On local colorings of Cartesian product graphs

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Abstract

A local coloring of a graph G is a function $c : V(G) \to \mathbb{N}$ such that for each $S \subseteq V(G)$, $2 \le |S| \le 3$, there exist $u, v \in S$ with $|c(u) - c(v)|$ at least the size of the subgraph induced by S. The maximum color assigned by c is the value $\chi_\ell(c)$ of c, and the local chromatic number of G is $\chi_\ell(G) = \min\{\chi_\ell(c) : c \text{ local coloring of } G\}$. In this note the local chromatic number is determined for Cartesian products of 3-chromatic graphs which in part corrects an error from [Omoomi, Pourmiri, On the local colorings of graphs, Ars Combin. 86 (2008) 147–159]. It is also proved that if G and H are graphs such that $\chi(G) \le \lfloor \chi_\ell(H)/2 \rfloor$ and H is triangle-free, then $\chi_\ell(G \,\square\, H) = \chi_\ell(H)$.

Key words: Local chromatic number; Chromatic number; Cartesian product

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1 Introduction

A *local coloring* of a graph G is a function $c : V(G) \to \mathbb{N}$ having the property that for each set $S \subseteq V(G)$ with $2 \le |S| \le 3$, there exist vertices $u, v \in S$ such that the colors of u and v differ by at least the size of the subgraph induced by S. Note that a local coloring is in particular a proper usual vertex coloring. The maximum color assigned by a local coloring c to a vertex of G is called the *value* of c and denoted by $\chi_\ell(c)$. The *local chromatic number* $\chi_\ell(G)$ of G is the minimum value over all local colorings c of G.

A local coloring is thus a usual coloring with two additional conditions: any induced path of length 2 must contain two vertices with colors differing by at least

2 and any triangle contains two vertices with colors that differ by at least 3. If the latter condition is dropped, one speaks of the so-called *semi-matching colorings* which were studied in [4]. In the class of triangle-free graphs, local colorings and semi-matching colorings thus form the same concept. We add that local colorings are of similar nature as $L(p,q)$-labelings in which labels of adjacent vertices differ by at least p and labels of vertices at distance 2 differ by at least q. More precisely, local colorings are similar to $L(1,2)$-labelings. For more information on $L(p,q)$-labelings we refer to recent papers [3, 10] in which graph products were studied.

Local colorings were introduced by Chartrand et al. in [1] (although the next paper written on the topic [2] was eventually published two years earlier). In the seminar paper it was shown that the study of the local chromatic number cannot be reduced to 2-connected graphs. Several exact values of the invariant were obtained, as for complete multipartite graphs, and so in particular for complete graphs. In the subsequent paper [2] the emphasize was on regular graphs, where Cartesian products with one factor being a hypercube played the central role. In [7] it was proved that $\chi_\ell(G) \leq \Delta(G)-2$ holds for any graph with $\Delta(G) \geq 3$ and different from K_4 and K_5. This result in particular confirms Conjecture 4.2 from [1] asserting that $\chi_\ell(G) = 4$ holds for cubic, non-bipartite, and non-complete graphs. In [8] local colorings of Kneser graphs were studied, this line or research was continued in [4] through the perceptive of semi-matching colorings.

In this note we are interested in the local coloring of Cartesian products of graphs, primarily motivated with investigations in [7] where exact local chromatic numbers were determined for some specific products. We proceed as follows. In the next section concepts, definitions, and results needed are recalled. In Section 3 the local chromatic number is determined for Cartesian products of 3-chromatic graphs. In particular, the local chromatic number of products of cycles is extracted, a result that in one part corrects an assertion from [7]. In the final section we then prove that if G and H are graphs such that $\chi(G) \leq \lfloor \chi_\ell(H)/2 \rfloor$ and H is triangle-free, then $\chi_\ell(G \,\square\, H) = \chi_\ell(H)$.

2 Preliminaries

We will use the notation $[n]$ for the set $\{1, \ldots, n\}$. Graphs considered here are simple. If $G = (V(G), E(G))$ is a graph, then the *order* of G is $|V(G)|$ and the *size* of G is $|V(G)|$. As usual, $\chi(G)$ is the chromatic number of G. If a coloring uses k colors then it is a *k-coloring* and if $k = \chi(G)$ then it is a *χ-coloring*. G is called *k-chromatic* if $\chi(G) \leq k$. Similarly, a local coloring that uses k colors is a *k-local coloring* and if $k = \chi_\ell(G)$ then it is a *χ_ℓ-coloring*.

The *Cartesian product* of graphs G and H is the graph $G \,\square\, H$ with the vertex set $G \times H$, and $(g,h)(g',h') \in E(G \,\square\, H)$ if either $gg' \in E(G)$ and $h = h'$, or $hh' \in E(H)$ and $g = g'$. The Cartesian product is commutative and associative, having the one vertex graph as a unit. The subgraph of $G \,\square\, H$ induced by $g \times V(H)$,

where $g \in V(G)$, is isomorphic to H. It is called an *H-layer* (over g) and denoted ${}^gH$. Similarly, the subgraph of $G \,\square\, H$ induced by $V(G) \times h$, where $h \in V(H)$, is isomorphic to G, called a *G-layer* (over h) and denoted G^g. For more information on the Cartesian product of graphs see [5].

We now recall some results on the local chromatic number. Note first that if G is a subgraph of H, then $\chi_\ell(G) \leq \chi_\ell(H)$. In the rest we will use the fact that a labeling c of $V(G)$ is a local coloring if and only if (i) c is a (usual) vertex coloring, (ii) every induced P_3 contains two vertices with colors at least two apart, and (iii) every induced triangle contains two vertices with colors at least three apart. We will also use the following:

Proposition 1 ([1]) *If G is a connected bipartite graph of order at least 3, then $\chi_\ell(G) = 3$.*

It is easy to prove Proposition 1: the lower bound follows because G contains at least one induced P_3, the upper bound is obtained by coloring each vertex of one bipartition set of G with 1 and each vertex of the other bipartition set with 3. We will also need the following result which follows by replacing color i with $2i - 1$ for any $1 \leq i \leq \chi(G)$ in a χ-coloring of G.

Proposition 2 ([1]) *For any graph G, $\chi(G) \leq \chi_\ell(G) \leq 2\chi(G) - 1$.*

As already mentioned, the local chromatic number of complete multipartite graphs was determined in [2], for further use we state the following special case:

Theorem 3 *If $n \geq 1$, then $\chi_\ell(K_n) = \lfloor (3n-1)/2 \rfloor$.*

We will also often use (without explicitly mentioning it) the 1957 Sabidussi's result [9] asserting that $\chi(G \,\square\, H) = \max\{\chi(G), \chi(H)\}$, cf. [5, Theorem 26.1].

3 Products of 3-chromatic graphs

Theorem 4 *Let G and H be 3-chromatic graphs with at least one edge.*

(i) If $\chi(G) = \chi(H) = 2$, then $\chi_\ell(G \,\square\, H) = 3$.

(ii) If $\chi(G) = 2$, $\chi(H) = 3$, and $\chi_\ell(H) = 3$, then $\chi_\ell(G \,\square\, H) = 4$.

(iii) If $\chi(G) = 2$, $\chi(H) = 3$, and $\chi_\ell(H) = 4$, then $4 \leq \chi_\ell(G \,\square\, H) \leq 5$.

(iv) If $\chi(G) = 2$, $\chi(H) = 3$, and $\chi_\ell(H) = 5$, then $\chi_\ell(G \,\square\, H) = 5$.

(v) If $\chi(G) = \chi(H) = 3$, then $\chi_\ell(G \,\square\, H) = 5$.

Proof. (i) Since G and H each have at least one edge, $G \,\square\, H$ contains at least one C_4 and since G and H are bipartite, $G \,\square\, H$ is bipartite as well. Hence $\chi_\ell(G \,\square\, H) = 3$ follows by Proposition 1.

(ii) Let c_G be a χ-coloring of G (so $c_G : V(G) \to [2]$) and let c_H be a χ_ℓ-coloring of H (so $c_H : V(H) \to [3]$). Then define $c : V(G \,\square\, H) \to [4]$ as follows:

$$c(g,h) = \begin{cases} 1; & c_G(g) = 1, c_H(h) = 1 \text{ or } c_G(g) = 2, c_H(h) = 3, \\ 2; & c_G(g) = 1, c_H(h) = 2, \\ 3; & c_G(g) = 2, c_H(h) = 1 \text{ or } c_G(g) = 1, c_H(h) = 3, \\ 4; & c_G(g) = 2, c_H(h) = 2. \end{cases}$$

Note that the endvertices of any edge from a G-layer receive colors that differ by at least 2. Moreover, a possible triangle of $G \,\square\, H$ can only lie in an H-layer, but H is triangle-free because $\chi_\ell(H) = 3$. It now readily follows that c is a 4-χ_ℓ-coloring of $G \,\square\, H$ and thus $\chi_\ell(G \,\square\, H) \leq 4$. To see that $\chi_\ell(G \,\square\, H) \geq 4$, consider a subgraph $X = K_2 \,\square\, C_{2k+1}$, $k \geq 1$. (Such a subgraph exists because $\chi(H) = 3$.) Assume for a moment that $\chi_\ell(X) = 3$. Then each of the 4-cycles of X must be colored with consecutive colors $1, 3, 1, 3$. But this clearly leads to a contradiction after considering all the 4-cycles above the edges of C_{2k+1}. Therefore $\chi_\ell(X) \geq 4$ and hence also $\chi_\ell(G \,\square\, H) \geq 4$. We conclude that c is a χ_ℓ-coloring of $G \,\square\, H$.

(iii) and (iv) If $\chi_\ell(H) = 4$, then $\chi_\ell(G \,\square\, H) \geq 4$ and if $\chi_\ell(H) = 5$, then $\chi_\ell(G \,\square\, H) \geq 5$. On the other hand, since $\chi(G \,\square\, H) = 3$, Proposition 1 implies that $\chi_\ell(G \,\square\, H) \leq 5$.

(v) Since $\chi(G) = \chi(H) = 3$, we have $\chi(G \,\square\, H) = 3$ and hence Proposition 2 implies that $\chi_\ell(G \,\square\, H) \leq 5$. It thus remains to prove that $\chi_\ell(G \,\square\, H) \geq 5$. For this sake it suffices to prove that $\chi_\ell(C_{2m+1} \,\square\, C_{2n+1}) \geq 5$ holds for any $m, n \geq 1$. Indeed, since $\chi(G) = \chi(H) = 3$, G contains an (induced) odd cycle C_{2m+1} and H contains an (induced) odd cycle C_{2n+1}, hence $G \,\square\, H$ contains an induced $C_{2m+1} \,\square\, C_{2n+1}$.

Set $X = C_{2m+1} \,\square\, C_{2n+1}$, and let $V(C_k) = [k]$ with natural adjacencies, so that $V(X) = [2m+1] \times [2n+1]$. Suppose on the contrary that c is a 4-local-coloring of X.

We first claim that if $e = xy \in E(X)$, then $c(\{u, v\}) \neq \{1, 2\}$ and $c(\{u, v\}) \neq \{3, 4\}$. We may assume without loss of generality that e lies in a C_{2m+1}-layer, that is, $x = (i, j)$ and $y = (i+1, j)$ for some $i \in [2m+1]$ and some $j \in [2n+1]$ (indices modulo $2m+1$ and $2n+1$, respectively). Suppose that $c(i, j) = 1$ and $c(i+1, j) = 2$. Then by the definition of the local coloring, $c(\{(i, j+1), (i+1, j+1)\}) = \{3, 4\}$. In the same way we get $c(\{(i, j+2), (i+1, j+2)\}) = \{1, 2\}$. Continuing the argument and having in mind that C_{2n+1} is an odd cycle, we arrive at $\{c(i, j-1), c(i+1, j-1)\} = \{1, 2\}$, which is a clear contradiction. We analogously arrive at a contradiction if $c(i, j) = 3$ and $c(i+1, j) = 4$. This proves the claim.

Let $A_i = (c^{-1}(1) \cup c^{-1}(2)) \cap C^i_{2m+1}$ and $B_i = (f^{-1}(3) \cup f^{-1}(4)) \cap C^i_{2m+1}$. Then $|A_i| = |B_{i+1}|$ and $|B_i| = |A_{i+1}|$ hold for any $1 \leq i \leq 2n+1$. Indeed, this follows from the above claim because $c(i, j) \in \{1, 2\}$ if and only if $c(i+1, j) \in \{3, 4\}$.

Clearly, $|A_i| + |B_i| = 2m + 1$ and hence $|A_i| \neq |B_i|$. Assume without loss of generality that $|A_1| > |B_1|$. Then $|A_i| \geq m + 1$ and $|B_i| \leq m$ hold for $i \in \{1, 3, \ldots, 2m+1\}$. It follows that $2m+1 = |A_1| + |B_1| = |A_1| + |A_m| \geq 2m+2$, the final contradiction. □

Note that by the commutativity of the Cartesian product, Theorem 4 covers all possible products with 3-chromatic factors.

If $\chi(G) = 2$, $\chi(H) = 3$, and $\chi_\ell(H) = 4$, Theorem 4 offers two possibilities: $\chi_\ell(G \,\square\, H) = 4$ or $\chi_\ell(G \,\square\, H) = 5$. Both can indeed happen.

To see that the first possibility can happen, recall that $\chi_\ell(K_3) = 4$ and observe that $\chi_\ell(K_2 \,\square\, K_3) = 4$.

Consider now the product $P_3 \,\square\, K_4 - e$, where $K_4 - e$ is the graph obtained from K_4 by deleting one of its edges. Note first that $\chi_\ell(K_4 - e) = 4$. Moreover, since K_3 has exactly two χ_ℓ-colorings (with colors 1, 2, 4 and 1, 3, 4, respectively), it follows that $K_4 - e$ has an essentially unique χ_ℓ-coloring: the vertices of degree 3 must be colored with 1 and 4, while the remaining two vertices are colored with 2 and 3. Therefore, the two non-adjacent $(K_4 - e)$-layers in a 4-local coloring of $P_3 \,\square\, K_4 - e$ must have identical coloring and consequently one P_3-layer is colored with colors 2, 3, 2, which is not possible. We conclude that $\chi_\ell(P_3 \,\square\, K_4 - e) > 4$ and so $\chi_\ell(P_3 \,\square\, K_4 - e) = 5$.

Corollary 5 *If $m, n \geq 3$, then*

$$\chi_\ell(C_m \square C_n) = \begin{cases} 3; & m, n \text{ are both even,} \\ 4; & \text{exactly one of } m \text{ and } n \text{ is even,} \\ 5; & m, n \text{ are both odd.} \end{cases}$$

Proof. Recall that $\chi_\ell(C_n) = 3$ holds for any $n \geq 4$. Then Theorem 4 (i), (ii), and (v) cover all the cases, except the case $C_3 \,\square\, C_{2n}$, $n \geq 2$, because $\chi_\ell(C_3) = 4$ and we thus have two possibilities due to Theorem 4 (iii). Now color the first C_3-layer of $C_3 \,\square\, C_{2n}$ with 1, 2, 4, the second layer with 3, 4, 1, and alternately continue with this coloring. This is a local coloring and hence $\chi_\ell(C_3 \,\square\, C_{2n}) = 4$. □

Corollary 5 corrects [8, Theorem 5] where it is claimed that $\chi_\ell(C_{2m+1} \square C_{2n+1}) = 4$. A coloring is proposed there for which it is claimed that it is easy to see to be a local coloring of value 4. An example of the proposed coloring is shown in Fig. 1, where it can be seen that the problem is the most outer 4-cycle which is colored 1, 2, 1, 2.

4 More exact values

For a long time (cf. [6]) it is known that there exist triangle-free graphs with an arbitrary high chromatic number. Consequently, there exist triangle-free graphs with an arbitrary high local chromatic number. In this section we prove:

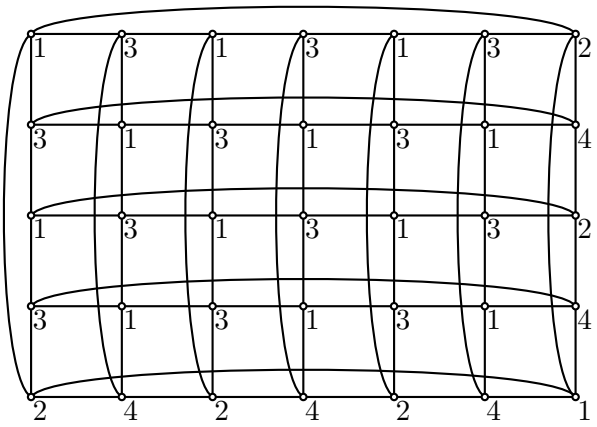

Figure 1: A coloring of $C_7 \,\square\, C_5$ which is not local

Theorem 6 *If G and H are graphs such that $\chi(G) \leq \left\lfloor \frac{\chi_\ell(H)}{2} \right\rfloor$ and H is triangle-free, then $\chi_\ell(G \,\square\, H) = \chi_\ell(H)$.*

Proof. $\chi_\ell(G \,\square\, H) \geq \chi_\ell(H)$ holds because H is an induced subgraph of $G \,\square\, H$. To prove the reverse inequality, let c_G be a χ-coloring of G, let c_H be a χ_ℓ-coloring of H, and define a coloring c of $G \,\square\, H$ as

$$c(g,h) = 2c_G(g) + c_H(h) - 2\,,$$

where computations are done modulo $\ell = \chi_\ell(H)$. As $c : V(G \,\square\, H) \to [\ell]$ it suffices to prove that c is a local coloring.

Consider vertices (g,h) and (g',h), $g \neq g'$, such that $c_G(g) \neq c_G(g')$. Suppose first that $c_H(h) = 1$. Since $\chi(G) \leq \lfloor \chi_\ell(H)/2 \rfloor$, we have

$$c(g,h) = 2c_G(g) - 1 \leq 2 \left\lfloor \frac{\ell}{2} \right\rfloor - 1 \leq \ell - 1\,.$$

By the same argument, $c(g',h) \leq \ell - 1$. Since $c_G(g) \neq c_G(g')$ it follows that $|c(g,h) - c(g',h)| \geq 2$. Moreover, because $c(g,h), c(g',h) \leq \ell - 1$ we also have $|c(g,h) - c(g',h)| \leq \ell - 2$. By the definition of the coloring c it follows that

$$|c(g,h) - c(g',h)| \geq 2 \tag{1}$$

holds for any vertex $h \in V(H)$.

Consider now an arbitrary triangle T of $G \,\square\, H$. By the definition of the Cartesian product, T is an induced subgraph in either a G-layer or in an H-layer. Moreover, since H is triangle-free, we infer that $T \subseteq G^h$ for some $h \in V(H)$. Then by (1), the vertices of T are colored with colors that are pairwise at least 2 apart. So T is properly locally colored.

Let P be an induced subgraph of $G \,\square\, H$ isomorphic to P_3. If $P \subset {}^{g}H$ or $P \subseteq G^h$, then we can argue as above for T that P is properly locally colored. Suppose now that P is induced on vertices (g,h), (g',h), and (g',h'), where $g \neq g'$ and $h \neq h'$. Considering the edge $(g,h)(g',h)$ and applying (1) yield the required conclusion. $\square$

Consider the product $K_4 \,\square\, K_6$. Since $\chi_\ell(K_6) = \lfloor (3 \cdot 6 - 1)/2 \rfloor = 8$, the condition $\chi(K_4) \leq \left\lfloor \frac{\chi_\ell(K_6)}{2} \right\rfloor$ is satisfied. On the other hand, by [8, Theorem 7], $\chi_\ell(K_4 \,\square\, K_6) > \chi_\ell(K_6)$. This demonstrates that the condition for H to be triangle-free cannot be dropped in general.

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