Convex sets in lexicographic products of graphs

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Abstract

Geodesic convex sets, Steiner convex sets, and J-convex (alias induced path convex) sets of lexicographic products of graphs are characterized. The geodesic case in particular rectifies [4, Theorem 3.1].

Key words: lexicographic product of graphs; convex sets; Steiner sets

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1 Introduction

Several types of convexity are present in graphs, the most prominent among them are convexities derived from path properties: the geodesic convexity, the induced path convexity, and the all-paths convexity. For a survey on convexities in graphs related to

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path properties we refer to [10], see also the proceedings [6] for several related papers on convexities in graphs as well as on other discrete structures.

The *geodesic interval* $I(u,v)$ of a graph G is the set consisting of vertices lying on all geodesics (shortest paths) between u and v. The interval function I and the associated geodesic convexity is a fundamental tool in metric graph theory and is well studied, see [2, 19] and references therein.

The notion of a Steiner tree of a multiset of vertices W can be considered as a generalization of the geodesic because when W consists of two different vertices, then a Steiner tree of W is a geodesic between these two vertices. Thus the concept of the Steiner interval of W which contains of the set of all vertices lying on all Steiner trees of W generalizes the interval function I. Steiner intervals on multisets in this sense were introduced in [3]. In this paper certain betweenness properties of the interval function I are extended to Steiner intervals. Steiner intervals on ordinary vertex subsets have been studied in several papers, see [13, 17, 21].

Induced paths of a given graph also lead to a graph convexity. The *induced path interval* between vertices u and v of a graph G is denoted with $J(u,v)$, that is,

$$J(u,v) = \{z \in V(G) \mid z \text{ lies on some induced } u,v\text{-path in } G\}\,.$$

The induced path convexity is also known as monophonic convexity [13] and as minimal path convexity [12]. The latter paper gives a characterization of the convex hull with respect to induced path convexity. The induced path convex sets are called *J-convex sets* for short. The betweenness and other similar properties are discussed in [7, 8, 18], for the hull number and an algorithm for the J-convex hull see [1].

In this paper we are interested in the convexity properties of the lexicographic product of graphs. This graph operation is one of the four standard graph products [15] and is of continuous interest, see the recent investigations [11, 14, 16, 20, 22].

The natural question to consider is what are convex subsets of $G \circ H$. For the geodesic convexity, the question was considered in [4] where it was proved (at least in one direction) that if neither of G and H is complete, then the only proper subgraphs that are convex are complete subgraphs. What about the cases when G or H is complete? No answer was given for the first case, and for the second case, that is, for the lexicographic products $G \circ K_m$, it was claimed [4, Theorem 3.1] that $Y \subseteq G \circ K_m$ is geodesic convex if and only if $p_G(Y)$ is geodesic convex. However, any G-layer G^u, where u is an arbitrary vertex of K_m, projects onto a convex subgraph of G but G^u is not convex in $G \circ K_m$. (The error in the proof of [4, Theorem 3.1] appears in this part "The converse of the theorem is obvious.") In Section 2 we clarify the situation and give the complete solution to this problem. In Sections 3 and 4 we respectively characterize Steiner convex sets and J-convex sets of lexicographic product, while in the remaining of this section we recall the graph distance and basic properties of the lexicographic product of graphs.

All graphs considered in this paper are connected, simple, and finite. The *distance* $d_G(u,v)$ between vertices u and v of a graph G is the length of a shortest path between u and v in G. If the graph G is clear from the context, we simply write $d(u,v)$.

The *lexicographic product* of graphs G and H is the graph $G \circ H$ (also denoted with $G[H]$) with the vertex set $V(G) \times V(H)$, vertices (g_1, h_1) and (g_2, h_2) are adjacent if either $g_1 g_2 \in E(G)$ or $g_1 = g_2$ and $h_1 h_2 \in E(H)$. $G \circ H$ is called *nontrivial* if both factors are graphs on at least two vertices. The lexicographic product is associative but not commutative. The one vertex graph is the unit for the operation and $G \circ H$ is connected if and only if G is connected. For more fundamental properties of the lexicographic product see [15].

For a vertex $h \in V(H)$, set $G^h = \{(g, h) \in V(G \circ H) \mid g \in V(G)\}$. The set G^h is called a *G-layer* of $G \circ H$. By abuse of notation we will also consider G^h as the corresponding induced subgraph. Then G^h is isomorphic to G. For $g \in V(G)$, the *H-layer* ${}^gH$ is defined as ${}^gH = \{(g, h) \in V(G \circ H) \mid h \in V(H)\}$. We will again also consider ${}^gH$ as an induced subgraph and note that it is isomorphic to H. A map $p_G : V(G \circ H) \to V(G)$, $p_G((g, h)) = g$, is the *projection* onto G and $p_H : V(G \circ H) \to V(H)$, $p_H((g, h)) = h$, the *projection* onto H.

2 Geodesic convexity

In order to characterize convex subgraphs of lexicographic products, the following concepts will be useful. A vertex u of a graph G is a Λ-*vertex* if u is adjacent to two nonadjacent vertices. (The notation reflects the fact that u is the middle vertex of an induced path on three vertices.) Note that all the other vertices of G have complete neighborhoods. An induced subgraph Y of the lexicographic product $G \circ H$ will be called Λ-*complete* if ${}^gH \cap Y = {}^gH$ holds for any Λ-vertex g of $p_G(Y)$.

Theorem 2.1 *Let $G \circ H$ be a nontrivial, connected lexicographic product. Then a proper, non-complete induced subgraph Y of $G \circ H$ is (geodesic) convex if and only if the following conditions hold:*

(i) $p_G(Y)$ is convex in G,
(ii) Y is Λ-complete, and
(iii) H is complete.

Proof. Suppose first that (i)-(iii) hold. Since H is complete, any vertices (g, h_1) and (g, h_2) that belong to Y are adjacent and hence clearly $I_{G \circ H}((g, h_1), (g, h_2)) \subseteq V(Y)$. Consider next vertices $(g_1, h_1), (g_2, h_2) \in V(Y)$, where $g_1 \neq g_2$. Let P be a $(g_1, h_1), (g_2, h_2)$-shortest path in $G \circ H$. Then $p_G(P)$ is a shortest g_1, g_2-path in G. Let (g, h) be an arbitrary inner vertex of P. Since $p_G(Y)$ is convex, ${}^gH \cap Y \neq \emptyset$. Moreover, since (g, h) is an inner vertex of the shortest path P and Y is Λ-complete, ${}^gH \cap Y = {}^gH$. Therefore $(g, h) \in I_{G \circ H}((g_1, h_1), (g_2, h_2))$ and hence $P \subseteq Y$. We conclude that Y is convex.

Conversely, let Y be a convex subgraph of $G \circ H$. Since Y is not complete (by theorem's assumption), Y contains an induced path $P = (g_1, h_1), (g_2, h_2), (g_3, h_3)$. We distinguish two cases.

Case 1: $g_1 = g_3$.
Since $d_Y((g_1,h_1),(g_3,h_3)) = 2$, H is not complete in this case. Let g be an arbitrary neighbor of g_1 in G. Then ${}^gH$ is a subset of $I_{G\circ H}((g_1,h_1),(g_3,h_3))$ and hence, since Y is convex, ${}^gH \subseteq Y$. Clearly, $d_{G\circ H}((g,h_1)(g,h_3)) = 2$, and again for any neighbor x of $g \in N_G(g_1)$, the layer ${}^xH$ is a subset of Y. Since $G \circ H$ is connected and thus G is connected, by induction of the distance from g_1 in G, we conclude that $Y = G \circ H$, a contradiction with theorem's assumption.

Case 2: $g_1 \neq g_3$.
Since P is induced and g_2 is a common neighbor of g_1 and g_3 in G, we have $d_G(g_1,g_3) = 2$. In particular, G is not complete. Clearly ${}^{g_2}H$ is included in $I_{G\circ H}((g_1,h_1),(g_3,h_3))$ and thus ${}^{g_2}H \subseteq Y$. If H is not complete there exists an induced path of length 2 in ${}^{g_2}H$ and we can continue as in Case 1 to conclude that $Y = G \circ H$. Thus we have proved that H is complete.

Consider next $p_G(Y)$ and let g be a Λ-vertex of $p_G(Y)$. Then there exists an induced path g_1, g, g_2 in $p_G(Y)$ and vertices $(g_1,h_1),(g,h),(g_2,h_2)$ that all belong to Y. Since Y is convex, ${}^gH \subseteq Y$. We conclude that Y is Λ-complete.

Finally, if $p_G(Y)$ would not be convex in G, Y would not be convex. $\square$

A subgraph X of a graph G is *2-convex* if, together with any vertices u and v of X with $d_G(u,v) = 2$, all common neighbors of u and v belong to X. The above proof might suggest that we could replace convexity of $p_G(Y)$ in condition (i) with 2-convexity. To see that this is not the case, consider any subgraph H of a graph G that is 2-convex but not convex (for instance, P_n as the subgraph of C_{2n}, $n \geq 3$) and the lexicographic product $G \circ K_n$, $n \geq 1$. Then $H \circ K_n$ is a 2-convex subgraph of $G \circ K_n$ that fulfills conditions (ii) and (iii) of the theorem, but it is not convex.

3 Steiner convexity

Let us formally define a Steiner tree and k-Steiner intervals, for ($k \geq 2$). In a connected graph $G = (V,E)$, a *Steiner tree* of a (multi)set $W \subseteq V$, is a minimum order tree in G that contains all vertices of W. The number of edges in a Steiner tree T of W is called *the Steiner distance of* W, denoted $d(W)$, while *the size* of T describes the number of vertices in T (i.e. $d(W)+1$). The *k-Steiner interval* is a mapping $S : V \times \cdots \times V \longrightarrow 2^V$ such that $S(u_1,u_2,\ldots,u_k)$ consists of all vertices in G that lie on some Steiner tree with respect to $\{u_1,\ldots,u_k\}$, where $u_1,\ldots,u_k$ are not necessarily distinct vertices of G (in this way S is an extension of I, as $S(u,v,\ldots,v) = I(u,v)$). A subset W of V is *k-Steiner convex* if $S(u_1,u_2,\ldots,u_k) \subseteq W$ for every k-tuple $(u_1,u_2,\ldots,u_k)$ in $W \times \cdots \times W$. We say that W is *Steiner convex* if W is k-Steiner convex for every $k \geq 2$. Clearly V and K, with $\langle K\rangle \cong K_n$, are Steiner convex and are called *trivial Steiner convex sets*.

In this section we characterize Steiner convex sets in lexicographic products. The result is parallel to Theorem 2.1 and the proof will use this theorem. However, the results are not equivalent as we will demonstrate with a family of examples at the end of the section. First a lemma:

Lemma 3.1 *Let $g_1, g_2, \ldots, g_k$ be different vertices of a connected graph G. Then for any (not necessarily different) vertices $h_1, h_2, \ldots, h_k$ of a graph H, a Steiner tree of $g_1, g_2, \ldots, g_k$ (in G) and a Steiner tree of $(g_1, h_1), (g_2, h_2), \ldots, (g_k, h_k)$ (in $G \circ H$) have the same size.*

Proof. Let $g_1, g_2, \ldots, g_m$, $m \geq k$, be the vertices of a Steiner tree T of $g_1, g_2, \ldots, g_k$. Select arbitrary vertices $h_{k+1}, \ldots, h_m$ of H and define the subgraph $\widehat{T}$ of $G \circ H$ as follows. Its vertices are $(g_1, h_1), (g_2, h_2), \ldots, (g_m, h_m)$, and (g_i, h_i) is adjacent to (g_j, h_j) if and only if g_i is in T adjacent to g_j. Since $g_1, g_2, \ldots, g_m$ are different vertices, $\widehat{T}$ is a tree of the same size as T. Clearly, $\widehat{T}$ connects the vertices $(g_1, h_1), (g_2, h_2), \ldots, (g_k, h_k)$. Moreover, $\widehat{T}$ is a Steiner tree, for otherwise $p_G(\widehat{T})$ would yield a smaller size tree in G than T. (Note that since all g_i's are different, no Steiner tree contains an edge of an H-layer.) □

Theorem 3.2 *Let $G \circ H$ be a nontrivial, connected lexicographic product. Then a proper, non-complete induced subgraph Y of $G \circ H$ is Steiner convex if and only if the following conditions hold:*

(i) $p_G(Y)$ is Steiner convex in G,
(ii) Y is Λ-complete, and
(iii) H is complete.

Proof. Suppose (i)-(iii) hold for a proper, non-complete induced subgraph Y of $G \circ H$. Let $S = \{(g_1, h_1), (g_2, h_2), \ldots, (g_k, h_k)\}$ be an arbitrary set of vertices of Y. We may assume that the notation is selected such that the vertices $g_1, \ldots, g_j$, $j \leq k$, form a largest set of different vertices of $p_G(Y)$. Then

$$p_G(\{(g_1, h_1), (g_2, h_2), \ldots, (g_j, h_j)\} = p_G(\{(g_1, h_1), (g_2, h_2), \ldots, (g_k, h_k)\})$$

and $SI_{p_G(Y)}(g_1, g_2, \ldots, g_j) \subseteq V(p_G(Y))$ since $p_G(Y)$ is Steiner convex in G. Let T be a Steiner tree (in G) of $g_1, g_2, \ldots, g_j$ and let its size be ℓ. By Lemma 3.1, the size of a Steiner tree $\widehat{T}$ of $(g_1, h_1), (g_2, h_2), \ldots, (g_j, h_j)$ is also ℓ. Furthermore, every Steiner tree for these vertices is contained in Y because of the Steiner convexity of $p_G(Y)$, Lemma 3.1, and the assumption that Y is Λ-complete. For every additional vertex (g_i, h_i), $j < i \leq k$, we need to add exactly one edge either of the type $(g_i, h_i)(g_i, h_p)$ or of the type $(g_i, h_i)(g_t, h_t)$, where $g_i g_t \in E(G)$. This tree T' has size $\ell + k - j$ and $p_G(T') = p_G(\widehat{T})$.

Let T'' be an arbitrary Steiner tree of S. For any g_i such that ${}^{g_i}H \cap T'' \neq \emptyset$, contract all the vertices from ${}^{g_i}H \cap T''$ into a single vertex. Then the size of the contraction is at least ℓ. But then the size of T'' must be at least $\ell + k - j$ and is hence equal to $\ell + k - j$. Therefore every Steiner tree of Y must be of the same form as T' described above. We conclude that Y is Steiner convex.

Conversely, let Y be a Steiner convex subset of $G \circ H$. Then Y is also geodesic convex. Hence by Theorem 2.1, $p_G(Y)$ is geodesic convex in G, Y is Λ-complete, and

H is complete. We thus only need to see that $p_G(Y)$ is also Steiner convex in G. Suppose not. Let $g \notin p_G(Y)$ be a vertex from a Steiner tree T of size ℓ of vertices $g_1, g_2, \ldots, g_k \in p_G(Y)$. By Lemma 3.1 there exists a Steiner tree of size ℓ on vertices $(g_1, h_1), (g_2, h_2), \ldots, (g_k, h_k)$ in $G \circ H$ that contains the vertex (g, h) for some $h \in V(H)$. Since Y is Steiner convex, $(g, h) \in Y$ and consequently $g \in p_G(Y)$—a contradiction and the proof is complete. □

In [9], a family of graphs G_k, $k \geq 2$, was constructed with the property that G_k has a k-Steiner convex subset that is not $k+1$-Steiner convex, see Figure 1 for G_2. Note that the outer cycle C of G_2 is geodesic convex but not 3-Steiner convex and thus not Steiner convex. Hence by Theorems 2.1 and 3.2, $K = C \circ K_n$ is geodesic convex but not Steiner convex in $G_2 \circ K_n$ for any $n \geq 2$.

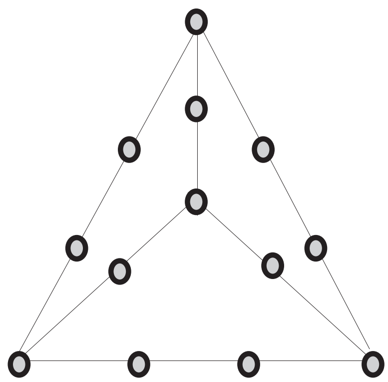

Figure 1: Graph G_2

4 Induced path convexity

In this final section we characterize the J-convex sets of lexicographic products. The characterization is parallel but not equivalent to Theorems 2.1 and 3.2.

Theorem 4.1 *Let $G \circ H$ be a nontrivial, connected lexicographic product. Then a proper, non-complete induced subgraph Y of $G \circ H$ is J-convex if and only if the following conditions hold:*

(i) *$p_G(Y)$ is J-convex in G,*
(ii) *Y is Λ-complete, and*
(iii) *H is complete.*

Proof. Suppose first that (i)-(iii) hold. Since H is complete, any vertices (g, h_1) and (g, h_2) that belong to Y are adjacent and hence clearly $J_{G \circ H}((g, h_1), (g, h_2)) \subseteq V(Y)$. Consider next vertices $(g_1, h_1), (g_2, h_2) \in V(Y)$, where $g_1 \neq g_2$. Let P be a $(g_1, h_1), (g_2, h_2)$-induced path in $G \circ H$. Then no edge of P lies in an H-layer and hence

$p_G(P)$ is an induced g_1, g_2-path in G. Let (g, h) be an arbitrary inner vertex of P. Since $p_G(Y)$ is J-convex, ${}^gH \cap Y \neq \emptyset$. Moreover, since (g, h) is an inner vertex of the induced path P and Y is Λ-complete, ${}^gH \cap Y = {}^gH$. Therefore $(g, h) \in J_{G\circ H}((g_1, h_1), (g_2, h_2))$ and hence $P \subseteq Y$. We conclude that Y is J-convex.

Conversely, let Y be a J-convex subgraph of $G \circ H$. Since Y is not complete (by theorem's assumption), Y contains an induced path on three vertices (g_1, h_1), (g_2, h_2), (g_3, h_3). Note that induced paths of length 2 are also isometric and we can use the same proof as for Theorem 2.1 to see that H is complete (and that G is not).

Consider next $p_G(Y)$ and let g be a Λ-vertex of $p_G(Y)$. Then there exist an induced path g_1, g, g_2 in $p_G(Y)$ and vertices h_1, h_2 in $V(H)$, such that $(g_1, h_1)(g, h)(g_2, h_2)$ is an induced path in Y for every $h \in V(H)$. Since Y is J-convex, ${}^gH \subseteq Y$. We conclude that Y is Λ-complete.

Finally, if $p_G(Y)$ would not be J-convex in G, Y would not be J-convex. □

To round off the paper, we conclude with yet another path convexity in graphs. The *all-paths interval*

$$A(u, v) = \{w \in V \mid w \text{ lies on some } u, v\text{-path in } G\}$$

naturally leads to the all-paths convexity [5, 10]. If G is 2-connected the only non-trivial all-paths convex sets are $V(G)$ and the singletons. Since a non-trivial lexicographic product $G \circ H$ is 2-connected as soon as G is connected, the all-paths convexity is trivial in $G \circ H$. Hence we can give the following concluding observation.

Remark 4.2 *Let G be a connected graph and H a graph on at least two vertices. Then the only all-paths convex sets in $G \circ H$ are the empty set, the singletons, and $V(G \circ H)$.*

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