It is therefore hoped that our results will have an impact on efficient resource placement in diagonal meshes.

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