The distinguishing number of Cartesian products of complete graphs*

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Abstract

The distinguishing number $D(G)$ of a graph G is the least integer d such that G has a labeling with d labels that is preserved only by a trivial automorphism. We prove that Cartesian products of relatively prime graphs whose sizes do not differ too much can be distinguished with a small number of colors. We determine the distinguishing number of the Cartesian product $K_k \,\square\, K_n$ for all k and n, either explicitly or by a short recursion. We also introduce column-invariant sets of vectors and prove a switching lemma that plays a key role in the proofs.

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1 Introduction

The distinguishing number is a symmetry related graph invariant that was introduced a decade ago by Albertson and Collins [2]. For its motivation we refer to [14]. Given a graph G its *distinguishing number* $D(G)$ is the least integer d such that G has a d-distinguishing labeling, where a labeling $\ell : V(G) \to \{1, \dots, d\}$ is *d-distinguishing* if it is invariant only under the trivial automorphism.

This concept has been studied continually since its introduction, see [6, 7, 15]. In the last couple of years the area really flourished. Numerous respectable results were obtained and several generalizations and variations proposed. For instance, in [8] and [12], an analogue of Brooks Theorem was recently obtained. It asserts that $D(G) \leq \Delta(G) + 1$ holds for any connected graph, where equality is attained exclusively for $K_{\Delta+1}$, $K_{\Delta,\Delta}$, and C_5. As to generalizations we note that Tymoczko [16] generalized the notion of the distinguishing number to group actions on sets, see also [4, 5, 12], and that Collins and Trenk [8] introduced and studied distinguishing labelings that are proper colorings.

Bogstad and Cowen [3] determined the distinguishing number of hypercubes. One way of looking at the n-cube is to consider it as the Cartesian product of n factors, all isomorphic to K_2. As it turned out, the result of Bogstad and Cowen was the tip of an iceberg, as has first been made evident by Albertson [1]. He proved that for a connected prime graph G, $D(G^r) = 2$ for all $r \geq 4$, and, if $|V(G)| \geq 5$, then $D(G^r) = 2$ for all $r \geq 3$. (Recall that a graph is *prime* if it cannot be represented as the Cartesian product of two nontrivial graphs.) Then, in [13], it was shown that $D(G^r) = 2$ for any connected graph $G \neq K_2$ and any $r \geq 3$. Lastly, the distinguishing number of all Cartesian powers was determined in [11] by proving that $D(G^k) = 2$ for any connected graph G and any $k \geq 2$, with the following three exceptions: $D(K_2^2) = D(K_2^3) = D(K_3^2) = 3$.

The present paper is closely related to the paper of Chan [5] in which she studies the distinguishing number of the action of a group G on a set X denoted by $D_G(X)$. More precisely, one searches for the smallest number of labels (or colors) such that there exists a labeling of X, where no nontrivial group element induces a label preserving permutation of X. In a special case, every element of the group $S_k \times S_n$ acts on the $k \times n$ grid ($\mathbb{N}_k \times \mathbb{N}_n$) as a permutation of the rows followed by a permutation of the columns. Hence for $k \neq n$ this action coincides with the action of the automorphism group of the graph $K_k \Box K_n$ on the set $V(K_k \Box K_n)$ and consequently $D(K_k \Box K_n) = D_{S_k \times S_n}(\mathbb{N}_k \times \mathbb{N}_n)$ for every $k \neq n$. To determine these numbers Chan [5, Theorem 3.2] recursively defines sets $T_{k,n}$ such that $D_{S_k \times S_n}(\mathbb{N}_k \times \mathbb{N}_n) = \min\{T_{k,n}\}$.

In this paper we begin with the investigation of products of relatively prime graphs and prove that $D(G \,\Box\, H) \leq d$ provided that $k \leq |G| \leq |H| \leq d^k - k + 1$.

Then we turn to products of complete graphs and prove our main result:

Theorem 1.1 *Let k, n, d be integers so that $d \geq 2$ and $(d-1)^k < n \leq d^k$. Then*

$$D(K_k \,\square\, K_n) = \begin{cases} d, & \text{if } n \leq d^k - \lceil \log_d k \rceil - 1; \\ d+1, & \text{if } n \geq d^k - \lceil \log_d k \rceil + 1. \end{cases}$$

If $n = d^k - \lceil \log_d k \rceil$ then $D(K_k \,\square\, K_n)$ is either d or $d+1$ and can be computed recursively in $O(\log^*(n))$ time.

This also provides a good upper bound on the distinguishing number of products of relatively prime graphs. (Recall that two graphs G and H are *relatively prime* if there is no nontrivial graph that is a factor of both G and H. Clearly, two prime graphs are relatively prime.)

After submission of our paper we learned that Theorem 1.1 had independently been discovered in the setting of edge labelings by Fisher and Issak [9]. They determined the values of k and n for which there is a labelling of the edges of the complete biparite graph $K_{k,n}$ that is preserved only by a trivial automorphism. Since the line graph of $K_{k,n}$ is isomorphic to $K_k \square K_n$ and $\mathrm{Aut}(K_{k,n})$ coincides with $\mathrm{Aut}(K_k \square K_n)$, their results on the distinguishing edge colorings of complete bipartite graphs can be translated to distinguishing vertex colorings of Cartesian product of complete graphs. Of course, Theorem 1.1 also implies their result on distinguishing edge colorings of $K_{k,n}$. Theorem 1.1 is almost the same as [9, Corollary 9], except that we do not need recursion for $K_3 \,\square\, K_6$, (this case is covered by Proposition 3.3) nor for $K_6 \,\square\, K_{61}$ or $K_{d^2-1} \,\square\, K_{d^{d^2-1}-2}$, $d \geq 3$ (which is covered by Proposition 3.10).

Our methods rely heavily on the structure of the automorphism group of the Cartesian product and hold for all other products with the same structure of the automorphism group. For example, all the results about the distinguishing number of Cartesian products of complete graphs also hold for the distinguishing number of the direct product of complete graphs.

It is tempting to replace the term *automorphism* in the definition of the distinguishing number by *endomorphism*, *retraction*, or *weak retraction*. For all structures where these morphisms are well understood one can expect general and interesting results.

This also holds for the distinguishing number of Cartesian products of infinite graphs, which we touch at the end of the paper.

For terms not defined here, in particular for the Cartesian product of graphs and its properties, we refer to [10].

2 Products of relatively prime graphs

In this section we consider Cartesian products of relatively prime graphs. The main result of the section, Theorem 2.2, asserts that the distinguishing number of such products is small provided that the sizes of the factors do not differ too much. We begin with the following lemma.

Lemma 2.1 *Let $k \geq 2$, $d \geq 2$, G a connected graph on k vertices, and H a connected graph on $d^k - k + 1$ vertices that is relatively prime to G. Then $D(G \,\Box\, H) \leq d$.*

Proof. Since G and H are relatively prime every automorphism maps G-fibers into G-fibers and H-fibers into H-fibers.

Denote the set of vectors of length k with integer entries between 1 and d by $\mathbb{N}_d^k$, and let S be the set of the following $k-1$ vectors from $\mathbb{N}_d^k$:

$$\begin{array}{l}(1,1,1,\ldots,1,1,1,2)\\(1,1,1,\ldots,1,1,2,2)\\(1,1,1,\ldots,1,2,2,2)\\\quad\vdots\\(1,2,2,\ldots,2,2,2,2).\end{array}$$

Consider the $d^k - k + 1$ vectors from $\mathbb{N}_d^k \setminus S$ and label the G-fibers with them. Then the number of 1's in the H-fibers is $d^{k-1} - k + 1, \ldots, d^{k-1} - 1, d^{k-1}$. Hence any label preserving automorphism φ of $G \,\Box\, H$ preserves these fibers individually, so φ can only permute the G-fibers. But since they are all different, it follows that φ is the identity. Hence, the described labeling is d-distinguishing. $\Box$

Before stating the next theorem we wish to remark, as one of the referees commented, that we could have defined a d-labeling of $G \,\Box\, H$ as a matrix L with entries $\{1, 2, \ldots, d\}$ whose rows/columns are indexed by vertices of G/H. This would have allowed different, somehow shorter proofs, of several of the results, for example the next one. We have decided to do without matrices and wish to apologize to those readers who would have preferred the other approach.

Theorem 2.2 *Let $k \geq 2$, $d \geq 2$, and G and H connected, relatively prime graphs with $k \leq |G| \leq |H| \leq d^k - k + 1$. Then $D(G \,\Box\, H) \leq d$.*

Proof. For $d = 2$ this is proved in [11, Theorem 4.2].

Let $d \geq 3$ and suppose that $|G| = k$. Call vectors $(a_1, a_2, \ldots, a_k)$ and $(b_1, b_2, \ldots, b_k)$ of $\mathbb{N}_2^k$ a *complementary pair* if $a_i + b_i = 3$ for every $1 \leq i \leq k$.

Let S be the set of $k-1$ vectors as in the proof of Lemma 2.1. Set $B = \mathbb{N}_2^k \setminus S$ and $C = \mathbb{N}_d^k \setminus \mathbb{N}_2^k$. Note that there are $2^{k-1} - k + 1$ complementary pairs in B. Let $B_s = \{\mathbf{v}^1, \mathbf{v}^2, \ldots, \mathbf{v}^{2^k-k+1-2s}\}$ be the set obtained from B by removing s, $0 \le s \le 2^{k-1} - k + 1$, complementary pairs and let C_t be the set obtained from C by removing t, $0 \le t \le d^k - 2^k$, vectors. It follows from the construction of B_s that the vectors $\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^k$, where $\mathbf{u}^i$ is defined as $\mathbf{u}^i = (v_i^1, v_i^2, \ldots, v_i^{2^k-k+1-2s}) \in \mathbb{N}_2^{2^k-k+1-2s}$ and v_i^j denotes the i-th coordinate of the vector $\mathbf{v}^j$ from B_s, have pairwise different numbers of ones.

For every $d \ge 3$ we can write $|H| = |B_s| + |C_t|$ for some s and t. Let ℓ be the labeling of $G \,\square\, H$ defined as follows. Arbitrarily select $2^k - k + 1 - 2s$ G-fibers and label them with the vectors of B_s. Then label the remaining G-fibers with vectors of C_t. Let G_1 denote the first and G_2 the second set of G-fibers. We claim that ℓ is d-distinguishing.

Let α be an automorphism of $G \,\square\, H$ that preserves ℓ. Then α can only permute some labels of the G-fibers inside G_1 and some labels of the G-fibers inside G_2. Since all of these labels are different, α must permute the H-fibers also. Let φ be the a non-trivial permutation of the H-fibers induced by α. Then φ induces a non-trivial permutation of the vectors $\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^k$. Since they have pairwise different numbers of ones, α is the identity.

Suppose next that $|G| > k$ (and $|G| \le |H|$). Select a subgraph G' of G with k vertices and use the above labeling for $G' \,\square\, H$. This labeling leads to at most k different numbers of ones in the H-fibers of $G' \,\square\, H$. Let K be the set of these numbers. Now label the H-fibers of $(G \setminus G') \,\square\, H$ arbitrarily with vectors from $\mathbb{N}_d^{|H|}$ such that every fiber has a distinct number of ones from the set $\{0, 1, \ldots, |H|\} \setminus K$. As before the G-fibers and the H-fibers are fixed by every automorphism. $\square$

To conclude this section we observe that distinguishing numbers of products of complete graphs are upper bounds for distinguishing numbers of products of relatively prime graphs. As we show in the next section, the bounds are good in most cases.

Proposition 2.3 *Let G and H be connected, relatively prime graphs with $|G| \neq |H|$. Then $D(G \,\square\, H) \le D(K_{|G|} \,\square\, K_{|H|})$.*

Proof. Since G and H are relatively prime, every automorphism preserves the set of G-fibers and the set of H-fibers, see [10, Corollary 4.17]. Since $|K_{|G|}| \neq |K_{|H|}|$ the same conclusion holds for $K_{|G|} \,\square\, K_{|H|}$ as well. Therefore, considering $G \,\square\, H$ as a spanning subgraph of $K_{|G|} \,\square\, K_{|H|}$ we infer that $\mathrm{Aut}(G \,\square\, H) \subseteq \mathrm{Aut}(K_{|G|} \,\square\, K_{|H|})$ and consequently $D(G \,\square\, H) \le D(K_{|G|} \,\square\, K_{|H|})$. $\square$

3 Proof of Theorem 1.1

It is already known that $D(K_n \,\square\, K_n) = 3$ for $n = 2, 3$, and $D(K_n \,\square\, K_n) = 2$ for $n > 3$, see [11]. Since $K_1 \,\square\, K_n$ is isomorphic to K_n, $D(K_1 \,\square\, K_n) = n$, we still have to determine the distinguishing numbers of $K_k \,\square\, K_n$ for $2 \le k \le n$. Clearly we may assume that $k < n$.

As K_k and K_n are relatively prime for $k \ne n$, Theorem 2.2, in the special case of complete factors, reads as:

Lemma 3.1 *Let $k, d \ge 2$ and let $k < n \le d^k - k + 1$. Then $D(K_k \,\square\, K_n) \le d$.*

On the other hand, if n is large enough, the distinguishing number also has to be large, as the next result asserts.

Lemma 3.2 *Let $k, d \ge 2$ and $n > d^k$. Then $D(K_k \,\square\, K_n) \ge d + 1$.*

Proof. Let ℓ be an arbitrary d-labeling of $K_k \,\square\, K_n$. Since there are more than d^k K_k-fibers, at least two of them have identical labels. Since $\mathrm{Aut}(K_k \,\square\, K_n)$ acts transitively on the K_k-fibers we infer that ℓ is not distinguishing. Hence $D(K_k \,\square\, K_n) \ge d + 1$. $\square$

Combining the above two results we can already determine the distinguishing number in many cases.

Proposition 3.3 *Let $k, d \ge 2$ and $(d-1)^k < n \le d^k - k + 1$ (and $n > k$). Then $D(K_k \,\square\, K_n) = d$.*

Proof. The assertion follows from Lemma 3.1, Lemma 3.2, and the fact that $D(K_k \,\square\, K_n) = 1$ if and only if $k = n = 1$. $\square$

Hence, we still have to determine $D\left(K_k \,\square\, K_{d^k - r}\right)$ for $0 \le r \le k - 2$, where $k, d \ge 2$. In particular, if $k = 2$ the only missing cases of Proposition 3.3 are those where n is a perfect square. By Lemmas 3.1 and 3.2 these numbers can only be d or $d + 1$. To this end the following concept is useful:

Let π be a permutation from S_k and $\mathbf{v} = (v_1, \ldots, v_k) \in \mathbb{N}_d^k$. Define $\pi\mathbf{v}$ by

$$\pi\mathbf{v} = \left(v_{\pi^{-1}(1)}, \ldots, v_{\pi^{-1}(k)}\right).$$

Then we say that the set $X = \{\mathbf{v}^1, \ldots, \mathbf{v}^r\}$ is *column-invariant* if there exists a nontrivial $\pi \in S_k$ such that

$$\{\mathbf{v}^1, \ldots, \mathbf{v}^r\} = \pi\{\mathbf{v}^1, \ldots, \mathbf{v}^r\},$$

where $\pi\{\mathbf{v}^1,\ldots,\mathbf{v}^r\} = \{\pi\mathbf{v}^1,\ldots,\pi\mathbf{v}^r\}$. In other words, π induces a permutation $\varphi_\pi \in S_r$ such that $\pi\mathbf{v}^i = \mathbf{v}^{\varphi_\pi(i)}$ for all $i, 1 \le i \le r$.

We can interpret the $\mathbf{v}^i$ as a d-labeling of $K_k \,\Box\, K_r$, where every $\mathbf{v}^i$ labels a K_k-fiber, and π as a permutation of the K_{d^k-r}-fibers. Column-invariance thus means that application of π to the K_{d^k-r}-fibers and successive application of φ_π to the K_k-fibers is a labeling preserving automorphism of $K_k \,\Box\, K_{d^k-r}$.

It should be noted though that not all d labels may be used by the vectors $\mathbf{v}^1,\ldots,\mathbf{v}^r$.

Lemma 3.4 *Let $k, d \ge 2$. Then $D\,(K_k \,\Box\, K_{d^k}) = d + 1$.*

Proof. Since $\mathbb{N}_d^k$ is column invariant $D\,(K_k \,\Box\, K_{d^k})$ must be larger than d. Furthermore, an application of Lemma 3.1 for $n = d^k$ and $d+1$ in the place of d shows that it is at most $d+1$. $\Box$

We wish to remark that Lemma 3.4 provides the missing cases of Proposition 3.3 for $k = 2$. Thus we know all distinguishing numbers of $K_k \,\Box\, K_n$ for $k = 1, 2$.

Lemma 3.5 (Switching Lemma) *Let $k, d \ge 2$ and $1 \le r < d^k$. Then every set of r vectors from $\mathbb{N}_d^k$ is column invariant if and only if every set of $d^k - r$ vectors from $\mathbb{N}_d^k$ is also column invariant.*

Proof. Let $\mathbf{v}^1,\ldots,\mathbf{v}^r$ be a set of r vectors from $\mathbb{N}_d^k$ that is column-invariant and $\mathbf{u}^1,\ldots,\mathbf{u}^{d^k-r}$ be the remaining $d^k - r$ vectors from $\mathbb{N}_d^k$.

By assumption the set $\mathbf{v}^1,\ldots,\mathbf{v}^r$ is column-invariant. Thus, there is a permutation π in S_k such that

$$\{\mathbf{v}^1,\ldots,\mathbf{v}^r\} = \pi\{\mathbf{v}^1,\ldots,\mathbf{v}^r\} = \{\pi\mathbf{v}^1,\ldots,\pi\mathbf{v}^r\}.$$

Since $\pi\mathbb{N}_d^k = \mathbb{N}_d^k$ we infer that $\pi\{\mathbf{u}^1,\ldots,\mathbf{u}^{d^k-r}\} = \{\mathbf{u}^1,\ldots,\mathbf{u}^{d^k-r}\}$. In other words, $\{\mathbf{u}^1,\ldots,\mathbf{u}^{d^k-r}\}$ is also column invariant.

By the same argument the column invariance of $\{\mathbf{u}^1,\ldots,\mathbf{u}^{d^k-r}\}$ entails that of $\{\mathbf{v}^1,\ldots,\mathbf{v}^r\}$. $\Box$

Proposition 3.6 *Let $k, d \ge 2$ and $1 \le r \le k - 2$. Then*

(i) $D(K_r \,\Box\, K_k) \ge d + 1 \Rightarrow D(K_k \,\Box\, K_{d^k-r}) = d + 1$ *and*

(ii) $D(K_r \,\Box\, K_k) \le d \Rightarrow D(K_k \,\Box\, K_{d^k-r}) = d$.

Proof. $D(K_r \,\square\, K_k) \geq d+1$ implies that there is no d-distinguishing labeling of $K_r \,\square\, K_k$. It follows that every set consisting of r vectors from $\mathbb{N}_d^k$ is column-invariant. By the Switching Lemma 3.5 this is possible only if every set consisting of $d^k - r$ vectors from $\mathbb{N}_d^k$ is column-invariant. We can thus conclude that there is no d-distinguishing labeling of $K_k \,\square\, K_{d^k-r}$ and hence $D\left(K_k \,\square\, K_{d^k-r}\right) \geq d+1$. The assertion (i) follows since we already know that $D\left(K_k \,\square\, K_{d^k-r}\right)$ is either d or $d+1$.

The proof of (ii) is similar. $\square$

Proposition 3.7 *Let $k, d \geq 2$ and $1 \leq r \leq k-2$. Then $D(K_k \,\square\, K_{d^k-r}) = d+1$ if and only if every set consisting of r vectors from $\mathbb{N}_d^k$ is column-invariant.*

Proof. If every set of r vectors from $\mathbb{N}_d^k$ is column-invariant, then $D(K_k \,\square\, K_r) \geq d+1$, and thus $D(K_k \,\square\, K_{d^k-r}) = d+1$ by Proposition 3.6 (i).

On the other hand, if there is a set of r vectors from $\mathbb{N}_d^k$ that is not column-invariant, then $D(K_k \,\square\, K_r) \leq d$, and thus $D(K_k \,\square\, K_{d^k-r}) \neq d+1$ by Proposition 3.6 (ii). $\square$

Proposition 3.8 *Let $d \geq 2$, $3 \leq k \leq d$. Then $D\left(K_k \,\square\, K_{d^k-1}\right) = d$.*

Proof. Let $\mathbf{v} = (v_1, \ldots, v_k) \in \mathbb{N}_d^k$, where $v_i \neq v_j$ for every $i \neq j$ and let $\pi \in S_k$. Then $\pi\mathbf{v} = \mathbf{v}$ if and only if $\pi = \mathrm{id}$. Hence Proposition 3.7 implies that $D\left(K_k \,\square\, K_{d^k-1}\right) = d$. $\square$

Proposition 3.9 *Let $k, d \geq 2$ and $0 \leq r < \log_d k$. Then $D(K_k \,\square\, K_{d^k-r}) = d+1$.*

Proof. The case $r = 0$ is covered by Lemma 3.4. Thus, let $r \geq 1$, and $\{\mathbf{v}^1, \ldots, \mathbf{v}^r\}$ be a set of r vectors from $\mathbb{N}_d^k$. For every $1 \leq i \leq k$ define $\mathbf{u}^i = (v_i^1, \ldots, v_i^r)$. Since $k > d^r$, at least two vectors from the set $\{\mathbf{u}^1, \ldots, \mathbf{u}^k\} \subseteq \mathbb{N}_d^k$ are the same. Suppose $\mathbf{u}^i = \mathbf{u}^j$ where $i < j$. In other words, $v_i^\ell = v_j^\ell$ for $1 \leq \ell \leq r$. Let $\pi \in S_k$ be the transposition $(i\,j)$. Then for any ℓ, $1 \leq \ell \leq r$,

$$\begin{aligned}
\pi\mathbf{v}^\ell &= (v^\ell_{\pi^{-1}(1)}, \ldots, v^\ell_{\pi^{-1}(i)}, \ldots, v^\ell_{\pi^{-1}(j)}, \ldots, v^\ell_{\pi^{-1}(k)}) \\
&= (v^\ell_{\pi(1)}, \ldots, v^\ell_{\pi(i)}, \ldots, v^\ell_{\pi(j)}, \ldots, v^\ell_{\pi(k)}) \\
&= (v^\ell_1, \ldots, v^\ell_j, \ldots, v^\ell_i, \ldots, v^\ell_k) \\
&= (v^\ell_1, \ldots, v^\ell_i, \ldots, v^\ell_j, \ldots, v^\ell_k) \\
&= \mathbf{v}^\ell\,.
\end{aligned}$$

Hence $\{\pi\mathbf{v}^1, \ldots, \pi\mathbf{v}^r\} = \{\mathbf{v}^1, \ldots, \mathbf{v}^r\}$. Since π is nontrivial, $\{\mathbf{v}^1, \ldots, \mathbf{v}^r\}$ is column-invariant. By Proposition 3.7 the assertion follows. $\square$

Proposition 3.10 *Let $d, r \geq 2$ and $r+2 \leq k \leq d^r - r + 1$. Then $D\left(K_k \,\square\, K_{d^k - r}\right) = d$.*

Proof. Since the assumptions of Lemma 3.1 are fulfilled we have $D(K_r \,\square\, K_k) \leq d$. The assertion then follows by Proposition 3.6 (ii). $\square$

From the above results we can complete the proof of the non-recursive part of Theorem 1.1 as follows. Let us call an integer r *good* if we have established a closed formula for $D\left(K_k \,\square\, K_{d^k - r}\right)$ (implicitly assuming $d^k - r > (d-1)^k$). Proposition 3.3 and Lemma 3.4 state that if r is not good then $1 \leq r \leq k-2$. By Proposition 3.9, if r is not good then $\lceil \log_d k \rceil \leq r \leq k-2$.

For $r = 1$, Propositions 3.8 and 3.9 yield the result. Let $r \geq 2$. Choosing k so that $r + 2 \leq k \leq d^{r-1}$ fits Proposition 3.10. This implies that only $r = \lceil \log_d k \rceil$ may not be good. This case can be treated by the following algorithm:

DISTINGUISHING(k, n)

1. $d = \lfloor n^{\frac{1}{k}} \rfloor + 1$
2. **if** $n \neq d^k - \lceil \log_d k \rceil$
3. **then** determine $D(K_k \,\square\, K_n)$ from Theorem 1.1
4. **else** determine $D(K_k \,\square\, K_n)$ from $D(K_{d^k - n} \,\square\, K_k)$ by an application of Proposition 3.6

We note that Step 3 returns the distinguishing number and that the recurrence step, Step 4, is executed only if $d^k - k + 1 < n$. Since $d \geq 2$ we infer

$$\begin{aligned} 2^k - k + 1 &< n, \\ 2^k &< 2n, \\ k - 1 &< \log_2 n. \end{aligned}$$

Hence $d^k - n < k - 1 < \log_2 n$. This means, instead of $K_k \,\square\, K_n$ we have to consider $K_{k_1} \,\square\, K_k$, where $k_1 = d^k - n < \log_2 n$. If DISTINGUISHING(k_1, k) also enters the recursive step, then with a call of DISTINGUISHING(k_2, k_1), where $k_2 < \log_2 k$. Since $k_i \geq 1$ the number of recursive steps cannot be more than the *iterated logarithm*

$$\log_2^* n.$$

Note that $\log_2^* 2 = 1$, $\log_2^* 4 = 2$, $\log_2^* 16 = 3$, $\log_2^* 65536 = 4$, and $\log_2^*(2^{65536}) = 5$.

For $d = 3$ we need a recursion for $r = 3$ and $k = 26$. It pertains to the product $K_{26} \,\square\, K_{3^{26} - 3}$. DISTINGUISHING$(26, 3^{26} - 3)$ leads to Step 4, tells us to find $D(K_3 \,\square\, K_{26})$, and to apply Proposition 3.6. We thus have to check whether $D(K_3 \,\square\, K_{26})$ is $\geq d + 1$ or $\leq d$. In the first case $D\left(K_{26} \,\square\, K_{3^{26} - 3}\right)$ is 4, in the other 3.

By Proposition 3.8 we infer that $D(K_3 \,\square\, K_{26}) = D\left(K_3 \,\square\, K_{3^3 - 1}\right) = 3$. Thus $D\left(K_{26} \,\square\, K_{3^{26} - 3}\right)$ is also 3.

4 Concluding remarks

The Cartesian product of finitely many relatively prime, connected infinite graphs behaves very much as in the finite case. Thus, the results of this paper have analogues in the infinite case. For example,

$$D(K_{\aleph_0} \,\square\, K_{\aleph_0}) = 2.$$

For a proof one simply labels with the vectors

$$\mathbf{s}^k = (1, 1, 1, \ldots, 1, 2, 2, 2 \ldots), \quad k = 1, 2, \ldots,$$

where $\mathbf{s}^k$ has k 1's and infinitely many 2's.

In general, however, the proofs are more complicated, in particular for large cardinals, and will be the subject of a subsequent paper.

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