Almost self-centered graphs

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Abstract

The center of a graph is the set of vertices with minimum eccentricity. Graphs in which all vertices are central are called self-centered graphs. In this paper almost self-centered (ASC) graphs are introduced as the graphs with exactly two non-central vertices. The block structure of these graphs is described and constructions for generating such graphs are proposed. Embeddings of arbitrary graphs into ASC graphs are studied. In particular it is shown that any graph can be embedded into an ASC graph of prescribed radius. Embeddings into ASC graphs of radius two are studied in more detail. ASC index of a graph G is introduced as the smallest number of vertices needed to add to G such that G is an induced subgraph of an ASC graph.

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1 Introduction

The center of a graph/network is one of the central concepts in location theory; many different center-related concepts were introduced. The reason is that we want to distribute sources within a graph/network such that they can be used elsewhere as efficiently as possible. Hence, we put them into central vertices. Another point of view is that every firm would prefer to establish in such a location where abundant natural and human resource are available. At the same time, every individual or consumer would like to be in a location, which is convenient to access all the basic amenities.

In many circumstances an ideal situation would be that resources can be placed at any location. In such cases we wish to have a graph in which any of its vertices is central. Such graphs are called self-centered (SC) graphs and they have been extensively studied, see [1, 3, 8, 12] and the survey [2]. For instance, these graphs are 2-connected and for any finite group F there is a SC graph whose automorphism group is isomorphic to the group F [10]. Recently, several algorithms for constructing SC graphs were described in [9]. We also mention that there is a related concept of eccentric graphs [4] as well as of eccentric digraphs [5].

In this paper we are interested in graphs that are not self-centered but as close to them as possible. If G is not SC, then it has at least two non-central vertices. We introduce and study the so-called almost self-centered (ASC) graphs in which all but two vertices are central. Of course, the remaining two vertices are diametrical. (A variant of this concept in which all vertices are peripheral except one is studied in [15].) See Figure 1 for three examples of ASC graphs, where each vertex is equipped with its eccentricity.

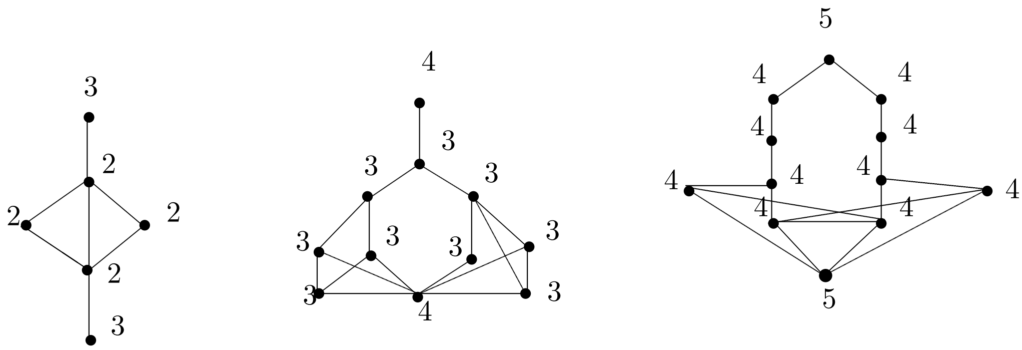

Figure 1: 2-ASC graph, 3-ASC graph, and 4-ASC graph

Situation where exactly two specific locations are desired has been studied earlier. See for instance [7, 13, 16], where in the later paper a problem of deploying two servers in a tree network is studied. ASC graphs may in particular serve as a network model in which we plan to install two expensive resources (we need both to be sure that the system also works even if one is corrupted) and the resources need to be far away due to an interference reason.

The paper is organized as follows. In the rest of this section the concepts needed are formally introduced. Then, in Section 2, we describe the block structure of ASC graphs and we give two constructions that generate such graphs. In the subsequent section we realize ASC graphs on any admissible number of vertices and embed an arbitrary graph into an ASC graph of prescribed radius. In Section 4 we introduce the ASC index of a graph G as the smallest number of vertices we need to add to G such that G is an induced subgraph of the supergraph. We give a closer look to the case where the supergraph is of radius 2. In the concluding remarks we propose several problems for further study.

Let $G = (V(G), E(G))$ be a connected graph. A *block* of G is an inclusion maximal 2-connected subgraph of G or a bridge. (For graph whose all blocks are odd cycles see [11].) The *distance* $d_G(u, v)$ between vertices u and v is the length of a shortest path between u and v. The notation will be simplified to $d(u, v)$ if the graph will be clear from the context. The *eccentricity* $e_G(v)$ or $e(v)$ of a vertex v is the distance to a farthest vertex from v. A vertex v is said to be an *eccentric vertex of* u if $d_G(u, v) = e(v)$. The *radius* $r(G)$ of G and the *diameter* $d(G)$ of G are the minimum and the maximum eccentricity, respectively. The *center* $C(G)$ and the *periphery* $P(G)$ consists of the set of vertices of minimum and maximum eccentricity, respectively. More formally,

$$C(G) = \{u \in V(G) \mid e(u) = r(G)\}$$

and

$$P(G) = \{u \in V(G) \mid e(u) = d(G)\} .$$

Vertices in $C(G)$ are called *central vertices* and those in $P(G)$ *diametrical*. A graph G of order n for which $|C(G)| = n$ (equivalently, $|P(G)| = n$) holds is called a *self-centered graph*, *SC graphs* for short. G is an *almost self-centered graph*, *ASC graphs* for short, if $|C(G)| = |V(G)| - 2$. If r is the radius of an SC graph or an ACS graph G, we will also say that G is an *r-SC graph* or *r-ASC graph*, respectively.

2 Two constructions

In this section we first describe the block structure of ASC graphs. We follow with two constructions that generate ASC graphs. The first construction uses an ASC graph as a basis and the second one uses an SC graph.

Clearly, P_3 is the unique ASC graph on at most 3 vertices. In general, each ASC contains a block with at most two pendant vertices attached to it. More precisely:

Proposition 2.1. *Let G be an ASC graph of order at least four. Then*

(i) G has at most two pendant vertices u and v.

(ii) The neighbor of a vertex u is different than the corresponding for the vertex v.

(iii) $G \setminus \{u, v\}$ is 2-connected.

Proof. It is not difficult to prove the result by considering an arbitrary cut vertex x of G and a shortest $x-y$ path, where $d(x, y) = e(x)$. Instead, we recall from [6] that the center of a graph lies in a single block. Let B be the block of G containing $C(G)$. If at least one of the two diametrical vertices x or y of G lies in B there is nothing to be proved. Suppose, that $x, y \notin B$. Since x and y are diametrical vertices, they cannot be adjacent. Hence x is adjacent to a vertex $x' \in B$ and y to a vertex $y' \in B$. If $x' = y'$ then $d(x, y) = 2$ and B must be a complete subgraph of G. But since G has at least four vertices, G would contain another vertex $z \neq x, y$ with $d(x, z) = 2$. Therefore $x' \neq y'$ and we are done. □

To characterize unicyclic ASC graphs is now (using Proposition 2.1) straightforward: a triangle with two pendant vertices attached (to different vertices of the triangle), and even cycles with one vertex attached.

In order to construct new ASC graphs we introduce the following operation. Let G and H be graphs and $u \in V(G)$. Then let

$$G \oplus_u H$$

be the graph obtained from the disjoint union of G and H by joining each vertex of H to all vertices in the closed neighborhood of u in G.

Theorem 2.2. *Let G be an r-ACS graph, $r \geq 2$, and let $u \in V(G)$ be a vertex with $e(u) = r$. Then for any graph H, $G \oplus_u H$ is an r-ASC graph.*

Proof. Set $K = G \oplus_u H$. Let x and y be vertices of G with $e_G(x) = e_G(y) = r + 1$ and let P be a shortest $x - y$ path in K. If P contains no vertex of H then P has length $r + 1$. Assume P contains a vertex w of H. By the construction and since P is a shortest path, P contains a subpath $w' \to w \to w''$, where $w', w'' \in V(G)$. Replacing this subpath with $w' \to u \to w''$ we get a shortest path between x and y of the same length. In conclusion, $e_K(x) = e_K(y) = r + 1$.

Analogously we infer that $e_K(w) = r$ holds for any vertex $w \in V(G)$ with $e_G(w) = r$.

Consider now an arbitrary vertex w of H (as a vertex of K). Since $e_G(u) = r$, there is a vertex $u' \in V(G)$ and a shortest path $u \to z \to \cdots \to u'$ of length r. Then $w \to z \to \cdots \to u'$ is a shortest path and hence $e_K(w) \geq r$. Similarly we also get that $e_K(w) \leq r$. So, all vertices of K but x and y have eccentricity r. □

Clearly, the only 1-ASC graphs are complete graphs with one edge removed. Note that Theorem 2.2 does not work for $r = 1$ because in $G \oplus_u H$ the eccentricity of any nonadjacent vertices is 2. Hence H must be a complete graph and the graph obtained is again a complete graph with an edge removed.

We next give another construction of ASC graphs that in particular yields many 2-connected ASC graphs. The key idea is use SC graphs.

Theorem 2.3. *Let G be an r-SC graph, u an arbitrary vertex of G, and X the set of eccentric vertices of u. Let H be a graph obtained from G by joining a new vertex x to all vertices of X. If the subgraph of G induced by X is of diameter at most 2, then H is an r-ASC graph.*

Proof. Note first that $e_H(u) = e_H(x) = r + 1$. Let w be an arbitrary vertex of H different from u and x. Since G is an r-SC graph, there is a vertex $w' \in V(G)$ such that $d_G(w, w') = r$. Let P be a shortest $w - w'$ path in H. If P does not contain x, it is clearly of length r. Otherwise P contains a subpath of the form $x' \to x \to x''$, where $x', x'' \in X$. Since X induces a subgraph of G of diameter at most 2, there exists a vertex $x''' \in X$ adjacent to x' and x''. Replacing x with x''' in P yields a shortest $w - w'$ path of length r in G. Therefore $e_H(w) = r$ and we are done. $\square$

We note here that graphs of diameter 2 are a highly nontrivial family of graphs, cf. [14].

Let us consider some special cases of the construction from Theorem 2.3. If G is a complete graph, the constructed graph is a complete graph minus an edge. If $G = C_{2r}$, the constructed graph is C_{2r} with a pendant vertex attached to it. We already know these two families of ASC graphs. If $G = C_{2r+1}$, the constructed graph is C_{2r+1} together with a vertex attached to two adjacent vertices of the cycle. For $r = 1$ we get K_4 minus an edge, but for $r \geq 2$ all the constructed graphs are new.

To conclude the section we pose:

Problem 2.4. *Which SC graphs G and which vertices u of G satisfy the condition of Theorem 2.3? More precisely, when the eccentric vertices of u induce a subgraph of diameter at most 2?*

3 Consequences of the constructions

In this section we give some applications and consequences derived from the constructions given in the previous section.

Lemma 3.1. *Let G be an r-ASC graph of order $n \geq 5$. Then $n \geq 2r + 1$.*

Proof. Let x and y be diametrical vertices of G and $P : x \to x_1 \to \cdots \to x_r \to y$ a diametrical path. Then $x_1, \ldots, x_r$ are from $C(G)$ and consequently belong to the same block B of G. If at least one of x and y, say x, lies in B, then there exists a path between x and x_r that is internally disjoint from P and of length at least r. So G has at least $|V(P)| + r - 1 = 2r + 1$ vertices. Suppose next $x, y \notin B$. Then there is a path between x_1 and x_r internally disjoint from P (in particular, not containing x and y) of length at least $r - 1$. If the length is $r - 1$, the graph constructed so far is an even cycle with two pendant vertices attached and thus not an ASC graph. Hence G contains at least one more vertex. In any case, G has at least $2r + 1$ vertices. $\square$

Theorem 2.2 implies that the bound of Lemma 3.1 is best possible:

Corollary 3.2. *For any $r \geq 1$ and any $n \geq 2r+1$ there exists an r-ASC graph of order n.*

Proof. For $r = 1$, the complete graphs with an edge deleted does the job. In particular, the smallest such graph is P_3 (obtained from K_3 by removing an edge).

Let $r \geq 2$. Then C_{2r} with one vertex attached is an r-ASC graph on $2r+1$ vertices. To construct r-ASC graphs with more than $2r+1$ vertices apply Theorem 2.2. $\square$

In Figure 2 two additional infinite families of 2-ASC and 3-ASC graphs are presented.

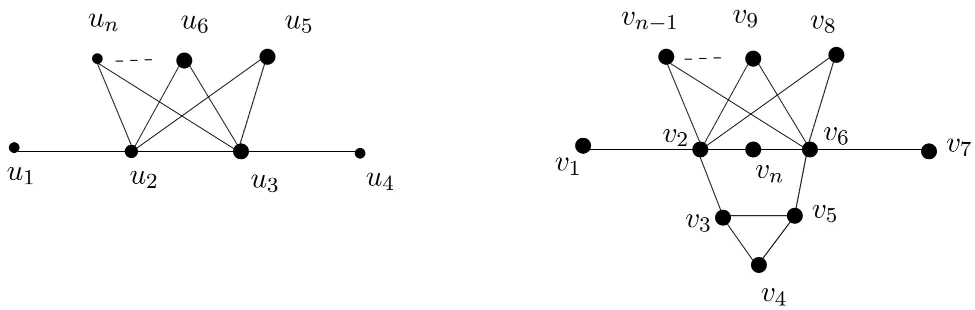

Figure 2: 2-ASC and 3-ASC graphs

Another consequence of Theorem 2.2 is:

Corollary 3.3. *Let H be a graph. Then for any $r \geq 2$ there exists an r-ACS graph X such that H is an induced subgraph of X.*

Proof. Let G be an arbitrary r-ASC graph. Then by Theorem 2.2 we can select $X = G \oplus_u H$, where $u \in V(G)$ is a vertex with $e(u) = r$. $\square$

Thus r-ASC graphs cannot be characterized in terms of forbidden subgraphs.

4 Embeddings into 2-ASC graphs

From Corollary 3.3 we know that for any graph G and any $r \geq 2$ there exits an r-ASC graph containing G as an induced subgraph. Hence, a natural optimization problem consists to find a smallest r-ASC graph containing G. More precisely, let

$$\theta_r(G) = \min\{|V(H)| - |V(G)| \ ; \ H \text{ is } r\text{−ASC}, G \text{ induced in } H\}\,,$$

be the *r-ASC index* of the graph G. So $\theta_r(G)$ is the minimum number of vertices needed to add to G in order to construct an r-ASC graph containing G as an induced subgraph.

As an example consider the tree T from Figure 3 that is induced by vertices $u_1, \ldots, u_{10}$. Construct the graph H by adding a new vertex x and connected it to all vertices of T but to u_4, see the figure again where the new edges are indicated with dashed lines. Then $e(u_1) = e(u_4) = 3$ and the eccentricity of any other vertex is 2, hence H is a 2-ASC graph. Since T is not a 2-ASC graph we conclude that $\theta_2(T) = 1$.

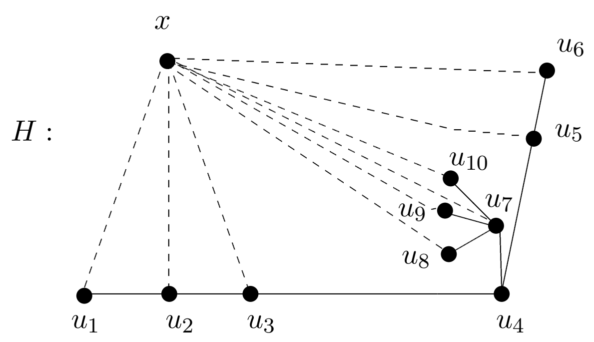

Figure 3: Tree T with $\theta_2(T) = 1$

Note that Corollary 3.3 asserts:

Corollary 4.1. *For any graph G and any $r \geq 2$, $0 \leq \theta_r(G) \leq 2r + 1$.*

Since 1-ASC graphs are the complete graphs with an edge removed, the first interesting case is $r = 2$. By Corollary 4.1, $0 \leq \theta_2(G) \leq 5$. However, more can be shown:

Theorem 4.2. *For any graph G, $\theta_2(G) \leq 2$.*

Proof. Throughout the proof, H is a graph obtained from G by adding two new vertices x and y, that is,

$$V(H) = V(G) \cup \{x, y\} \ .$$

We need to define the edge set of H such that H is a 2-ASC graph containing G as an induced subgraph. We distinguish the following cases depending on the diameter of G.

Case 1: $d(G) \leq 2$.
Suppose first that G is a complete graph (that is, $d(G) = 1$). Let u be an arbitrary vertex of G and set

$$E(H) = E(G) \cup \{xy\} \cup \{xw \ ; \ w \in V(G), w \neq u\} \, .$$

Then it is straightforward to see that $e_H(u) = e_H(y) = 3$ and that the eccentricity of all the other vertices is 2. Hence H is a 2-ACS graph. (Note that the construction works also for $G = K_2$, in which case $H = P_4$.)

If $d(G) = 2$ select u to be an arbitrary diametrical vertex of G. Then the same construction as above yields a 2-ASC graph H.

Case 2: $d(G) \geq 3$.
There exist vertices u and v with $d_G(u, v) = 3$. Then define the edge set of H as follows

$$E(H) = E(G) \cup \{xz, z \in V(G), z \neq u\} \cup \{yz, z \in V(G), z \neq v\} \cup \{xy\}\,.$$

Note first that $e_H(u) = e_H(v) = 3$. Since vertex x is adjacent to all vertices of G but vertex u, we get $e_H(x) = 2$. Similarly, $e_H(y) = 2$. Let finally z be a vertex of $V(G)$, $z \neq u, v$. Then vertex z is not adjacent to both u and v, since otherwise we would have $d_G(u, v) \leq 2$. Assume without loss of generality $zu \notin E(G)$. Then since $zy, yu \in E(H)$, $\mathrm{d}_H(z, u) = 2$. Similarly, $d_H(z, v) \leq 2$. Moreover, any other vertex z' is also adjacent to vertex x, hence $d_H(z, z') \leq 2$. We conclude that $e_H(z) = 2$ and the proof is complete. □

Next, we find exact values of the 2-ASC index for some well-known classes of graphs. By Theorem 4.2 we need to decide between three possibilities: 0, 1, and 2.

Proposition 4.3. *(i) For any $n \geq 2$, $\theta_2(K_n) = 2$.*

(ii) $\theta_2(K_{m,n}) = \begin{cases} 2, & m = n = 1; \\ 1, & \text{otherwise.} \end{cases}$

(iii) $\theta_2(C_n) = \begin{cases} 1, & n = 4, 5, 6; \\ 2, & \text{otherwise.} \end{cases}$

(iv) $\theta_2(P_n) = \begin{cases} 0, & n = 4; \\ 1, & n = 3, 5, 6; \\ 2, & \text{otherwise.} \end{cases}$

Proof. (i) K_n is not an ASC graph. Moreover, adding a vertex and connect it to some vertices of K_n results in a graph with diameter at most 2. Hence $\theta(K_n) > 1$.

(ii) $K_{1,1} = K_2$, hence $\theta_2(G) = 2$ by (i). Now assume that $K_{m,n}$ has at least three vertices, say $m \geq 2$. Let H be a graph obtained from $K_{m,n}$ by joining a new vertex x to all vertices but one, say u, from the m-partite set. Then H is a 2-ASC graph where $e_H(u) = e_H(x) = 3$ and the remaining vertices of H have eccentricity 2.

(iii) No cycle is an ASC graph, so θ_2 is either 1 or 2. $\theta_2(C_n) = 1$ holds for $n = 4, 5, 6$ as it can be seen in Figure 4, where the 2-ASC graphs H_1, H_2, and H_3 contain C_4, C_5, and C_6, respectively.

$\theta_2(C_3) = 2$ follows from (i). It remains to consider the case $n \geq 7$. Let $u_1, u_2, \ldots, u_n$ be the vertices of C_n with the natural adjacencies.

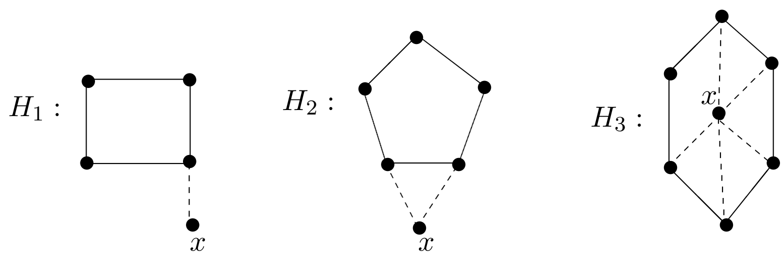

Figure 4: Embedding C_4, C_5, and C_6 into H_1, H_2, and H_3

Suppose that $\theta_2(C_n) = 1$. Then there exists a 2-ASC graph H consisting of C_n and a vertex x adjacent to some vertices of C_n. Suppose first that x is a diametrical vertex of H. We assume without loss of generality that $d_H(x, u_4) = 3$. Then it follows easily that $d_H(u_1, u_4) = 3$ as well, a contradiction. Hence we may assume without loss of generality that $d_H(u_1, u_4) = 3$. Then x is not adjacent to both u_1 and u_4, we may assume that $xu_4 \notin E(H)$. Since $n \geq 7$, we infer that $d_H(u_n, u_4) \geq 3$, the final contradiction.

(iv) P_4 is a 2-ASC graph. P_3 embeds into P_4 thus $\theta_2(P_3) = 1$. For the fact that $\theta_2(P_n) = 1$ holds also for $n = 5, 6$ see Figure 5.

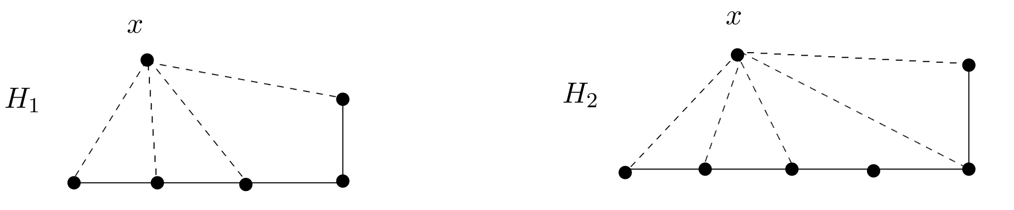

Figure 5: Embedding P_5 and P_6 into H_1 and H_2

Suppose now $n \geq 7$. Let $u_1, u_2, \ldots, u_n$ be the vertices of P_n with the natural adjacencies and assume that $\theta_2(P_n) = 1$. Let H be a 2-ASC graph consisting of P_n and a vertex x adjacent to some vertices of P_n. Assume $d_H(u_1, u_4) = 3$. Then $d_H(u_1, u_5) = 2$ and hence u_1 (as well as u_5) is adjacent to x. Similarly, because $d_H(u_4, u_7) = 2$ we get that u_4 is also adjacent to x. But this would mean that $d_H(u_1, u_4) = 2$ which is not possible. Analogously we find that u_n is not a diametrical vertex of H. Suppose $d_H(u_i, u_j) = 3$, where $i > 1$ and $i + 3 \leq j < n$. Since $d_H(u_i, u_{j+1}) = 2$, u_i is adjacent to x. It follows (because $d_H(u_i, u_j) = 3$) that x is not adjacent to u_j. But then $d_H(u_{i-1}, u_j) \geq 3$, a contradiction. Finally, suppose

$d_H(x, u_i) = 3$ for some i. If $i = 1$, the it follows that $d_H(u_1, u_4) = 3$. By symmetry we may thus assume that $1 < i < n$. Then x is adjacent to none of u_{i-1}, u_i, and u_{i+1}. But then at least one of $d_H(u_{i-2}, u_{i+1}) = 3$ if $i > 2$ and $d_H(u_{i-1}, u_{i+2}) = 3$ if $i < n-1$ holds, the final contradiction. □

5 Concluding remarks

In this paper we have introduced ASC graphs and the corresponding ASC index. These two concepts offer numerous problems, here is a selection of topics that could be investigated.

Problem 5.1. *Could ASC graphs be recognized faster than computing all the eccentricities of the graph?*

Problem 5.2. *Which is the computational complexity of determining $\theta_r(G)$? In particular, can we decide in polynomial time whether $\theta_r(G) = 1$?*

Problem 5.3. *Are there some well-known classes of graphs for which the ASC index can be determined efficiently?*

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