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Research Article

Local colorings of Cartesian product graphs

Sandi Klavžar[a,b,c] and Zehui Shao[d,e]

[a] *Faculty of Mathematics and Physics, University of Ljubljana, Slovenia*
[b] *Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia*
[c] *Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia*
[d] *Key Laboratory of Pattern Recognition and Intelligent Information Processing, Institutions of Higher Education of Sichuan Province, China*
[e] *School of Information Science and Technology, Chengdu University, Chengdu, 610106, China*

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A local coloring of a graph G is a function $c : V(G) \to \mathbb{N}$ such that for each $S \subseteq V(G)$, $2 \le |S| \le 3$, there exist $u, v \in S$ with $|c(u) - c(v)|$ at least the number of edges in the subgraph induced by S. The maximum color assigned by c is the value $\chi_\ell(c)$ of c, and the local chromatic number of G is $\chi_\ell(G) = \min\{\chi_\ell(c) : \ c \text{ is a local coloring of } G\}$. In this note the local chromatic number is determined for Cartesian products $G \,\Box\, H$, where G and GH are 3-colorable graphs. This result in part corrects an error from [Omoomi, Pourmiri, On the local colorings of graphs, Ars Combin. 86 (2008) 147–159]. It is also proved that if G and H are graphs such that $\chi(G) \le \lfloor \chi_\ell(H)/2 \rfloor$, then $\chi_\ell(G \,\Box\, H) \le \chi_\ell(H) + 1$.

Keywords: local chromatic number; chromatic number; Cartesian product

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1. Introduction

A *local coloring* of a graph G is a function $c : V(G) \to \mathbb{N}$ having the property that for each set $S \subseteq V(G)$ with $2 \le |S| \le 3$, there exist vertices $u, v \in S$ such that the colors of u and v differ by at least the size of the subgraph induced by S. Note that a local coloring is in particular a proper usual vertex coloring. The maximum color assigned by a local coloring c to a vertex of G is called the *value* of c and denoted by $\chi_\ell(c)$. The *local chromatic number* $\chi_\ell(G)$ of G is the minimum value over all local colorings c of G.

A local coloring is thus a usual coloring with two additional conditions: any induced path of length 2 must contain two vertices with colors differing by at least 2 and any triangle contains two vertices with colors that differ by at least 3. If the latter condition is dropped, one speaks of the so-called *semi-matching colorings* which were studied in [4]. In the class of triangle-free graphs, local colorings and semi-matching colorings thus form the same concept. We add that local colorings are of similar nature as $L(p, q)$-labelings in which labels of adjacent vertices differ by at least p and labels of vertices at distance 2 differ by at least q. From this point of view, local colorings are similar to $L(1, 2)$-labelings. For more information on $L(p, q)$-labelings we refer to recent papers [1, 6, 10, 11] in which graph products were studied.

Local colorings were introduced by Chartrand et al. in [2] (although the next paper written on the topic [3] was eventually published two years earlier). In the seminar paper it was shown that the study of the local chromatic number cannot be reduced to 2-connected graphs. More precisely, the local chromatic number of a graph can be bigger than the maximum of the local chromatic numbers of its blocks. Several exact values of the invariant were obtained, including that of complete graphs and complete multipartite graphs. In the subsequent paper [3] the emphasize was on regular graphs, where Cartesian products with one factor being a hypercube played the central role. In [7] it was proved that $\chi_\ell(G) \leq 2\Delta(G) - 2$ holds for any graph with $\Delta(G) \geq 3$ and different from K_4 and K_5. This result in particular confirms Conjecture 4.2 from [2] asserting that $\chi_\ell(G) = 4$ holds for cubic, non-bipartite, and non-complete graphs. In [8] local colorings of Kneser graphs were studied, this line or research was continued in [4] through the perceptive of semi-matching colorings.

In this note we are interested in the local coloring of Cartesian products of graphs, primarily motivated with investigations in [7] where exact local chromatic numbers were determined for some specific products. We proceed as follows. In the next section concepts, definitions, and results needed are recalled. In Section 3 the local chromatic number is determined for Cartesian products of 3-colorable graphs. In particular, the local chromatic number of products of cycles is extracted, a result that in one part corrects an assertion from [7]. In the final section we then prove that if G and H are graphs such that $\chi(G) \leq \lfloor \chi_\ell(H)/2 \rfloor$, then $\chi_\ell(G \,\square\, H) \leq \chi_\ell(H) + 1$.

2. Preliminaries

We will use the notation $[n]$ for the set $\{1, \ldots, n\}$. Graphs considered here are simple. If $G = (V(G), E(G))$ is a graph, then the *order* of G is $|V(G)|$ and the *size* of G is $|E(G)|$. As usual, $\chi(G)$ is the chromatic number of G. If a coloring uses k colors then it is a *k-coloring* and if $k = \chi(G)$ then it is a *χ-coloring*. G is called *k-colorable* if $\chi(G) \leq k$. Similarly, a local coloring that uses k colors is a *k-local coloring* and if $k = \chi_\ell(G)$ then it is a *χ_ℓ-coloring*.

The *Cartesian product* of graphs G and H is the graph $G \,\square\, H$ with the vertex set $G \times H$, and $(g, h)(g', h') \in E(G \,\square\, H)$ if either $gg' \in E(G)$ and $h = h'$, or $hh' \in E(H)$ and $g = g'$. The Cartesian product is commutative and associative, having the one vertex graph as a unit. The subgraph of $G \,\square\, H$ induced by $g \times V(H)$, where $g \in V(G)$, is isomorphic to H. It is called an *H-layer* (over g) and denoted ${}^gH$. Similarly, the subgraph of $G \,\square\, H$ induced by $V(G) \times h$, where $h \in V(H)$, is isomorphic to G, called a *G-layer* (over h) and denoted G^h. For more information on the Cartesian product of graphs see [5].

We now recall some results on the local chromatic number. Note first that if G is a subgraph of H, then $\chi_\ell(G) \leq \chi_\ell(H)$. In the rest we will use the fact that a labeling c of $V(G)$ is a local coloring if and only if (i) c is a (usual) vertex coloring, (ii) every induced P_3 contains two vertices with colors at least two apart, and (iii) every induced triangle contains two vertices with colors at least three apart. We will also use the following:

Proposition 1. ([2]) *If G is a connected bipartite graph of order at least 3, then $\chi_\ell(G) = 3$.*

It is easy to prove Proposition 1: the lower bound follows because G contains at least one induced P_3, the upper bound is obtained by coloring each vertex of one bipartition set of G with 1 and each vertex of the other bipartition set with 3. We will also need the following result which follows by replacing color i with $2i - 1$ for any $1 \leq i \leq \chi(G)$ in a χ-coloring of G.

Proposition 2. ([2]) *For any graph G, $\chi(G) \leq \chi_\ell(G) \leq 2\chi(G) - 1$.*

As already mentioned, the local chromatic number of complete multipartite graphs was determined in [3], for further use we state the following special case:

Theorem 3. *If $n \geq 1$, then $\chi_\ell(K_n) = \lfloor (3n-1)/2 \rfloor$.*

We will also often use (without explicitly mentioning it) the 1957 Sabidussi's result [9] asserting that $\chi(G \,\Box\, H) = \max\{\chi(G), \chi(H)\}$, cf. [5, Theorem 26.1].

3. Products of 3-colorable graphs

Theorem 4. *Let G and H be 3-colorable graphs with at least one edge.*

(i) If $\chi(G) = \chi(H) = 2$, then $\chi_\ell(G \,\Box\, H) = 3$.
(ii) If $\chi(G) = 2$, $\chi(H) = 3$, and $\chi_\ell(H) = 3$, then $\chi_\ell(G \,\Box\, H) = 4$.
(iii) If $\chi(G) = 2$, $\chi(H) = 3$, and $\chi_\ell(H) = 4$, then $4 \leq \chi_\ell(G \,\Box\, H) \leq 5$.
(iv) If $\chi(G) = 2$, $\chi(H) = 3$, and $\chi_\ell(H) = 5$, then $\chi_\ell(G \,\Box\, H) = 5$.
(v) If $\chi(G) = \chi(H) = 3$, then $\chi_\ell(G \,\Box\, H) = 5$.

Proof. (i) Since G and H each have at least one edge, $G \,\Box\, H$ contains at least one C_4 and since G and H are bipartite, $G \,\Box\, H$ is bipartite as well. Hence $\chi_\ell(G \,\Box\, H) = 3$ follows by Proposition 1.

(ii) Let c_G be a χ-coloring of G (so $c_G : V(G) \to [2]$) and let c_H be a χ_ℓ-coloring of H (so $c_H : V(H) \to [3]$). Then define $c : V(G \,\Box\, H) \to [4]$ as follows:

$$c(g,h) = \begin{cases} 1;\, c_G(g) = 1, c_H(h) = 1 \text{ or } c_G(g) = 2, c_H(h) = 3, \\ 2;\, c_G(g) = 1, c_H(h) = 2, \\ 3;\, c_G(g) = 2, c_H(h) = 1 \text{ or } c_G(g) = 1, c_H(h) = 3, \\ 4;\, c_G(g) = 2, c_H(h) = 2. \end{cases}$$

Note that the endvertices of any edge from a G-layer receive colors that differ by at least 2. Moreover, a possible triangle of $G \,\Box\, H$ can only lie in an H-layer, but H is triangle-free because $\chi_\ell(H) = 3$. It now readily follows that c is a 4-χ_ℓ-coloring of $G \,\Box\, H$ and thus $\chi_\ell(G \,\Box\, H) \leq 4$. To see that $\chi_\ell(G \,\Box\, H) \geq 4$, consider a subgraph $X = K_2 \,\Box\, C_{2k+1}$, $k \geq 1$. (Such a subgraph exists because $\chi(H) = 3$.) Assume for a moment that $\chi_\ell(X) = 3$. Then each of the 4-cycles of X must be colored with consecutive colors $1,3,1,3$. But this clearly leads to a contradiction after considering all the 4-cycles above the edges of C_{2k+1}. Therefore $\chi_\ell(X) \geq 4$ and hence also $\chi_\ell(G \,\Box\, H) \geq 4$. We conclude that c is a χ_ℓ-coloring of $G \,\Box\, H$.

(iii) and (iv) If $\chi_\ell(H) = 4$, then $\chi_\ell(G \,\Box\, H) \geq 4$ and if $\chi_\ell(H) = 5$, then $\chi_\ell(G \,\Box\, H) \geq 5$. On the other hand, since $\chi(G \,\Box\, H) = 3$, Proposition 2 implies that $\chi_\ell(G \,\Box\, H) \leq 5$.

(v) Since $\chi(G) = \chi(H) = 3$, we have $\chi(G \,\Box\, H) = 3$ and hence Proposition 2 implies that $\chi_\ell(G \,\Box\, H) \leq 5$. It thus remains to prove that $\chi_\ell(G \,\Box\, H) \geq 5$. For this sake it suffices to prove that $\chi_\ell(C_{2m+1} \,\Box\, C_{2n+1}) \geq 5$ holds for any $m, n \geq 1$. Indeed, since $\chi(G) = \chi(H) = 3$, G contains an (induced) odd cycle C_{2m+1} and H contains an (induced) odd cycle C_{2n+1}, hence $G \,\Box\, H$ contains an induced $C_{2m+1} \,\Box\, C_{2n+1}$.

Set $X = C_{2m+1} \,\Box\, C_{2n+1}$, and let $V(C_k) = [k]$ with natural adjacencies, so that $V(X) = [2m+1] \times [2n+1]$. Suppose on the contrary that c is a 4-local-coloring of X.

If x and y are vertices of X, then let $c(\{x,y\})$ denote the set $\{c(x), c(y)\}$. We first claim that if $e = xy \in E(X)$, then $c(\{x,y\}) \neq \{1,2\}$ and $c(\{x,y\}) \neq \{3,4\}$. We may assume without loss of generality that e lies in a C_{2m+1}-layer, that is, $x = (i,j)$ and $y = (i+1,j)$ for some $i \in [2m+1]$ and some $j \in [2n+1]$ (indices modulo $2m+1$ and $2n+1$, respectively). Suppose that $c(i,j) = 1$ and $c(i+1,j) = 2$. Then by the definition of the local coloring, $c(\{(i,j+1),(i+1,j+1)\}) = \{3,4\}$. In the same way we get $c(\{(i,j+2),(i+1,j+2)\}) = \{1,2\}$. Continuing the argument and having in mind that

C_{2n+1} is an odd cycle, we arrive at $\{c(i,j-1), c(i+1,j-1)\} = \{1,2\}$, which is a clear contradiction. We analogously arrive at a contradiction if $c(i,j) = 3$ and $c(i+1,j) = 4$. This proves the claim.

Let $A_i = (c^{-1}(1) \cup c^{-1}(2)) \cap C^i_{2m+1}$ and $B_i = (c^{-1}(3) \cup c^{-1}(4)) \cap C^i_{2m+1}$. Then $|A_i| = |B_{i+1}|$ and $|B_i| = |A_{i+1}|$ hold for any $1 \le i \le 2n+1$. Indeed, this follows from the above claim because $c(i,j) \in \{1,2\}$ if and only if $c(i+1,j) \in \{3,4\}$.

Clearly, $|A_i| + |B_i| = 2m+1$ and hence $|A_i| \neq |B_i|$. Assume without loss of generality that $|A_1| > |B_1|$. Then $|A_i| \ge m+1$ and $|B_i| \le m$ hold for $i \in \{1,3,\ldots,2m+1\}$. It follows that $2m+1 = |A_1| + |B_1| = |A_1| + |A_m| \ge 2m+2$, the final contradiction. □

Note that by the commutativity of the Cartesian product, Theorem 4 covers all possible products with 3-colorable factors.

If $\chi(G) = 2$, $\chi(H) = 3$, and $\chi_\ell(H) = 4$, Theorem 4 offers two possibilities: $\chi_\ell(G \,\square\, H) = 4$ or $\chi_\ell(G \,\square\, H) = 5$. To see that the first possibility can happen, recall that $\chi_\ell(K_3) = 4$ and observe that $\chi_\ell(K_2 \,\square\, K_3) = 4$. For the other possibility consider the graph G and the product $P_3 \,\square\, G$ from Fig. 1.

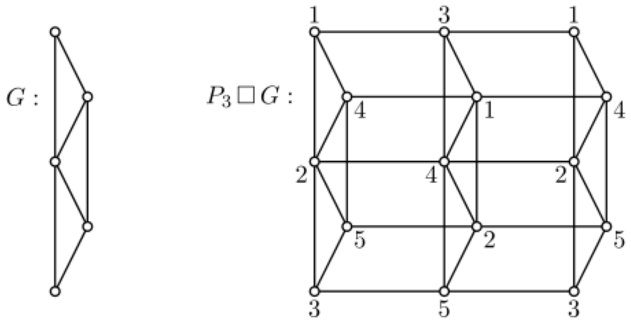

Figure 1. Graph G and the product $P_3 \,\square\, G$ with a 5-local coloring

Note first that $\chi(G) = 3$ and $\chi_\ell(G) = 4$. In addition, by a case analysis (or using computer) we can show that $\chi_\ell(P_3 \,\square\, G) > 4$. Here the case analysis can by simplified by frequently using the fact that K_3 has exactly two χ_ℓ-colorings (with colors 1, 2, 4 and 1, 3, 4, respectively), and by the fact that in any 4-local coloring of G, the vertex of degree 4 receives either color 1 or color 4. Hence $\chi_\ell(P_3 \,\square\, G) \ge 5$. On the the other hand $\chi_\ell(P_3 \,\square\, G) \le 5$ by Theorem 4 (iii); see also Fig. 1 for a 5-local coloring. In conclusion, $\chi_\ell(P_3 \,\square\, G) = 5$.

Corollary 5. *If* $m, n \ge 3$, *then*

$$\chi_\ell(C_m \square C_n) = \begin{cases} 3; & m, n \textit{ are both even}, \\ 4; & \textit{exactly one of } m \textit{ and } n \textit{ is even}, \\ 5; & m, n \textit{ are both odd}. \end{cases}$$

Proof. We recall from [3, Theorem E] that $\chi_\ell(C_n) = 3$ holds for any $n \ge 4$. Then Theorem 4 (i), (ii), and (v) cover all the cases, except the case $C_3 \,\square\, C_{2n}$, $n \ge 2$, because $\chi_\ell(C_3) = 4$ and we thus have two possibilities due to Theorem 4 (iii). Now color the first C_3-layer of $C_3 \,\square\, C_{2n}$ with 1, 2, 4, the second layer with 3, 4, 1, and alternately continue with this coloring. This is a local coloring and hence $\chi_\ell(C_3 \,\square\, C_{2n}) = 4$. □

Corollary 5 corrects [8, Theorem 5] where it is claimed that $\chi_\ell(C_{2m+1} \square C_{2n+1}) = 4$. A coloring is proposed there for which it is claimed that it is easy to see to be a local coloring of value 4. An example of the proposed coloring is shown in Fig. 2, where it can be seen that the problem is the most outer 4-cycle which is colored 1, 2, 1, 2.

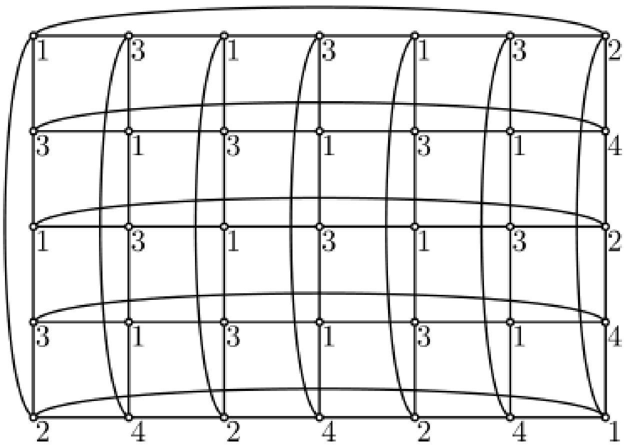

Figure 2. A coloring of $C_7 \,\square\, C_5$ which is not local

4. More bounds

In this section we prove the following result. Note that it in particular extends Theorem 4 (iii).

Theorem 6. *If G and H are graphs such that $2 \leq \chi(G) \leq \left\lfloor \frac{\chi_\ell(H)}{2} \right\rfloor$, then*

$$\chi_\ell(H) \leq \chi_\ell(G \,\square\, H) \leq \chi_\ell(H) + 1 \,.$$

Proof. $\chi_\ell(G \,\square\, H) \geq \chi_\ell(H)$ holds because H is an induced subgraph of $G \,\square\, H$. To prove the upper bound, let c_G be a χ-coloring of G, let c_H be a χ_ℓ-coloring of H, and define a coloring c of $G \,\square\, H$ as

$$c(g, h) = 2c_G(g) + c_H(h) - 2 \,,$$

where computations are done modulo $\ell + 1$, where $\ell = \chi_\ell(H)$. As $c : V(G \,\square\, H) \to [\ell + 1]$ it suffices to prove that c is a local coloring.

Consider any two vertices (g, h) and (g', h), $g \neq g'$, such that $c_G(g) \neq c_G(g')$. If $2c_G(g) + c_H(h) - 2 \leq \ell + 1$ and $2c_G(g') + c_H(h) - 2 \leq \ell + 1$ or $2c_G(g) + c_H(h) - 2 \geq \ell + 2$ and $2c_G(g') + c_H(h) - 2 \geq \ell + 2$, then $|c(g, h) - c(g', h)| = 2|c_G(g) - c_G(g')| \geq 2$. If $2c_G(a) + c_H(h) - 2 \leq \ell + 1$ and $2c_G(b) + c_H(h) - 2 \geq \ell + 2$, where $\{a, b\} = \{g, g'\}$, then $|c(g, h) - c(g', h)| = |2c_G(a) - 2c_G(b) + (\ell + 1)| \geq 3$. Therefore, we have

$$|c(g, h) - c(g', h)| \geq 2 \,. \tag{1}$$

Similarly, we can obtain $|c(g, h) - c(g, h')| \geq 1$ for any two vertices (g, h) and (g, h') with $c_H(h) \neq c_H(h')$.

Consider now an arbitrary triangle T of $G \,\square\, H$. By the definition of the Cartesian product, T is an induced subgraph in either a G-layer or in an H-layer. Assume first that $T \subseteq G^h$ for some $h \in V(H)$. Then by (1), the vertices of T are colored with colors that are pairwise at least 2 apart. So T is properly locally colored. Suppose next that $T \subseteq^g H$ for some $g \in V(G)$. Since c_H is a χ_ℓ-coloring of H, there exist two vertices $(g, h), (g, h') \in V(T)$ such that $3 \leq |c_H(h) - c_H(h')| \leq \ell - 2$. Otherwise, $4 \leq \ell \leq 5$ and the result is clear. If $2c_G(g) + c_H(h) - 2 \leq \ell + 1$ and $2c_G(g) + c_H(h') - 2 \leq \ell + 1$ or $2c_G(g) + c_H(h) - 2 \geq \ell + 2$ and $2c_G(g) + c_H(h') - 2 \geq \ell + 2$, then $|c(g, h) - c(g, h')| =$

$|c_H(h) - c_H(h')| \geq 3$. If $2c_G(g) + c_H(a) - 2 \leq \ell + 1$ and $2c_G(g) + c_H(b) - 2 \geq \ell + 2$, where $\{a, b\} = \{h, h'\}$, then $|c(g, h) - c(g, h')| = |c_H(a) - c_H(b) + (\ell + 1)| \geq 3$. So T is properly locally colored.

Let P be an induced subgraph of $G \,\square\, H$ isomorphic to P_3. If $P \subseteq {}^g H$ or $P \subseteq G^h$, then we can argue as above for T that P is properly locally colored. Suppose now that P is induced on vertices (g, h), (g', h), and (g', h'), where $g \neq g'$ and $h \neq h'$. Considering the edge $(g, h)(g', h)$ and applying (1) yield the required conclusion. □

The second example after Theorem 4 (iii) demonstrates that the upper bound $\chi_\ell(H)+1$ in Theorem 6 is sharp. For another example consider the product $K_4 \,\square\, K_6$. By Theorem 3, $\chi_\ell(K_6) = \lfloor (3 \cdot 6 - 1)/2 \rfloor = 8$, hence the condition $\chi(K_4) \leq \left\lfloor \frac{\chi_\ell(K_6)}{2} \right\rfloor$ is satisfied. On the other hand, by [7, Theorem 7], $\chi_\ell(K_4 \,\square\, K_6) > \chi_\ell(K_6)$. Using Theorem 6 we conclude that $\chi_\ell(K_4 \,\square\, K_6) = 9$.

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