$L(2,1)$-labeling of direct product of paths and cycles

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Abstract

An $L(2,1)$-labeling of a graph G is an assignment of labels from $\{0,1,\ldots,\lambda\}$ to the vertices of G such that vertices at distance two get different labels and adjacent vertices get labels that are at least two apart. The λ-number $\lambda(G)$ of G is the minimum value λ such that G admits an $L(2,1)$-labeling. Let $G \times H$ denote the direct product of G and H. We compute the λ-numbers for each of $C_{7i} \times C_{7i}$, $C_{11i} \times C_{11j} \times C_{11k}$, $P_4 \times C_m$, and $P_5 \times C_m$. We also show that for $n \geq 6$ and $m \geq 7$, $\lambda(P_n \times C_m) = 6$ if and only if $m = 7k$, $k \geq 1$. The results are partially obtained by a computer search.

Key words: $L(2,1)$-labeling; Vertex labeling; λ-number; Direct product of graphs; Rotagraph; Fasciagraph; Channel assignment; Dynamic algorithm

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1 Introduction

Consider the problem of assigning frequencies to radio transmitters at various nodes in a territory. Transmitters that are close must receive frequencies that are sufficiently apart, for otherwise they may be at the risk of interfering with each other. The spectrum of frequencies is a very important resource on which there are increasing demands, both civil and military. This calls for an efficient management of the spectrum. It is assumed that transmitters are all of identical type and that signal propagation is isotropic. Further, since frequencies themselves are quantized in practice, there is no loss of generality in assuming that they admit integer values.

The foregoing problem, with the objective of minimizing the span of frequencies, was first placed on a graph-theoretical footing in 1980 by Hale [11] who established its equivalence to generalized vertex coloring problem that is known to be computationally hard. (Vertices correspond to transmitter locations and their labels to radio frequencies, while adjacencies are determined by geographical "proximity" of the transmitters.) Roberts [23] subsequently proposed a variation to the problem in which distinction is made between transmitters that are "close" and those that are "very close." This enabled Griggs and Yeh [10] to formulate the $L(2,1)$-labeling of graphs that has since been an object of extensive research [2]–[9], [13, 14, 16, 18], [20]–[30].

Formally, an $L(2,1)$*-labeling* of a graph G is an assignment f of non-negative integers to vertices of G such that

$$|f(u) - f(v)| \geq \begin{cases} 2; & d_G(u,v) = 1, \\ 1; & d_G(u,v) = 2\,. \end{cases}$$

⟦q-(2,1)-coloring⟧

The difference between the largest label and the smallest label assigned by f is called the *span* of f, and the minimum span over all $L(2,1)$-labelings of G is called the λ-*number* of G, denoted by $\lambda(G)$. The general problem of determining $\lambda(G)$ is NP-hard [9]. Moreover, determining $\lambda(G)$ is an NP-complete problem even for graphs G with diameter 2 [10]. On the other hand, if the graph is known to be a tree, then there is an efficient solution [2]. This result has been extended in [4] to k-almost trees (for any fixed k). For additional information concerning related complexity issues, we refer to [4].

The following result constitutes a useful lower bound.

Lemma 1.1 *(Griggs & Yeh [10]) Let G be a graph with maximum degree $\Delta \geq 2$. If G contains three vertices of degree Δ such that one of them is adjacent to the other two, then $\lambda(G) \geq \Delta + 2$.*

The foregoing lower bound is achievable in many cases [9, 20, 24, 30]. In particular, this is true with respect to Cartesian products as well as strong products of finitely many cycles, where there are certain conditions on lengths of individual cycles [13, 14]. Indeed, graphs G exist for which $\lambda(G)$ is strictly larger than the lower bound suggested by Lemma

1.1 [30]. The present paper presents sharp bounds on λ-number of direct product (defined below) of cycles and paths.

By a graph is meant a finite, simple and undirected graph having at least two vertices. Unless otherwise indicated, graphs are also connected. Let P_m (resp. C_m) denote a path (resp. a cycle) on m vertices, where $V(P_m) = V(C_m) = \{0, \ldots, m-1\}$ and where adjacencies are defined in a natural way.

For graphs $G = (V, E)$ and $H = (W, F)$, the *direct product* $G \times H$ of G and H is defined as follows: $V(G \times H) = V \times W$ and $E(G \times H) = \{\{(a, x), (b, y) : \{a, b\} \in E \text{ and } \{x, y\} \in F\}$. This product (that is commutative and associative in a natural way) is one of the most important graph products with potential applications in engineering, computer science and related disciplines. For example, the diagonal mesh studied by Tang and Padubirdi [26] with respect to multiprocessor network is representable as $\times$-product of two odd cycles that has several attractive properties, viz., low diameter, high independence number and high odd girth [12]. Ramirez and Melhem [22] present a fault-tolerant computational array whose underlying graph is isomorphic to a connected component of $P_{2i+1} \times P_{2i+1}$.

The following statements are relevant with respect to $C_m \times C_n$, $C_m \times P_n$, and $P_m \times P_n$, and will be (implicitly) used in the sequel:

(i) $C_{2i+1} \times C_{2j+1}$ is nonbipartite while each of the rest is bipartite, and

(ii) each of $C_{2i+1} \times C_n$ and $C_{2i+1} \times P_n$ is connected, while each of the rest consists of two connected components.

(iii) $C_{2i+1} \times P_n$ is isomorphic to a connected component of $C_{2(2i+1)} \times P_n$.

Let $P = v_1, v_2 \ldots v_n$ and $Q = u_1, u_2 \ldots u_n$ be disjoint paths on n vertices. Then Z_n denotes the graph with the set of vertices $V(Z) := V(P) \cup V(Q)$. The set of edges of Z_n is for $i = 1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor$ defined with:

$$E(Z) := \begin{cases} E(P) \cup E(Q) \cup \{v_{2i}u_{2i-1}, v_{2i}u_{2i+1}\}; & n \text{ odd}, \\ E(P) \cup E(Q) \cup \{v_{2i}u_{2i-1}, v_{2i}u_{2i'1}\} \cup \{v_n u_{n-1}\}; & n \text{ even}. \end{cases}$$

Let f and g be $(2,1)$-labelings of P_n and let $f \circ g$ be the assignment to the vertices of Z_n such that the restriction of $f \circ g$ to the first (second) P_n in Z_n equals f (g).

We now define graph denoted $D_{n,q}$ as follows. Its vertices are q-(2,1)-labelings of P_n. Vertices $f, g \in D_{n,q}$ are adjacent if and only if $f \circ g$ is a $(2,1)$-labeling in Z_n.

The next theorem can now be very easily derived from the concepts and results presented in [18].

Theorem 1.2 *(i) $C_{2i} \times P_n$ admits a q-(2,1)-labeling if and only if $D_{n,q}$ contains a closed walk of length i.*

(ii) $C_{2i+1} \times P_n$ admits a q-(2,1)-labeling if and only if $D_{n,q}$ contains a closed walk of length $2i+1$.

2 Preliminaries

Let $G = (V(G), E(G))$ be a graph. A *walk* is a sequence of vertices $v_1, v_2, \ldots, v_n$ and edges v_iv_{i+1}, $1 \leq i \leq n-1$. A *path* on n vertices is a walk on n different vertices and denoted P_n. A walk is *closed* if $v_1 = v_n$. A closed walk in which all vertices (except the first and the last) are different, is a *cycle*. The cycle on n vertices is denoted C_n. For $u, v \in V(G)$, $d_G(u,v)$ or $d(u,v)$ denotes the length of a shortest walk (i.e., the number of edges on a shortest walk) in G from u to v. These definitions extend naturally to directed graphs.

Let $G_0, G_1, \ldots, G_{n-1}$ be disjoint graphs and $X_0, X_1, \ldots, X_{n-1}$ a sequence of sets of edges such that an edge of X_i joins a vertex of G_i with a vertex of G_{i+1} (indices modulo n). A *polygraph*

$$\Omega_n = \Omega_n(G_0, G_1, \ldots, G_{n-1};\ X_0, X_1, \ldots, X_{n-1})$$

is defined in the following way:

$$V(\Omega_n) = V(G_0) \cup V(G_1) \cup \cdots \cup V(G_{n-1}),$$
$$E(\Omega_n) = E(G_0) \cup X_0 \cup E(G_1) \cup X_1 \cup \cdots \cup E(G_{n-1}) \cup X_{n-1}.$$

Polygraphs were introduced in chemical graph theory as a model for polymers, cf. [1], and studied in, for instance, [17, 19, 31]. Assume that for $0 \leq i \leq n-1$, G_i is isomorphic to a fixed graph G. Let, in addition, the sets X_i, $0 \leq i \leq n-1$, be equal to a fixed edge set X. Then we call the polygraph Ω_n a *rotagraph* and denote it $\omega_n(G;X)$. We will also say that $\omega_n(G;X)$ is a rotagraph with consecutive *fibers* $G_0, G_1, \ldots, G_{n-1}$. A *fasciagraph* $\psi_n(G;X)$ is a rotagraph $\omega_n(G;X)$ *without* edges between the fibers G_{n-1} and G_0.

In the rest of this section we recall concepts and results that were recently introduces in [18] and are essential for the present work. For a graph G set

$$\mathcal{F}_q(G) = \{f : V(G) \to \{0, 1, \ldots, q-1\}\}\,.$$

A subset of $\mathcal{F}_q(G)$ will be called a *graph q-property*. If q will be clear from the context or not essential, we will shortly say a *graph property*.

Let $\mathcal{L}_q(G) \subseteq \mathcal{F}_q(G)$ be the set of functions f with the following property: Let $f \in \mathcal{L}_q(G)$, then if $uv \in E(G)$ we have $|f(u) - f(v)| \geq 2$, and if $d(u,v) = 2$ we have $|f(u) - f(v)| \geq 1$. Clearly, $\mathcal{L}_q(G)$ describes the admissible $(2,1)$-labelings of G.

Let $\omega_n(G;X)$ be a rotagraph with consecutive fibers $G_0, G_1, \ldots, G_{n-1}$. Then the restriction of $f \in \mathcal{F}_q(\omega_n(G;X))$ to consecutive fibers $X_i, X_{i+1}, \ldots, X_{i+k}$ (indices modulo n) will be denoted f_i^{i+k}. We say that a graph property $\mathcal{P}_q$ is *hereditary* (for rotagraphs), if for any rotagraph $\omega_n(G;X)$ with consecutive fibers $G_0, G_1, \ldots, G_{n-1}$,

$$f \in \mathcal{P}_q(\omega_n(G;X)) \Rightarrow f_i^{i+k} \in \mathcal{P}_q(\Psi_{k+1}(G;X));\ i,k = 0, 1, \ldots, n-1\,.$$

Note that $\mathcal{L}_q$ is hereditary property.

A graph property $\mathcal{P}_q$ is called *d-local* (for rotagraphs), $d \geq 1$, if for any rotagraph $\omega_n(G;X)$, $n \geq 2d+1$, with consecutive fibers $G_0, G_1, \ldots, G_{n-1}$, and any $f \in \mathcal{F}_q(\omega_n(G;X))$,

$$f_i^{i+d} \in \mathcal{P}_q(\Psi_{d+1}(G;X)), 0 \leq i \leq n-1 \Rightarrow f \in \mathcal{P}_q(\omega_n(G;X))\,.$$

Note that $\mathcal{L}_q$ is a 2-local property.

Let $\mathcal{P}_q$ be a d-local property, and $\omega_n(G;X)$ a rotagraph with $n \geq 2d+1$. We define a directed graph $D_d(G;X)$ as follows. Its vertices are the functions from $\mathcal{P}_q(\Psi_2(G;X))$, while its arcs are of two types: the first type arcs will be simply called *arcs*, and the second type arcs will be *d-arcs*. Now, in $D_d(G;X)$ make an arc from f to g if and only if f restricted to the second fiber of $\Psi_2(G;X)$ equals to g restricted to the first fiber of $\Psi_2(G;X)$. In addition, if $d \geq 2$, then for any directed path (consisting of arcs) of length $d-1$, say $f_1 \to f_2 \to \ldots \to f_d$, we make a d-arc from f_1 to f_d whenever the composition of $f_1, f_2, \ldots, f_d$ belongs to $\mathcal{P}_q(\Psi_{d+1}(G;X))$. In the particular case when $d = 2$ we interpret this as follows: If the composition of f_1 and f_2 belongs to $\mathcal{P}_q(\Psi_3(G;X))$ then we leave the arc from f_1 to f_2, otherwise we remove it.

Theorem 2.1 ([18]) *Let $\mathcal{P}_q$ be a hereditary, d-local property, and $\omega_n(G;X)$ a rotagraph with $n \geq 2d+1$. Then $\mathcal{P}_q(\omega_n(G;X)) \neq \emptyset$ if and only if $D_d(G;X)$ contains (not necessarily different) vertices $f_0, f_1, \ldots, f_{n-1}$ connected with arcs (f_i, f_{i+1}) and d-arcs (f_i, f_{i+d-1}) for $i = 0, 1, \ldots, n-1$ (indices modulo n).*

Corollary 2.2 ([18]) *Let $\mathcal{P}_q$ be a hereditary, d-local property, $1 \leq d \leq 2$, and $\omega_n(G;X)$ a rotagraph with $n \geq 5$. Then $\mathcal{P}_q(\omega_n(G;X)) \neq \emptyset$ if and only if $D_d(G;X)$ contains a directed closed walk of length n.*

3 λ-numbers of $C_{7i} \times C_{7j}$ and $C_{11i} \times C_{11j} \times C_{11k}$

Determining $\lambda(C_m \times C_n)$ is important also because it yields analogous results for $\lambda(C_m \times P_n)$ and $\lambda(P_m \times P_n)$ in most cases. In the present section, we show that the lower bound of Lemma 1.1 is achieved for each of $C_{7i} \times C_{7j}$ and $C_{11i} \times C_{11j} \times C_{11k}$.

Theorem 3.1 *If $m \equiv 0 \, (mod\ 7)$ and $n \equiv 0 \, (mod\ 7)$, then $\lambda(C_m \times C_n) = 6$.*

Proof. By Lemma 1.1, $\lambda(C_m \times C_n) \geq 6$, since $C_m \times C_n$ is a regular graph of degree four. It, therefore, suffices to present a valid $L(2,1)$-labeling of $C_m \times C_n$ using the labels $0, \ldots, 6$, where m and n are as stated. Let a vertex (i,j) of $C_m \times C_n$ be assigned the integer $f(i,j) = (8i + 4j) \bmod 7$. The assignment is clearly well-defined.

A vertex adjacent to (i,j) is of the form $(i+a, j+b)$, where $a, b \in \{+1, -1\}$, and $i+a$ (resp. $j+b$) is modulo m (resp. n). Note that $f(i+a, j+b) = [(8i+4j) + (8a+4b)] \bmod 7$. For the four cases corresponding to a, b in $\{+1, -1\}$, $(8a+4b) \bmod 7$ is equal to exactly one of 2, 3, 4 and 5. Accordingly, $2 \leq |f(i,j) - f(i+a, j+b)| \leq 5$.

A vertex at a distance of two from (i,j) is of the form $(i+c, j+d)$, where $c, d \in \{+2, 0, -2\}$, and c, d are not both zero. Note that $f(i+c, j+d) = [(8i+4j) + (8c+4d)] \bmod 7$. Conditions on c and d are such that $8c+4d$ is necessarily nonzero. Further, $|8c+4d|$ is a multiple of 8 and at most equal to 24. Accordingly, $8c+4d$ is not a multiple of 7. It follows that $|f(i,j) - f(i+c, j+d)| \geq 1$.

Conclusions are valid even if i (resp. j) is of the form $m-2$ or $m-1$ (resp. $n-2$ or $n-1$), since m and n are themselves multiples of 7. $\square$

For $0 \leq a \leq 6$, let V_a be the set of vertices of a connected component of $C_m \times C_n$ that receive label a in the proof of Theorem 3.1. The sets $V_0, \ldots, V_6$ form a vertex partition into equal-size independent sets, where elements of each V_a dominate (5/7)th of the vertices (including themselves) in that component. Accordingly, elements of each V_a correspond to as many vertex-disjoint $K_{1,4}$'s. Also, vertices in each $(V_{2i} \cup V_{2i+1})$ correspond to as many edge-disjoint $K_{1,4}$'s, $0 \leq i \leq 2$.

Corollary 3.2 *If $m \geq 5$, $n \geq 4$ and $i \geq 1$, then $\lambda(P_m \times P_n) = \lambda(C_{7i} \times P_n) = 6$.*

Proof. Each of $P_m \times P_n$ and $C_{7i} \times P_n$ is of largest degree four, and satisfies Lemma 1.1. Further, (i) $P_m \times P_n$ is a subgraph of
$C_{7i} \times C_{7j}$ for some i and j, and (ii) $C_{7i} \times P_n$ is a subgraph of $C_{7i} \times C_{7j}$ for some j. □

Theorem 3.3 *If $r \equiv 0\,(mod\ 11)$, $s \equiv 0\,(mod\ 11)$ and $t \equiv 0\,(mod\ 11)$, then $\lambda(C_r \times C_s \times C_t) = 10$.*

Proof. By Lemma 1.1, $\lambda(C_r \times C_s \times C_t) \geq 10$ as $C_r \times C_s \times C_t$ is a regular graph of degree eight, so it suffices to present a valid $L(2,1)$-labeling of $C_r \times C_s \times C_t$ using the labels $0, \ldots, 10$. Let a vertex (i,j,k) of $C_r \times C_s \times C_t$ be assigned the integer $(24i + 12j + 6k)$ mod 11. The assignment is clearly well-defined.

Analogous to the proof of Theorem 3.1, it suffices to prove that (i) $2 \leq (24a+12b+6c)$ mod $11 \leq 9$, where $a,b,c \in \{+1,-1\}$, and (ii) $(24x + 12y + 6z)$ mod $11 > 0$, where $x,y,z \in \{+2,0,-2\}$ and x,y,z are not all zero.

There are a total of eight cases corresponding to $a,b,c \in \{+1,-1\}$. For each, the reader may check to see that $(24a + 12b + 6c)$ mod 11 is equal to exactly one of 2, 3, 4, 5, 6, 7, 8 and 9. It is next shown that $24x + 12y + 6z$ is nonzero and not a multiple of 11, where x, y and z are as stated.

If $x \neq 0$, then $24x + 12y + 6z$ is of the same sign as x; if $x = 0$ and $y \neq 0$, then $24x + 12y + 6z$ is of the same sign as y; if $x = y = 0$, then $z \neq 0$, and $24x + 12y + 6z$ is of the same sign as z. It follows that $24x + 12y + 6z \neq 0$.

"$24x + 12y + 6z$ is not a multiple of 11" is equivalent to "$|4x + 2y + z|$ is not a multiple of 11." If $x = 0$, then $|4x + 2y + z| \leq 6 < 11$. If $y = 0$, then $|4x + 2y + z| \leq 10 < 11$. If $z = 0$, then $|4x + 2y + z| = 2 \cdot |2x + y|$ that is not a multiple of 11 as $2x + y$ is not such. It follows that if $x = 0$ or $y = 0$ or $z = 0$, then $|4x + 2y + z|$ is not a multiple of 11.

If x, y and z are all nonzero and of the same sign, then $|4x + 2y + z| = 14$ that is not a multiple of 11. On the other hand, if x, y and z are all nonzero and not of the same sign, then $|4x + 2y + z| < 11$. □

Let V_a be the set of vertices of a connected component of $C_r \times C_s \times C_t$ that receive label a in the proof of Theorem 3.3, $0 \leq a \leq 10$. The sets $V_0, \ldots, V_{10}$ form a vertex partition into equal-size independent sets. Elements of each V_a correspond to as many vertex-disjoint $K_{1,8}$'s. Also, elements of each $(V_{2i} \cup V_{2i+1})$ correspond to as many edge-disjoint $K_{1,8}$'s, $0 \leq i \leq 4$.

We conclude this section with an upper bound on λ-number of finitely many cycles.

Theorem 3.4 *If $k \geq 2$ and $m_0, \ldots, m_{k-1}$ are each a multiple of $2^k + 1$, then $2^k + 2 \leq \lambda(C_{m_0} \times \cdots \times C_{m_{k-1}}) \leq 2^{k+1}$.*

Proof. Let k and $m_0, \ldots, m_{k-1}$ be as stated. The graph $C_{m_0} \times \cdots C_{m_{k-1}}$ is regular of degree 2^k. Accordingly, lower bound is immediate. Further, this graph admits of a vertex partition into equal-size (independent dominating) sets $V_0, \ldots, V_{2^k}$ such that the (shortest) distance between any two distinct elements of V_i is at least three [15]. Let a vertex v be assigned the integer label $2i$ if and only if $v \in V_i$, $0 \leq i \leq 2^k$. It is easy to see that the resulting labeling is a valid $L(2,1)$-labeling. Accordingly, $\lambda(C_{m_0} \times \cdots \times C_{m_{k-1}}) \leq 2^{k+1}$. □

4 λ-numbers of $P_4 \times C_m$

In Corollary 3.2 we have seen that $\lambda(P_n \times C_{7i}) = 6$, $n \geq 4$, $i \geq 1$. In this section we demonstrate that for $n = 4$, the result holds for any cycle C_m:

Theorem 4.1 *For any $m \geq 3$, $\lambda(P_4 \times C_m) = 6$.*

Proof. By Lemma 1.1, $\lambda(P_4 \times C_m) \geq 6$ for any $m \geq 3$. Hence we need to construct labelings with labels 0, 1, 2, 3, 4, 5, 6.

Case 1: $m = 4 + 4s$, $s \geq 0$.
In this case, we repeat the following labeling:

2	2	3	3	|	2	2	3	3
5	5	6	6	|	5	5	6	6
0	0	1	1	|	0	0	1	1
3	3	4	4	|	3	3	4	4

Case 2: $m = 9 + 4s$, $s \geq 0$.
Now we have the following repeated solution:

0	0	1	1	2	2	3	3	4	|	2	3	3	4
2	5	3	6	4	0	5	1	6	|	0	5	1	6
2	5	3	6	4	0	5	1	6	|	0	5	1	6
0	0	1	1	2	2	3	3	4	|	2	3	3	4

Case 3: $m = 14 + 4s$, $s \geq 0$.
In this case we have the following repeated solution:

0	0	1	1	2	2	3	3	4	|	4	5	5	6	6	|	2	3	3	4
2	5	3	6	4	0	5	1	6	|	2	0	3	1	4	|	0	5	1	6
2	5	3	6	4	0	5	1	6	|	2	0	3	1	4	|	0	5	1	6
0	0	1	1	2	2	3	3	4	|	4	5	5	6	6	|	2	3	3	4

Case 4: $m = 23 + 4s$, $s \geq 0$.
In this case we proceed as follows. First take two times the block with 9 columns and after the block with 5 columns from Case 3. This gives a solution for $m = 23$. Then repeat the block with 4 columns in order to get all the remaining solutions.

Hence we are left with the following sporadic cases:

Case 5: $m = 3, 5, 6, 7, 10, 11, 15, 19$.

For $m = 7$ we apply Theorem 3.1. For the other cases solutions are, respectively:

6	2	1
5	3	0
5	3	0
6	2	1

4	2	5	3	3
6	2	5	0	6
0	0	6	1	4
2	4	4	1	3

3	2	4	4	5	6
0	6	6	2	2	0
2	3	0	0	6	5
1	5	5	3	3	4

0	0	2	2	3	3	4	4	3	3
5	4	4	0	0	1	1	0	0	5
1	1	6	6	5	5	6	6	2	2
6	3	3	2	2	3	3	4	4	6

0	0	1	1	0	4	4	5	3	4	2
6	4	5	3	6	2	2	1	0	0	6
2	3	6	2	5	0	6	6	5	3	4
1	0	0	1	4	0	3	3	1	2	5

1	4	4	2	6	6	4	1	3	6	2	3	3	5	6
2	6	6	0	0	2	3	6	4	5	0	6	1	0	3
0	0	3	3	5	5	0	0	1	2	0	4	4	6	5
3	5	5	1	1	3	2	4	5	3	6	2	2	1	4

4	5	5	1	1	3	4	5	5	4	4	2	3	5	5	6	4	2	2
0	0	3	3	6	6	1	0	2	2	6	6	0	0	2	2	4	6	6
3	6	6	0	0	4	3	6	6	0	0	3	2	4	6	0	0	1	3
1	1	4	4	2	5	0	1	4	4	5	5	1	4	6	3	3	5	5

□

5 λ-numbers of $P_5 \times C_m$

The result of the previous section asserts that for any any $m \geq 3$, $\lambda(P_4 \times C_m) = 6$. For the direct products $P_5 \times C_m$ the situation is similar: For almost any m, $\lambda(P_5 \times C_m) = 6$. However, there are several exceptions that make our considerations a bit more involved. We are going to prove:

Theorem 5.1 *Let $m \geq 3$. Then*

$$\lambda(P_5 \times C_m) = \begin{cases} 7; & m = 3,4,5,6,8,9,10,12,13,17,18,20,24,26,34,40\,, \\ 6; & \text{otherwise}\,. \end{cases}$$

Proof. By Lemma 1.1, $\lambda(P_5 \times C_m) \geq 6$ for any $m \geq 3$.

We first present solutions for the products $P_5 \times C_{2k}$, $k \geq 22$. Any such graph contains two isomorphic connected components, thus we will give solutions for one component. First, the following blocks will be called *Block A* and *Block B*, respectively.

0	5	2	0	1	6	4	2
3	0	6	5	4	2	0	5
1	6	4	3	2	0	5	3
4	2	1	0	6	3	1	6
2	0	5	6	4	1	6	4

2	0	5	2	0	1	6	4
5	3	0	6	5	4	2	0
3	1	6	4	3	2	0	5
6	4	2	1	0	6	3	1
4	2	0	5	6	4	1	6

Case 1: $k = 22 + 8s$, $s \geq 0$.
$L(2,1)$-labelings are obtained from the following solution for $k = 22$ to which we add Block A as many times as necessary.

0	5	3	1	3	5	0	1	4	5	2	0	5	3	1	3	5	0	1	4	5	2
3	1	6	5	0	2	4	6	2	0	5	3	1	6	5	0	2	4	6	2	0	5
1	6	4	0	2	4	6	2	0	6	3	1	6	4	0	2	4	6	2	0	6	3
4	0	2	4	6	1	0	5	3	1	6	4	0	2	4	6	1	0	5	3	1	6
0	2	5	6	1	3	5	3	1	5	4	0	2	5	6	1	3	5	3	1	5	4

Case 2: $k = 23 + 8s$, $s \geq 0$.
$L(2,1)$-labelings are obtained from the following solution for $k = 23$ to which we add Block A as many times as necessary.

0	5	0	1	0	6	5	4	3	2	0	1	6	4	2	0	5	2	0	1	6	4	2
3	2	6	5	4	3	2	1	0	6	5	4	2	0	5	3	0	6	5	4	2	0	5
1	6	4	3	2	1	0	6	5	4	3	2	0	5	3	1	6	4	3	2	0	5	3
4	0	1	0	6	5	4	3	2	1	0	6	3	1	6	4	2	1	0	6	3	1	6
0	2	5	6	4	3	2	1	0	6	5	4	1	6	4	2	0	5	6	4	1	6	4

Case 3: $k = 24 + 8s$, $s \geq 0$.
$L(2,1)$-labelings are obtained from the following solution for $k = 24$ to which we add Block B as many times as necessary.

2	0	5	2	0	1	6	4	2	0	5	2	0	1	6	4	2	0	5	2	0	1	6	4
5	3	0	6	5	4	2	0	5	3	0	6	5	4	2	0	5	3	0	6	5	4	2	0
3	1	6	4	3	2	0	5	3	1	6	4	3	2	0	5	3	1	6	4	3	2	0	5
6	4	2	1	0	6	3	1	6	4	2	1	0	6	3	1	6	4	2	1	0	6	3	1
4	2	0	5	6	4	1	6	4	2	0	5	6	4	1	6	4	2	0	5	6	4	1	6

Case 4: $k = 25 + 8s$, $s \geq 0$.
$L(2,1)$-labelings are obtained from the following solution for $k = 25$ to which we add Block A as many times as necessary.

0 5 3 1 3 5 0 1 4 5 2 0 5 3 1 6 4 2 0 5 3 1 6 4 2
3 1 6 5 0 2 4 6 2 0 5 3 1 6 4 2 0 5 3 1 6 4 2 0 5
1 6 4 0 2 4 6 2 0 6 3 1 6 4 2 0 5 3 1 6 4 2 0 5 3
4 0 2 4 6 1 0 5 3 1 6 4 2 0 5 3 1 6 4 2 0 5 3 1 6
0 2 5 6 1 3 5 3 1 5 4 2 0 5 3 1 6 4 2 0 5 3 1 6 4

Case 5: $k = 26 + 8s$, $s \geq 0$.
L(2,1)-labelings are obtained from the following solution for $k = 26$ to which we add Block A as many times as necessary.

0 5 3 1 3 5 0 1 4 5 2 0 5 0 1 0 6 5 4 3 2 0 1 6 4 2
3 1 6 5 0 2 4 6 2 0 5 3 2 6 5 4 3 2 1 0 6 5 4 2 0 5
1 6 4 0 2 4 6 2 0 6 3 1 6 4 3 2 1 0 6 5 4 3 2 0 5 3
4 0 2 4 6 1 0 5 3 1 6 4 0 1 0 6 5 4 3 2 1 0 6 3 1 6
0 2 5 6 1 3 5 3 1 5 4 0 2 5 6 4 3 2 1 0 6 5 4 1 6 4

Case 6: $k = 27 + 8s$, $s \geq 0$.
L(2,1)-labelings are obtained from the following solution for $k = 27$ to which we add Block A as many times as necessary.

0 5 3 1 3 5 0 1 4 5 2 0 5 2 0 1 6 4 2 0 5 2 0 1 6 4 2
3 1 6 5 0 2 4 6 2 0 5 3 0 6 5 4 2 0 5 3 0 6 5 4 2 0 5
1 6 4 0 2 4 6 2 0 6 3 1 6 4 3 2 0 5 3 1 6 4 3 2 0 5 3
4 0 2 4 6 1 0 5 3 1 6 4 2 1 0 6 3 1 6 4 2 1 0 6 3 1 6
0 2 5 6 1 3 5 3 1 5 4 2 0 5 6 4 1 6 4 2 0 5 6 4 1 6 4

Case 7: $k = 28 + 8s$, $s \geq 0$.
L(2,1)-labelings are obtained from the following solution for $k = 28$ to which we add Block B as many times as necessary.

2 0 5 3 1 6 4 2 0 5 3 1 6 4 2 0 5 3 1 6 4 2 0 5 3 1 6 4
5 3 1 6 4 2 0 5 3 1 6 4 2 0 5 3 1 6 4 2 0 5 3 1 6 4 2 0
3 1 6 4 2 0 5 3 1 6 4 2 0 5 3 1 6 4 2 0 5 3 1 6 4 2 0 5
6 4 2 0 5 3 1 6 4 2 0 5 3 1 6 4 2 0 5 3 1 6 4 2 0 5 3 1
4 2 0 5 3 1 6 4 2 0 5 3 1 6 4 2 0 5 3 1 6 4 2 0 5 3 1 6

Case 8: $k = 29 + 8s$, $s \geq 0$.
L(2,1)-labelings are obtained from the following solution for $k = 29$ to which we add Block A as many times as necessary.

0 5 3 1 3 5 0 1 4 5 2 0 5 3 1 3 5 0 1 4 5 2 0 5 3 1 6 4 2
3 1 6 5 0 2 4 6 2 0 5 3 1 6 5 0 2 4 6 2 0 5 3 1 6 4 2 0 5
1 6 4 0 2 4 6 2 0 6 3 1 6 4 0 2 4 6 2 0 6 3 1 6 4 2 0 5 3
4 0 2 4 6 1 0 5 3 1 6 4 0 2 4 6 1 0 5 3 1 6 4 2 0 5 3 1 6
0 2 5 6 1 3 5 3 1 5 4 0 2 5 6 1 3 5 3 1 5 4 2 0 5 3 1 6 4

By the above cases we have $\lambda(P_5 \times C_{2k}) = 6$ for $k \geq 22$. If $2k = 4i + 2$, then each connected component of $P_5 \times C_{2k}$ is isomorphic to $P_5 \times C_k$, hence we also have $\lambda(P_5 \times C_k) = 6$ for k odd, and $k \geq 23$.

We next demonstrate that $P_5 \times C_k$ admits $L(2{,}1)$-labelings with 7 labels for $k = 11$, 15, 16, 19, 22, 30, 32, 36, 38. Since solutions for 22, 30, and 38 give also solutions for 11, 15, and 19, respectively, it is enough to present solutions for the cases 16, 22, 30, 32, 36 and 38. They are, respectively, given below.

2	5	6	1	3	5	3	1
0	2	4	6	1	0	5	4
6	4	0	2	4	6	2	0
1	6	5	0	2	4	6	2
5	3	1	3	5	0	1	4

0	5	3	1	3	5	0	1	4	5	2
3	1	6	5	0	2	4	6	2	0	5
1	6	4	0	2	4	6	2	0	6	3
4	0	2	4	6	1	0	5	3	1	6
0	2	5	6	1	3	5	3	1	5	4

0	5	0	1	0	6	5	4	3	2	0	1	6	4	2
3	2	6	5	4	3	2	1	0	6	5	4	2	0	5
1	6	4	3	2	1	0	6	5	4	3	2	0	5	3
4	0	1	0	6	5	4	3	2	1	0	6	3	1	6
0	2	5	6	4	3	2	1	0	6	5	4	1	6	4

2	0	5	2	0	1	6	4	2	0	5	2	0	1	6	4
5	3	0	6	5	4	2	0	5	3	0	6	5	4	2	0
3	1	6	4	3	2	0	5	3	1	6	4	3	2	0	5
6	4	2	1	0	6	3	1	6	4	2	1	0	6	3	1
4	2	0	5	6	4	1	6	4	2	0	5	6	4	1	6

0	5	3	1	3	5	0	1	4	5	2	0	5	3	1	6	4	2
3	1	6	5	0	2	4	6	2	0	5	3	1	6	4	2	0	5
1	6	4	0	2	4	6	2	0	6	3	1	6	4	2	0	5	3
4	0	2	4	6	1	0	5	3	1	6	4	2	0	5	3	1	6
0	2	5	6	1	3	5	3	1	5	4	2	0	5	3	1	6	4

0	5	3	1	3	5	0	1	4	5	2	0	5	2	0	1	6	4	2
3	1	6	5	0	2	4	6	2	0	5	3	0	6	5	4	2	0	5
1	6	4	0	2	4	6	2	0	6	3	1	6	4	3	2	0	5	3
4	0	2	4	6	1	0	5	3	1	6	4	2	1	0	6	3	1	6
0	2	5	6	1	3	5	3	1	5	4	2	0	5	6	4	1	6	4

By the above constructions and by Corollary 3.2 we conclude that $\lambda(P_5 \times C_m) = 6$ for all m except for $m = 3$, 4, 5, 6, 8, 9, 10, 12, 13, 17, 18, 20, 24, 26, 34, and 40. To complete the proof we must show that in these remaining cases $\lambda(P_5 \times C_m) = 7$ holds.

We first claim that there are no $L(2,1)$-labelings with 7 labels for $P_5 \times C_{2k}$ if $k < 7$ or $k =$ 9, 10, 12, 13, 17, 20. The graph $D_{5,6}$ consists of 1098 vertices (determined by a computer program). In order to search for cycles in $D_{5,6}$ exactly one strongly connected component (with 132 vertices) was detected. Using a simple backtracking in that component, we have established that $D_{5,6}$ does not contain cycles of length 2, 3, 4, 5, 6, 9, 10, 12, 13, 17 and 20. Therefore, by Theorem 1.2 there are also no $L(2,1)$-labelings with 7 labels for $P_5 \times C_k$ where $k = 3$, 4, 5, 6, 8, 9, 10, 12, 13, 17, 18, 20, 24, 26, 34, and 40.

Finally, we implemented the antivoter algorithm [21] adapted for $L(2,1)$-labelings. We have obtained labelings with 8 labels for $P_5 \times C_k$, where $k < 7$, and $k = 8$, 9, 10, 12, 13, 17, 18, 20, 24, 26, 34, and 40. Note that from a labeling with 8 labels of $P_5 \times C_k$ a labeling with 8 labels of $P_5 \times C_{2k}$ can be constructed easily, therefore we list only the cases with $k = 3, 4, 5, 9, 13, 17$.

1	6	5
2	7	4
0	7	4
0	6	3
1	5	2

4	7	5	2
4	0	5	1
7	0	6	1
7	2	6	3
0	3	5	4

2	2	7	3	1
6	0	0	4	7
4	4	6	5	0
7	1	1	3	2
7	3	5	6	1

2	6	5	7	4	2	5	3	7
2	7	3	2	5	1	7	0	0
4	0	0	1	7	0	3	5	4
6	6	7	4	4	5	3	7	1
3	3	1	2	2	6	1	7	2

2	7	6	2	2	5	1	6	2	2	6	3	5
2	4	5	0	7	6	3	4	4	0	7	1	6
0	0	7	1	4	0	0	1	7	0	3	1	4
7	2	3	1	4	2	5	3	6	5	5	6	3
5	5	6	7	5	2	6	3	1	2	2	7	1

6	6	5	7	1	3	5	2	3	3	5	2	4	7	5	1	1
4	1	2	7	1	3	6	1	5	7	6	0	0	2	5	3	4
7	0	3	5	0	4	7	1	4	0	2	4	7	3	6	0	7
5	0	3	6	0	4	7	0	3	6	2	5	1	1	7	0	2
5	1	2	7	1	2	6	5	3	6	1	7	3	5	4	4	3

□

The antivoter algorithm that we used at the end of the above proof and some of its generalizations have proved to be reasonably good heuristics for coloring various types of graphs including random k-colorable graphs, DIMACS challenge graphs [3], frequency assignment "realistic" graphs, and others [25, 27, 29]. For completeness of the presentation we briefly recall the algorithm:

get a random order of vertices;
run a greedy coloring algorithm;

while not stopping condition do
    if the coloring is proper then recolor vertices of the maximum color
    select a bad vertex v (randomly)
    assign a new color to v
end while

The greedy coloring always takes the minimal color which does not violate any constraints.

6 λ-numbers of $P_n \times C_m$, $n \geq 6$

In this section we prove that Corollary 3.2 finds all optimal solutions (with respect to Lemma 1.1) for $n \geq 6$. More precisely:

Theorem 6.1 *Let $n \geq 6$ and $m \geq 7$. Then $\lambda(P_n \times C_m) = 6$ if and only if $m = 7k$, $k \geq 1$.*

Proof. By Lemma 1.1, $\lambda(P_n \times C_m) \geq 6$. Hence, using Corollary 3.2, it suffices to show that $\lambda(P_6 \times C_m) \geq 7$ if $m \neq 7k$. For this sake we use our method of Theorem 1.2.

We know that $P_6 \times C_m$ admits a 6-(2,1)-labeling if and only if $D_{6,6}$ contains a closed walk of length m, if m is odd, or a closed walk of length $\frac{m}{2}$, if m is even. The graph $D_{6,6}$ consists of 3638 vertices (determined by a computer program). In order to search for cycles in $D_{6,6}$, exactly eight strongly connected components of $D_{6,6}$ were detected, each of them consisting of seven vertices and exactly one directed cycle. Therefore, all closed walks in $D_{6,6}$ are of length $7k$, $k \geq 1$, thus a 6-$L(2,1)$-labeling of 0 $P_6 \times C_m$ for $m \not\equiv 0 \pmod 7$ does not exist. □

By Theorem 6.1, $\lambda(P_n \times C_m) \geq 7$ for $m \neq 7k$. We believe that the equality holds, but were not able to cover all the cases. For instance, we can show that for any $n \geq 6$ and any $k \geq 1$ we have $\lambda(P_n \times C_{3k}) = 7$. In addition, for any $n \geq 6$ we also have $\lambda(P_n \times C_4) = 7$. In general, however, the above conjecture cannot be deduced from labelings of direct products of two cycles in the way as is Corollary 3.2 obtained from Theorem 3.1. Indeed, using backtracking we computed that there is no labeling with labels $0, 1, \ldots, 7$ for any of the graphs $C_4 \times C_4$, $C_4 \times C_5$, $C_5 \times C_5$, $C_5 \times C_6$, and $C_6 \times C_6$.

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