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On induced and isometric embeddings of graphs into the strong product of paths☆

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Abstract

The strong isometric dimension and the adjacent isometric dimension of graphs are compared. The concepts are equivalent for graphs of diameter 2 in which case the problem of determining these dimensions can be reduced to a covering problem with complete bipartite graphs. Using this approach several exact strong and adjacent dimensions are computed (for instance of the Petersen graph) and a positive answer is given to the Problem 4.1 of Fitzpatrick and Nowakowski [The strong isometric dimension of finite reflexive graphs, Discuss. Math. Graph Theory 20 (2000) 23–38] whether there is a graph G with the strong isometric dimension bigger that $\lceil |V(G)|/2\rceil$.

Keywords: Strong product of graphs; Adjacent isometric dimension; Strong isometric dimension

1. Introduction

Graph products offer a variety of possibilities to introduce different graph dimensions. Nešetřil and Rödl [9] presented a general framework that for any class of graphs and for any graph product gives a different dimension concept. Slightly more precisely, the dimension of G is defined as the minimum number of factor graphs (from a selected class of graphs and with respect to a selected graph product) such that G embeds as an *induced subgraph* into their product. Nešetřil and Rödl proved a nice general result that either a fixed dimension is equal to 1 or tends to infinity. Earlier, Poljak and Pultr [10] introduced three specific related dimensions: the dimension of bipartite graphs with respect to induced embeddings into the direct product of paths of length 3, the dimension with respect to induced embeddings into the strong product of paths of length 2, and the dimension with respect to induced embeddings into the direct product of complete graphs. The latter dimension was introduced by Nešetřil and Rödl [8], see also [3], while for the bipartite dimension we refer to [11].

Isometric embeddings of graphs into product graphs were also intensively studied, cf. [7]. A classical result of Graham and Winkler [6] asserts that any graph can be canonically isometrically embedded into the Cartesian product of graphs. Since this embedding is unique among all irredundant isometric embeddings with respect to the largest possible number of factors, the latter number is called the *isometric dimension* of a graph. We also add that four different dimensions

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(product dimension, isometric dimension, induced dimension, and dimension) with respect to the Cartesian product are treated in [1].

Back in 1938, Schönberg [12] proved that every connected graph admits an isometric embedding into the strong product of paths, cf. [7, Proposition 5.2]. Hence one can define the *strong isometric dimension*, $\mathrm{idim}(G)$, of a graph G as the least number k such that G embeds isometrically into the strong product of k paths. Recently, Fitzpatrick and Nowakowski [4] extensively studied this concept and obtained several interesting results, see also [5].

In the next section, we introduce the strong isometric dimension and the adjacent isometric dimension of a graph and note that the latter dimension was independently—and in different contexts—introduced in [2,10]. In Section 3, the adjacent and the strong dimension are compared and the computation of the strong (adjacent) isometric dimension for graphs of diameter 2 is reduced to a covering problem of their complements with complete bipartite graphs. In the last section, we use this approach to construct graph with large dimensions, thus in particular answering a question of Fitzpatrick and Nowakowski from [4].

2. Preliminaries

Let (M, d_1) and (N, d_2) be metric spaces. Then a mapping $f : M \to N$ is an *isometric embedding* if $d_2(f(x), f(y)) = d_1(x, y)$ for any $x, y \in M$. In particular, we say a subgraph H of a graph G is *isometric* if $d_H(u, v) = d_G(u, v)$ for all vertices u, v of H.

The *strong product* $G = \boxtimes_{i=1}^{k} G_i$ of graphs $G_1, \ldots, G_k$ is the graph defined on the Cartesian product of the vertex sets of the factors, two distinct vertices $(u_1, u_2, \ldots, u_k)$ and $(v_1, v_2, \ldots, v_k)$ being adjacent if and only if u_i is equal or adjacent to v_i in G_i for $i = 1, 2, \ldots, k$. The strong product $\boxtimes_{i=1}^{k} G$ is called the *kth strong power of* G and will be denoted $G^{\boxtimes k}$.

Let G be a graph, then by $d_G(u, v)$ we denote the standard graph distance, that is, the number of edges on a shortest u, v-path. The following result is well-known, cf. [7, Lemma 5.1].

Lemma 1. *Let $G = \boxtimes_{i=1}^{k} G_i$ be the strong product of connected graphs. Then*

$$d_G(u, v) = \max_{1 \leqslant i \leqslant k} d_{G_i}(u_i, v_i).$$

By P_n we denote the path of length n. We will always assume $V(P_n) = \{0, 1, \ldots, n\}$, where i is adjacent to $i + 1$ for $i = 0, \ldots, n - 1$.

The *strong isometric dimension*, $\mathrm{idim}(G)$, of a graph G is the least number k such that for some $n \geqslant 1$, G isometrically embeds into $P_n^{\boxtimes k}$. In fact, the length of the paths in the product can be bounded as follows. (Recall that the *diameter*, $\mathrm{diam}(G)$, of a connected graph G is the maximum distance between any two vertices of G.)

Lemma 2. *Let G be a graph of diameter d. If G can be isometrically embedded into a $P_n^{\boxtimes k}$ then it can be isometrically embedded into $P_d^{\boxtimes k}$.*

Proof. Let $f : V(G) \to V(P_n^{\boxtimes k})$ be an isometric embedding. Hence $f(u) = (u^{(1)}, \ldots, u^{(k)})$, where for $i = 1, \ldots, k$, $u^{(i)} \in \{0, 1, \ldots, n\}$. Set $M_i = \max\{u^{(i)} \mid u \in V(G)\}$ and $m_i = \min\{u^{(i)} \mid u \in V(G)\}$. Since $\mathrm{diam}(G) = d$, Lemma 1 implies that $M_i - m_i \leqslant d$ for $i = 1, \ldots, k$. Then the mapping $g : V(G) \to V(P_d^{\boxtimes k})$ defined by $g(u) = (u^{(1)} - m_1, \ldots, u^{(k)} - m_k)$ is an isometry. □

As we already mentioned, Poljak and Pultr [10] introduced a graph dimension (giving it no name) as the smallest number n such that G is an induced subgraph of $P_2^{\boxtimes n}$. Independently (and at the same time) Dewdney [2] proceeded as follows. For an arbitrary graph G, the *adjacency metric* $a : V(G) \times V(G) \to \{0, 1, 2\}$ is defined by $a(u, v) = 0$ if $u = v$; $a(u, v) = 1$ if $uv \in E(G)$; and $a(u, v) = 2$ otherwise. Then the *adjacent isometric dimension*, $\mathrm{adim}(G)$, of G is the smallest number n such that the metric space (G, a) isometrically embeds into the metric space $(\mathbb{Z}_3^n, d_\infty)$. Now, it is easy to see that $\mathrm{adim}(G)$ equals the smallest integer n such that G is an induced subgraph of $P_2^{\boxtimes n}$, hence both concepts are equivalent.

3. Strong and adjacent isometric dimension

In this section we compare the adjacent isometric dimension and the strong isometric dimension of a graph. Any of the two dimensions can be arbitrarily bigger than the other, consider the following examples. Clearly, for any n we have $\mathrm{idim}(P_n) = 1$, while $\mathrm{adim}(P_n) = \lceil \log_2 n \rceil$ as proved in [10]. On the other hand, $\mathrm{idim}(C_{2n}) = n$, see [4], but $\mathrm{adim}(C_{2n}) = \lceil \log_2 2n \rceil$, see [10].

Let $G + x$ be the graph obtained from G by adding the vertex x and joining it to every vertex of G. Then we have:

Proposition 3. *Let G be a graph. Then*

$$\mathrm{adim}(G) \leqslant \mathrm{idim}(G + x) \leqslant \mathrm{adim}(G) + 1.$$

Proof. Let $\mathrm{adim}(G)=k$ and let f be a corresponding embedding, so that for a vertex u of G, $f(u)=((f(u)_1, \ldots, f(u)_k)$ with $f(u)_i \in \{0, 1, 2\}$. Define now a mapping g from $G + x$ into the strong product of $k + 1$ paths of length 2 as follows. Set $g(x) = (1, \ldots, 1, 1)$ and $g(u) = ((f(u)_1, \ldots, f(u)_k, 0)$ for any $u \neq x$. Since $G + x$ is of diameter at most 2, it is straightforward to verify that g is an isometric embedding (with respect to the usual distance). We conclude that $\mathrm{idim}(G + x) \leqslant \mathrm{adim}(G) + 1$.

For the first inequality just observe that vertices u and v of G are not adjacent if and only if they are on distance 2 in $G + x$. Hence, a strong isometric embedding of $G + x$ is also an adjacent isometric embedding. □

Invoking Schönberg's result that idim is well-defined, Proposition 3 gives an alternative argument to the ones from [2,10] that adim is well-defined as well. Note also that an induced subgraph of diameter 2 is an isometric subgraph, hence $\mathrm{idim}(G) = \mathrm{adim}(G)$ holds for all graphs G of diameter (at most) 2. For such graphs we have the following theorem due to Dewdney. The proof's idea is also included here since it will be used later. $K_1 = K_{1,0}$ is treated as a complete bipartite graph.

Theorem 4. *Let G be a graph with* $\mathrm{diam}(G) = 2$. *Then* $\mathrm{idim}(G)$ *is equal to the smallest r for which the edges of* $\overline{G}$ *can be covered with complete bipartite subgraphs* $B_1, \ldots, B_r$ *of* $\overline{G}$, *such that for any edge e of G there exists a* B_i *with one end of e belonging to* B_i *but not the other.*

Proof (*Sketch*). Suppose that $\mathrm{idim}(G) = r$. By Lemma 2 there is an isometric embedding $f : V(G) \to V(H)$, where $H = P_2^{\boxtimes r}$. For $i = 1, 2, \ldots, r$ let B_i be a complete bipartite graph with the bipartition $X_i + Y_i$, where $X_i = \{u \in V(G) \mid (f(u))_i = 0\}$ and $Y_i = \{u \in V(G) \mid (f(u))_i = 2\}$. Then these B_i's form a required covering.

Conversely, assume that the edges of $\overline{G}$ can be covered with r complete bipartite graphs B_i with bipartitions $V(B_i) = X_i + Y_i$, $i = 1, 2, \ldots, r$, such that for any edge uv of G there is an i with $u \in B_i$ and $v \notin B_i$. Define a mapping $f : V(G) \to V(P_2^{\boxtimes r})$ with

$$(f(u))_i = \begin{cases} 0, & u \in X_i, \\ 2, & u \in Y_i, \\ 1 & \text{otherwise.} \end{cases}$$

The (sketch of the) proof is completed by noting that f is an isometry. □

Consider the complete graph on four vertices minus an edge $K_4 - e$. It is of diameter 2 and its complement consists of an edge and two isolated vertices so that its edge(s) can be covered with one complete bipartite graph $K_{1,1}$. Since $\mathrm{idim}(K_4 - e) = 2$, we see that the condition of Theorem 4 requiring that for any edge uv of G there is an i with $u \in B_i$ and $v \notin B_i$ cannot be dropped. Moreover, this example also shows that for an optimal embedding we may (and must) use a K_1 in a covering with complete bipartite graphs. However, it would be nice to simplify the conditions of Theorem 4. In many cases this can indeed be done as follows.

Theorem 5. *Let G be a graph with* $\mathrm{diam}(G) = 2$ *and let any edge of G be contained in an induced path on three vertices. Then* $\mathrm{idim}(G)$ *is equal to the smallest r such that the edges of* $\overline{G}$ *can be covered with r complete bipartite subgraphs.*

Proof. By Theorem 4 we only need to prove that if $\overline{G}$ is covered with r complete bipartite graphs B_i with bipartitions $V(B_i) = X_i + Y_i$, for $i = 1, 2, \ldots, r$, then G embeds isometrically into $H = P_2^{\boxtimes r}$. We define f as in the (sketch of the) proof of Theorem 4. If $d_G(u, v) = 2$, then uv is an edge of $\overline{G}$. Hence uv is covered with at least one graph B_i, thus $|(f(u))_i - (f(v))_i| = 2$ and so $d_H(f(u), f(v)) = 2$. Let now u and v be vertices with $d_G(u, v) = 1$. If for some i we have $u, v \in B_i$, then since u and v are not adjacent in $\overline{G}$, we have either $(f(u))_i = (f(v))_i = 0$ or $(f(u))_i = (f(v))_i = 2$. It follows that $\max_i \{|(f(u))_i - (f(v))_i|\} \leqslant 1$. To see that this maximum equals 1, let $u \to v \to w$ be an induced path that exists by the theorems assumption. Then $uw \in E(\overline{G})$. Let B_i be a complete bipartite graph that covers the edge uw. Then $v \notin B_i$, hence by Theorem 4 G isometrically embeds into H. □

4. Graphs with large isometric dimension

Fitzpatrick and Nowakowski [4, Question 4.1] asked whether there is a graph G with $\mathrm{idim}(G) \geqslant \lceil |V(G)|/2 \rceil$? Consider the following example from [2]. Let D be the graph obtained from $K_{3,3}$ by subdividing one of its edges. Then $\mathrm{adim}(D) = 5$ and since $\mathrm{diam}(D) = 2$ we also have $\mathrm{idim}(D) = 5$. In addition, note that the join of graphs of diameter 2 is a graph of the same diameter, hence we can take the join of an arbitrary number of copies of D to obtain graphs with $\mathrm{idim}(G) > \lceil |V(G)|/2 \rceil$. This construction is in a way trivial since $\overline{G}$ is disconnected. In this section we construct graphs with $\mathrm{idim}(G) > \lceil |V(G)|/2 \rceil$ such that $\overline{G}$ is 2-connected. We also give several exact dimensions, for instance the dimension of the Petersen graph is 5 and the dimension of the complement of the generalized Petersen graph $P(n, k)$ with n even, $k \geqslant 2$, and n and k being relatively prime equals n.

For the next theorem we recall the following concepts. A set X of vertices of a graph G is called a *vertex cover* if every edge of G is incident with a vertex of X. The minimum size of a vertex cover of G is denoted $\beta(G)$ and the size of a largest independent set by $\alpha(G)$. $\Delta(G)$ and $\delta(G)$ are the largest and the smallest degree of G, respectively. The minimum length of a cycle of G is the *girth* $g(G)$ of G.

Theorem 6. *Let G be a 2-connected graph on n vertices with $g(G) \geqslant 5$. Then*

$$\mathrm{idim}(\overline{G}) = \mathrm{adim}(\overline{G}) \geqslant \left\lceil \frac{n}{2} \frac{\delta(G)}{\Delta(G)} \right\rceil .$$

Moreover, the equality holds if and only if $\alpha(G) = n - \lceil (n/2)\delta(G)/\Delta(G) \rceil$.

Proof. We first show that $\mathrm{diam}(\overline{G}) = 2$. As G is connected, $\mathrm{diam}(\overline{G}) \geqslant 2$. So let u and v be arbitrary nonadjacent vertices of $\overline{G}$. Then uv is an edge of G and as G is 2-connected, uv lies in a cycle of G. A shortest such cycle is induced and of length at least 5, hence $d_{\overline{G}}(u, v) = 2$ and so $\mathrm{diam}(\overline{G}) \leqslant 2$.

Let $e = uv$ be an arbitrary edge of $\overline{G}$. We claim that e is contained in an induced path of $\overline{G}$ on three vertices. Let w be a neighbor of u in G. If wv is not an edge of G, then uvw induces a path on three vertices in $\overline{G}$. So let $vw \in E(G)$. Consider now another neighbor of u in G, say w'. (It exists as G is 2-connected.) As $g(G) \geqslant 5$, $vw' \notin E(G)$, but now uvw' is an induced path in $\overline{G}$.

By the above and Theorem 5 it follows that $\mathrm{idim}(\overline{G})$ is equal to the smallest r such that the edges of G can be covered with r complete bipartite subgraphs. As G is C_4-free, the complete bipartite graphs from such a covering can only be copies of the stars $K_{1,s}$, where $s \leqslant \Delta(G)$. Therefore,

$$\mathrm{idim}(\overline{G}) \geqslant \left\lceil \frac{|E(G)|}{|E(K_{1,\Delta(G)})|} \right\rceil \geqslant \left\lceil \frac{n\delta(G)}{2} \frac{1}{\Delta(G)} \right\rceil ,$$

which proves the first assertion.

Concerning the equality, note that the smallest number of stars that cover the edges of G is just the vertex cover number of G. Therefore, the equality holds if and only if $\beta(G) = \lceil (n/2)\delta(G)/\Delta(G) \rceil$. Since $\alpha(G) + \beta(G) = n$, cf. [13, Lemma 3.1.21], the second assertion follows. □

Corollary 7. *Let G be a 2-connected, regular graph with $g(G) \geqslant 5$. Then* $\mathrm{idim}(\overline{G}) = \mathrm{adim}(\overline{G}) \geqslant \lceil n/2 \rceil$, *where the equality holds if and only if* $\alpha(G) = \lfloor n/2 \rfloor$.

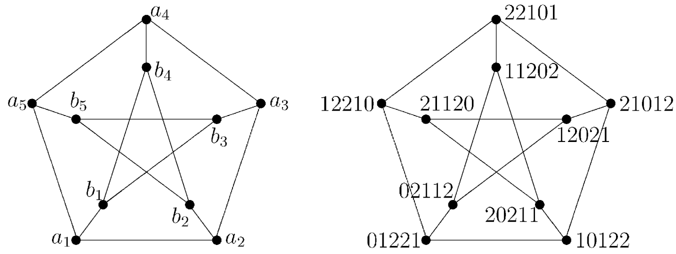

Fig. 1. The Petersen graph and its isometric embedding into $P_2^{\boxtimes 5}$.

Let G_k, $k \geqslant 1$, be the graph obtained from C_{6k+3} and a vertex w with connecting w to every third vertex of the C_{6k+3} (so that w is of degree $2k+1$). Then G_k has $6k+4$ vertices and it is easy to see that $\alpha(G_k) = 3k+1$. Then Theorem 6 implies that $\mathrm{idim}(\overline{G_k}) > 3k+2 = |V(\overline{G_k})|/2$, thus giving an infinite family of graphs with the strong isometric dimension bigger that half of its order.

Corollary 7 can also be used to obtain additional exact dimensions of graphs. For instance, in [4] is proved that for $n \geqslant 4$, $\mathrm{idim}(C_n) = \lceil n/2 \rceil$. Thus, Corollary 7 implies that for $n \geqslant 5$, $\mathrm{idim}(\overline{C_n}) = \lceil n/2 \rceil$. For another example consider generalized Petersen graphs $P(n,k)$ with n even, $k \geqslant 2$, and n and k being relatively prime. Then $\mathrm{idim}(\overline{P(n,k)}) = n$.

We conclude this note by computing the dimensions of the Petersen graph.

Proposition 8. *Let P be the Petersen graph, then* $\mathrm{idim}(P) = \mathrm{adim}(P) = 5$.

Proof. Let a_i and b_i, $1 \leqslant i \leqslant 5$, be the vertices of P as shown in Fig. 1. Set $A = \{a_i \mid i=1,\ldots,5\}$ and $B = \{b_i \mid i=1,\ldots,5\}$.

Clearly, any edge of P is contained in an induced path on three vertices, hence we may apply Theorem 5. Let $\mathscr{C}$ be a collection of complete bipartite subgraphs of $\overline{P}$ that cover the edges of $\overline{P}$. Suppose first that there is a copy of $K_{2,5}$ in $\mathscr{C}$. Let $V(K_{2,5}) = X + Y$, where $X = \{x, y\}$. If $xy \in E(P)$ then $|Y| \leqslant 4$. On the other hand, if x is not adjacent to y, then $y \cup \{Y\} = \{z \in V(P) \mid d_P(x,z) = 2\}$. But then y is in $\overline{P}$ not adjacent to all vertices of Y. It follows that $\mathscr{C}$ contains no copy of $K_{2,5}$.

Suppose next that there is a copy of $K_{3,3}$ in $\mathscr{C}$. Let $V(K_{3,3}) = X + Y$. Note that $|X \cap A| \leqslant 2$ and $|X \cap B| \leqslant 2$, for otherwise Y would contain less than three vertices. Similarly $|Y \cap A| \leqslant 2$ and $|Y \cap B| \leqslant 2$. So we may, without loss of generality, assume $|X \cap A| = 2$. If the two vertices of $X \cap A$ would not be adjacent in P, then we would have $|Y \cap B| = 3$, which is not possible. Hence, we may in addition without loss of generality assume $X \cap A = \{a_1, a_2\}$. Because of adjacencies in P we see that $b_1, b_2, a_3, a_5 \notin Y$ and therefore $a_4 \in Y$, for otherwise we would again have $|Y \cap B| = 3$. This implies that $a_3, a_5, b_4 \notin X$. Hence, the third vertex of X must be one of the vertices b_1, b_2, b_3, b_5. However, if X contains any of these vertices, then, using the adjacencies in P again, Y cannot contain three elements. For instance, if $b_1 \in X$ then besides a_4 only b_5 can be in Y.

We have thus shown that $\mathscr{C}$ contains only subgraphs isomorphic to $K_{2,2}$, $K_{2,3}$, $K_{2,4}$, $K_{1,6}$, and smaller ones. Since $|E(\overline{P})| = 30$, it follows that $|\mathscr{C}| \geqslant 4$. Moreover, if $|\mathscr{C}| = 4$, it necessarily contains at least three copies of $K_{2,4}$. So assume that $\mathscr{C}$ is indeed such and consider an arbitrary copy of $K_{2,4} =: Z$ with the bipartition $X + Y$, where $|X| = 2$. Then the two vertices of X must be adjacent in P. Moreover, the vertices of Y are uniquely determined and in P they induce two independent edges. It follows that the vertices of Z induce three independent edges of P. Hence, there are precisely five possibilities to select $X \cup Y$, and thus there are 15 different subgraphs Z isomorphic to $K_{2,4}$.

Let Z and Z' be two different copies of $K_{2,4}$ from $\mathscr{C}$. If $V(Z)$ contains two vertices of $V(Z')$ that are adjacent in P, then $V(Z) = V(Z')$. Then, as $Z \neq Z'$, Z and Z' have four common edges. But then the subgraphs from $\mathscr{C}$ cannot cover all the 30 edges of $\overline{P}$. So we may assume in the rest that $V(Z) \neq V(Z')$. Then it is straightforward to verify that $|V(Z) \cap V(Z')| = 3$. Let e, f, and g be the edges of P induced by $V(Z)$. Since $|V(Z) \cap V(Z')| = 3$, $V(Z')$ contains exactly one of the ends of each e, f, and g. Then Z and Z' have at least one edge in common. Consider now three copies of $K_{2,4}$ from $\mathscr{C}$: Z_1, Z_2, and Z_3. Then Z_1 covers eight edges of $\overline{P}$, Z_2 covers at most seven additional edges, and Z_3 covers at most six new edges. Hence any such three subgraphs cover at most 21 edges of $\overline{P}$. It follows that we cannot cover all the 30 edges of $\overline{P}$ with four complete bipartite graphs and therefore $\mathrm{idim}(P) \geqslant 5$.

To complete the proof we show that the edges of $\overline{P}$ are covered with the following complete bipartite subgraphs Z_i, $1 \leqslant i \leqslant 5$. Let $V(Z_i) = X_i + Y_i$, where $X_i = \{a_i, b_i\}$ and Y_i is the set of vertices that are at distance 2 from both a_i and b_i in P. For instance, $Y_1 = \{a_3, a_4, b_2, b_5\}$. Then it is straightforward to check that the subgraphs Z_i cover the edges of $\overline{P}$, hence Theorem 5 implies $\mathrm{idim}(P) \leqslant 5$. The corresponding embedding is shown on Fig. 1. □

Note that any independent set of edges of P of size 5 yields an isometric embedding into $P_2^{\boxtimes 5}$ similar to the one from Fig. 1.

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References

[1] G. Burosch, P.V. Ceccherini, On the Cartesian dimensions of graphs, J. Combin. Inform. System Sci. 19 (1994) 35–45.
[2] A.K. Dewdney, The embedding dimension of a graph, Ars Combin. 9 (1980) 77–90.
[3] N. Eaton, V. Rödl, Graphs of small dimensions, Combinatorica 16 (1996) 59–85.
[4] S.L. Fitzpatrick, R.J. Nowakowski, The strong isometric dimension of finite reflexive graphs, Discuss. Math. Graph Theory 20 (2000) 23–38.
[5] S.L. Fitzpatrick, R.J. Nowakowski, Copnumber of graphs with strong isometric dimension two, Ars Combin. 59 (2001) 65–73.
[6] R.L. Graham, P.M. Winkler, On isometric embeddings of graphs, Trans. Amer. Math. Soc. 288 (1985) 527–536.
[7] W. Imrich, S. Klavžar, Product Graphs: Structure and Recognition, Wiley, New York, 2000.
[8] J. Nešetřil, V. Rödl, A simple proof of the Galvin–Ramsey property of the class of all finite graphs and a dimension of a graph, Discrete Math. 23 (1978) 49–55.
[9] J. Nešetřil, V. Rödl, Three remarks on dimensions of graphs, Ann. Discrete Math. 28 (1985) 199–207.
[10] S. Poljak, A. Pultr, Representing graphs by means of strong and weak products, Comment. Math. Univ. Carolin. 22 (1981) 449–466.
[11] S. Poljak, V. Rödl, A. Pultr, On a product dimension of bipartite graphs, J. Graph Theory 7 (1983) 475–486.
[12] I.J. Schönberg, Metric spaces and positive definite functions, Trans. Amer. Math. Soc. 44 (1938) 522–536.
[13] D.B. West, Introduction to Graph Theory, second ed., Prentice-Hall, Upper Saddle River, NJ, 2001.