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INTERSECTION GRAPHS OF HALFLINES AND HALFPLANES*

Sandi KLAVŽAR and Marko PETKOVŠEK
Department of Mathematics, University of Ljubljana, 61111 Ljubljana, Yugoslavia

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We give three different characterizations of intersection graphs of halflines in $\boldsymbol{R}^1$ and determine the number of such graphs on n vertices. We also characterize intersection graphs of halfplanes in $\boldsymbol{R}^2$ in terms of forbidden subgraphs, and prove that sphericity of joins of triangulated graphs with bipartite complements is at most 2.

1. Introduction

If $\mathscr{F} = \{S_1, S_2, \ldots, S_n\}$ is a family of sets, then its *intersection graph* is the graph $G = (\mathscr{F}, E)$ where $S_iS_j \in E$ if and only if $S_i \cap S_j \neq \emptyset$ and $i \neq j$ (see [4]). Characterization of intersection graphs of unit spheres in $\boldsymbol{R}^3$ is a difficult problem [5]. An interesting subclass consists of intersection graphs of halfspaces (observe that replacing halfspaces with spheres of sufficiently large diameter preserves the intersection graph). It is easy to see that the classes of intersection graphs of halfspaces in $\boldsymbol{R}^n$ and $\boldsymbol{R}^m$ coincide if $n, m \geqslant 2$. Therefore it suffices to consider intersection graphs of halfplanes in $\boldsymbol{R}^2$. These in turn are built from intersection graphs of halflines in $\boldsymbol{R}^1$ in a simple way. We shall call intersection graphs of finite families of halflines *unbounded interval graphs.*

Lemma 1. *G is the intersection graph of a finite family of halfplanes in $\boldsymbol{R}^2$ if and only if G is a join of a finite family of unbounded interval graphs.*

Proof. A family $\mathscr{F}$ of parallel halfplanes in $\boldsymbol{R}^2$ has the same intersection graph as some family of halflines in $\boldsymbol{R}^1$ (obtained by projecting the elements of $\mathscr{F}$ orthogonally on the common normal of their boundaries). Because two nonparallel halfplanes have nonempty intersection, the lemma follows. □

In Section 2 we give three different characterizations of unbounded interval graphs and enumerate them, and in Section 3 we characterize intersection graphs of halfplanes in terms of forbidden subgraphs.

If G is a graph and $v \in V(G)$, then $\mathrm{Adj}(v) = \{w \in V(G) / vw \in E(G)\}$. The

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relation $\leqslant$ in $V(G)$, defined by:

$$v \leqslant w \Leftrightarrow \mathrm{Adj}(v) \subseteq \mathrm{Adj}(w) \cup \{w\},$$

is called the *vicinal preorder* of G. The *Dilworth number* of G is the size of a maximum antichain in $V(G)$ with respect to $\leqslant$, or equivalently, the size of a minimum cover of $V(G)$ by chains with respect to $\leqslant$ (see [1]). Other graph-theoretical definitions follow [3].

2. Unbounded interval graphs

Theorem 1. *For a graph G the following statements are equivalent*:

(1) *G is an unbounded interval graph,*

(2) *$V(G)$ can be partitioned into V_1 and V_2 such that V_1 and V_2 induce complete subgraphs in G and V_1 forms a chain in the vicinal preorder of G,*

(3) *G is triangulated and $\bar{G}$ is bipartite,*

(4) *G contains no induced subgraph isomorphic to C_4 or to any of $\bar{C}_{2n+1}$, for $n = 1, 2, \ldots$*

Proof. (1)$\Rightarrow$(2): Let G be an unbounded interval graph. We partition $V(G)$ into two subsets V_1 and V_2, with V_1 corresponding to intervals infinite to the left and V_2 corresponding to intervals infinite to the right. Clearly, V_1 and V_2 induce complete graphs in G. Let v_i and v_j from V_1 correspond to intervals $(-\infty, a_i]$ and $(-\infty, a_j]$ such that $a_i \leqslant a_j$. Then $\mathrm{Adj}(v_i)$ is contained in $\mathrm{Adj}(v_j) \cup \{v_j\}$, proving that V_1 forms a chain in the vicinal preorder of G.

(2)$\Rightarrow$(3): Suppose $C = v_1v_2 \cdots v_k$, $k \geqslant 4$, is a cycle in G. If at least three of the v_i's belong to the same complete subgraph of G then C has a chord. Otherwise $k = 4$ and each of V_1, V_2 contains two vertices of C. Wlg. assume that $v_1, v_2 \in V_1$, $v_3, v_4 \in V_2$ and $v_1 \leqslant v_2$ (recall that V_1 forms a chain for $\leqslant$). Then $v_2v_4 \in E(G)$ is a chord, and G is triangulated. Obviously $\bar{G}$ is bipartite with partition $V(G) = V_1 \cup V_2$.

(3)$\Rightarrow$(4): Because G is triangulated it contains no induced subgraph isomorphic to C_4. Because $\bar{G}$ is bipartite G contains none of $\bar{C}_{2n+1}$, for $n = 1, 2, \ldots$

(4)$\Rightarrow$(1): Since bipartite graphs are comparability graphs, G is an interval graph by Theorem 8.1, of [3]. Let $\mathcal{F}$ be a family of intervals on the real line whose intersection graph is G, and let I be an interval from $\mathcal{F}$. Denote by $\mathcal{D}(I)$ the set of all intervals from $\mathcal{F}$ which do not intersect I. As $\bar{G}$ contains no triangles, the size of the largest independent set in G cannot exceed 2. Therefore all the intervals from $\mathcal{D}(I)$ lie on the same side of I. By extending I on the other side into infinity the intersection graph of $\mathcal{F}$ is preserved. In this way we can replace all the intervals from $\mathcal{F}$ with unbounded intervals without changing the intersection graph of the family. $\square$

From (2) it follows that the Dilworth number of unbounded interval graphs is at most 2, and hence these graphs are permutation graphs (see [1]). However, there exist permutation graphs which are also interval graphs and have Dilworth number at most 2, but are not unbounded interval graphs. An example is furnished by the stars $K_{1,n}$, for $n \geq 3$. Also, it is easy to see that the class of unbounded interval graphs is properly contained in the class of unit interval graphs.

Theorem 2. *There are exactly* $2^{n-2} + 2^{[\frac{1}{2}n-1]}$ *pairwise nonisomorphic unbounded interval graphs on n vertices* ($n \geq 1$).

Proof. A family of unbounded intervals can be represented by a word over the alphabet $\{L, R\}$ by assigning L's and R's to intervals unbounded to the left and right, respectively, and ordering the letters coherently with the order of the endpoints of their corresponding intervals. Let W denote the collection of all finite words constructed on alphabet $\{L, R\}$, and W_n the collection of all those words from W which have length n. If $w \in W$, let $G(w)$ denote the intersection graph of the family represented by w. For example, if $w = LRLL$ then $G(w)$ is isomorphic to $K_4 - e$. Then $G(W_n)$ is the set of all unbounded interval graphs on n vertices. Let $r(w)$ be the word obtained from w by reversing it, and changing L's into R's and vice versa. Obviously r is an involution, and the words w and $r(w)$ yield the same graph. Hence $|G(W_n)| \leq e_n$, where e_n is the number of orbits of r in W_n. For n odd, the middle letters of w and $r(w)$ are different. Hence in this case r has no fixed point, and there are 2^{n-1} orbits of cardinality 2. For n even, r has $2^{\frac{1}{2}n}$ fixed points. Therefore there are $2^{\frac{1}{2}n}$ orbits of cardinality 1 and $\frac{1}{2}(2^n - 2^{\frac{1}{2}n})$ orbits of cardinality 2. Thus

$$e_n = \begin{cases} 2^{n-1}, & \text{for } n \text{ odd}, n \geq 1, \\ 2^{n-1} + 2^{\frac{1}{2}n-1}, & \text{for } n \text{ even}, n \geq 0. \end{cases} \tag{1}$$

Let

$$A_n := RW_{n-1} \cup W_{n-1}L, \quad \text{for } n \geq 1,$$
$$B_n := LW_{n-2}R, \quad \text{for } n \geq 2,$$

where juxtaposition indicates concatenation. Then $W_n = A_n \cup B_n$ for $n \geq 2$. The graphs from $G(A_n)$ have at least one vertex of degree $n-1$ while this is not true for graphs from $G(B_n)$. Hence

$$|G(W_n)| = |G(A_n)| + |G(B_n)|, \quad \text{for } n \geq 2. \tag{2}$$

The graphs of $G(A_n)$ are obtained from graphs of $G(W_{n-1})$ by joining them with K_1. It is easy to see that joins of nonisomorphic graphs with K_1 are themselves nonisomorphic, hence

$$|G(A_n)| = |G(W_{n-1})|, \quad \text{for } n \geq 1. \tag{3}$$

We claim that different words from B_n give rise to nonisomorphic graphs unless they belong to the same orbit of r. To see this, note first that complements of graphs from $G(B_n)$ are connected and bipartite, hence the partition of vertices of these graphs into two cliques is unique. Let $G=(V, E) \in G(B_n)$ and let $V=V_1 \cup V_2$ be the corresponding partition. Suppose we decide to represent vertices from V_1 with L's and vertices from V_2 with R's. Let $w \in B_n$ be such that $G(w)=G$. Then w contains $|V_1|$ L's. If $V_1=\{v_1, v_2, \ldots, v_k\}$ and $\deg(v_i) \leqslant \deg(v_{i+1})$ then there are $\deg(v_{i+1})-\deg(v_i)$ R's between the ith and the $i+1$st L in w, for $i=1,2, \ldots, k-1$, while the last L is followed by $n-1-\deg(v_k)$ R's. One sees that this w is unique. If we represent vertices from V_1 with R's and vertices from V_2 with L's, then we obtain $r(w)$. For instance, P_4 is represented by $LRLR$, which is a fixed point of r.

As a consequence, $|G(B_n)|$ equals the number of orbits of r in B_n. Since the first and the last letter of words from B_n are fixed we have

$$|G(B_n)|=e_{n-2}, \quad \text{for } n \geqslant 2. \tag{4}$$

From (2), (3) and (4) it follows that

$$|G(W_n)|=|G(W_{n-1})|+e_{n-2}, \quad \text{for } n \geqslant 2. \tag{5}$$

Since $|G(W_1)|=1$ the solution of (5) is, using (1),

$$|G(W_n)|=1+\sum_{k=0}^{n-2} e_k=2^{n-2}+2^{[\frac{1}{2}n-1]},$$

proving the theorem. $\square$

3. Intersection graphs of halfplanes

Theorem 3. *A graph G is the intersection graph of a finite family of halfplanes in $\mathbf{R}^2$ if and only if G contains no induced subgraph isomorphic to $\bar{P}_5$ or to any of $\bar{C}_{2n+1}$, for $n=1,2, \ldots$.*

Proof. If G is the intersection graph of a finite family of halfplanes, then by Lemma 1 each connected component of $\bar{G}$ is the complement of some unbounded interval graph. As P_5 is connected and $\bar{P}_5$ is not an unbounded interval graph (note that it contains C_4 as an induced subgraph), G contains no induced subgraph isomorphic to $\bar{P}_5$. As by Theorem 1 unbounded interval graphs have bipartite complements, G contains no induced subgraph isomorphic to any of $\bar{C}_{2n+1}$, for $n=1,2, \ldots$.

Conversely, suppose that G contains no induced subgraph isomorphic to $\bar{P}_5$ or to any of $\bar{C}_{2n+1}$, for $n=1,2, \ldots$. Then $\bar{G}$ is bipartite. Let $\bar{C}=(U_1, U_2, F)$ be a connected component of $\bar{G}$, and $u, v \in U_1$. Let $w_0 w_1 \cdots w_{2n}$, $n \geqslant 2$, $w_0=u$, $w_{2n}=v$, be a path in $\bar{C}$. Since the subgraph of $\bar{C}$ induced by the vertices w_0, w_1,

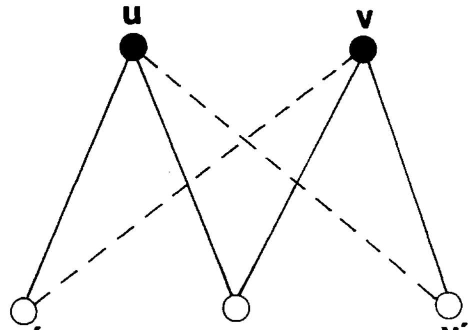

Fig. 1.

w_2, w_3 and w_4 is not P_5, at least one of w_0w_3 and w_1w_4 belongs to F. Repeating this argument we obtain a path uwv of length 2 connecting u and v in $\bar{C}$. Suppose u and v are incomparable in the vicinal preorder of $\bar{C}$. Then there exist $u', v' \in U_2$ such that $uu', vv' \in F$ and $uv', vu' \notin F$ (see Fig. 1). But then u', u, w, v, v' induce P_5 in $\bar{C}$, a contradiction. It follows that U_1 forms a chain in the vicinal preorder of $\bar{C}$ and hence of C. Then by Theorem 1 C is an unbounded interval graph, and by Lemma 1, G is the intersection graph of some family of halfplanes in $\boldsymbol{R}^2$. □

It is easy to see that joins of permutation graphs are permutation graphs. As unbounded interval graphs are permutation graphs, Lemma 1 implies that intersection graphs of halfplanes are permutation graphs, too. Therefore both unbounded interval graphs and intersection graphs of halfplanes are perfect.

The *sphericity* sph(G) of a graph G is defined to be the minimum number n such that G is isomorphic to some intersection graph of unit spheres in $\boldsymbol{R}^n$ (see [2]).

Corollary 1. *If G is a join of a finite family of triangulated graphs with bipartite complements, then* sph$(G) \leqslant 2$.

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