# The median function on graphs with bounded profiles

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**Abstract**

The median of a profile $\pi = (u_1, \ldots, u_k)$ of vertices of a graph $G$ is the set of vertices $x$ that minimize the sum of distances from $x$ to the vertices of $\pi$. It is shown that for profiles $\pi$ with diameter $d$ the median set can be computed within an isometric subgraph of $G$ that contains a vertex $x$ of $\pi$ and the $r$-ball around $x$, where $r > 2d - 1 - 2d / \mid \pi \mid$. The median index of a graph and $r$-joins of graphs are introduced and it is shown that $r$-joins preserve the property of having a large median index. Consensus strategies are also briefly discussed on graph with bounded profiles.

**Key words**: Consensus; Median function; Local property; Median graph; Consensus strategy.

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## 1 Introduction

The idea of consensus is present in many different fields, for instance in economics, sociology, and biology; we refer to [4] for a general mathematical formalization of the consensus theory. The general situation can be frequently presented as a group of clients that wish to achieve a consensus by some rational process which can in turn be modeled with consensus functions on some discrete structure. The most studied discrete structures in this respect are posets (see, for instance, [5, 8]) and graphs.

A natural way of achieving a consensus on a graph is by means of a median function that is defined as follows. A *profile* $\pi$ on a graph $G$ is a finite sequence of vertices of $G$. (Sequences are taken in order to enable possible repetitions.) For a vertex $x$ of $G$ let $D(x,\pi) = \sum_{v\in\pi} d(x,v)$, where $d$ is the usual shortest path distance. Then $x$ is a *median vertex for* $\pi$ if $D(x,\pi)$ is minimum. The *median function* $M$ is the function that for each profile $\pi$ on $G$ returns the set of its median vertices $M(\pi)$. The set $M(\pi)$ is called the *median* of $\pi$. The median function is also known as the *median procedure.*

The special case of median functions when the profile is the whole vertex set of a (vertex-weighted) graph has been extensively studied, see [3, 13] and references therein. On the other hand, a profile on a graph is often small with respect to the whole graph. For instance, it can consist of as few as three vertices of which two are at distance two [11]. In such cases the median of the profile might be found without processing the entire graph, and many algorithms which do not work globally may be locally feasible for such profiles. Hence it seems reasonable to consider profiles with bounded diameter.

In the last part of this section the remaining concepts and definitions needed are given. In the next section we consider profiles $\pi$ with bounded diameter and show that then the median of $\pi$ can be obtained locally, either in a properly bounded isometric subgraph or in an induced subgraph that contains $\pi$.

Mulder [11] proved that the majority strategy produces $M(\pi)$, for all $\pi$ if and only if the corresponding graph is median. (In fact, it is also equivalent to the fact that the majority strategy produces $M(\pi)$, for all $\pi$ of length 3.) For another characterization of median graphs in terms of the median function see [2]. Due to this central role of median graphs in the graph theoretical consensus theory we introduce in Section 3 a median index of a graph. We also introduced a $d$-join of graphs and prove that the $d$-join of graphs with median indices at least $d$ is a graph with the same property.

In the last section we discuss known consensus related to situations with bonded profiles and propose two new majority strategies. (For more information on consensus strategies on graphs consult [1].)

A subgraph $H$ of a graph $G$ is an *isometric subgraph* if $d_H(u,v) = d_G(u,v)$ for all vertices $u,v$ in $H$. The *diameter* $\mathrm{diam}(G)$ of a graph $G$ is $\max_{u,v\in V(G)} d(u,v)$, while the *diameter of a profile* $\pi$, $\text{diam}(\pi)$, is $\max_{u,v\in\pi} d(u,v)$. The *k-ball* at a vertex $x$ of a graph $G$ is $B_k(x) = \{v \in V \mid d(x,v) \leq k\}$. We will also use the same notation to denote the subgraph of $G$ induced by $B_k(x)$. Finally, a connected graph $G$ is a *median graph* if, for every triple $u$, $v$, $w$ of vertices, there exist a unique vertex $x$, called the median of $u$, $v$, $w$, such that $x$ lies simultaneously on shortest paths joining $u$ and $v$, $v$ and $w$, and $u$ and $w$. We refer to [7] for a survey on median graphs including their role in location theory.

## 2 The median function on profiles with bounded diameter

We begin by showing that the median of a profile $\pi$ on $G$ with $\text{diam}(\pi) = d$ can be obtained by restricting to a relatively small isometric subgraph of $G$.

**Theorem 1** *Let $G$ be a connected graph, $\pi$ a profile on $G$ with* $\text{diam}(\pi) = d$*, and $x \in \pi$. Let $H$ be an isometric subgraph of $G$ containing $B_r(x)$ where*

$$r > 2d - 1 - \frac{2d}{\mid \pi \mid}.$$

*Then the median of $\pi$ in $G$ is the same as the median of $\pi$ in $H$.*

**Proof.** We first show that the median of $\pi$ in $G$ is contained in $B_r(x)$, where $r > 2d - 1 - \frac{2d}{|\pi|}$. Let $w \in \pi$. Then

$$D(w,\pi) = \sum_{\substack{u\in\pi \\ u\neq w}} d(w,u) \leq (\mid \pi \mid -1)d.$$

Let $w' \in G \setminus B_r(x)$. Then $d(w',x) \geq r+1$ while for any other $w \in \pi, w \neq x$ we infer

$$r+1 \leq d(w',x) \leq d(w',w) + d(w,x) \leq d(w',w) + d,$$

therefore $d(w',w) \geq r+1-d$. Thus for $w' \in G \setminus B_r(x)$ we have:

$$\begin{aligned} D(w',\pi) &\geq r+1+(\mid \pi \mid -1)(r+1-d) \\ &= r \mid \pi \mid + \mid \pi \mid -d \mid \pi \mid +d \\ &> (2d-1-\frac{2d}{\mid \pi \mid}) \mid \pi \mid + \mid \pi \mid -d \mid \pi \mid +d \\ &= (\mid \pi \mid -1)d \\ &\geq D(w,\pi). \end{aligned}$$

It follows that the median of $\pi$ in $G$ is contained in $B_r(x)$.

Since $H$ is an isometric subgraph of $G$ containing $B_r(x)$ we have $D_H(w, \pi) = D_G(w, \pi)$ for any $w \in V(H)$. Hence by the above argument the median of $\pi \in H$ contained in $B_r(x)$. As $B_r(x)$ is a subgraph of $H$ the median of $\pi$ in $H$ is the same as the median of $\pi$ in $G$. □

Note that if $B_r(x)$ is an isometric subgraph of $G$ for some vertex $x$ of $\pi$ then we may set $H = B_r(x)$ in Theorem 1. This holds in particular for any tree $T$ and any vertex from a profile on $T$.

In the previous theorem the graph $H$ to which a computation can be restricted must be isometric. In some cases it might not be easy to find such a subgraph (or it might also not exist), hence we next wish to drop the isometry assumption. This can be done by extending the corresponding balls.

**Theorem 2** *Let $G, \pi, d, x$ and $r$ be as in Theorem 1. Let $H$ be an induced subgraph of $G$ containing $B_{r+d}(x)$. Then the median of $\pi$ in $G$ is the same as the median of $\pi$ in $H$.*

**Proof.** From the proof of Theorem 1 we know that the median of $\pi$ in $G$ lies in $B_r(x)$. Thus it suffices to prove that the distance between a vertex $x \in \pi$ and a vertex $y \in B_r(v)$ is unaltered in the induced subgraph $B_{r+d}(v)$. This distance cannot increase because every shortest path from $x$ to $y$ in $B_r(v)$ is also a path in $B_{r+d}(v)$. Also the distance from $x$ to $y$ in $B_{r+d}(v)$ is $\leq r + d$.

Consider an arbitrary $x, y$-path $P$ that does not lie completely in $B_{r+d}(v)$. Then $P$ can be decomposed to two subpaths, one from $y$ to a vertex $z$ outside $B_{r+d}(v)$ and then followed by a subpath from $z$ to $x$. The former path is of length at least $d + 1$, while the latter is of length at least $(r - d) + (d + 1) = r + 1$. Hence $P$ is of length at least $\geq r + d + 2$ and thus $P$ is not a geodesic. We conclude that $M(\pi)$ of $G$ is the same as $M(\pi)$ of $H$. □

Theorems 1 and 2 can be applied in all situations in which it is possible to detect some previously studied structure in the vicinity of a profile. As an example of such a result we state:

**Corollary 3** *Let $v$ be a vertex in $G$ such that $B_{3d}(v)$ is a tree. Then for any profile $\pi$ on $G$ containing $v$ with* $\mathrm{diam}(\pi) \leq d$*, $M(\pi)$ is either a single vertex or a path.*

**Proof.** Note that $3d \geq d + r$, set $H = B_{r+d}(x)$, recall that the median of a profile on a tree is either a single vertex or a path, and apply Theorem 2. □

## 3 Locally median graphs

As already mentioned, median graphs form one of the central graph classes in the graph theoretical consensus theory. Therefore we introduce the following concepts.

Let $v$ be a vertex of a graph $G$. Then the *median index of* $v$, $\mathrm{mx}_G(v)$, is the largest integer $k \leq \mathrm{diam}(G)$ such that $B_j(v)$ is a median graph for $0 \leq j \leq k$. The *median index of* $G$, $\mathrm{mx}(G)$, is the minimum of the median indices of the vertices of $G$. For instance, $\mathrm{mx}(C_n) = \lfloor n/2 \rfloor - 1$, while for a tree $T$, $\mathrm{mx}(T) = \mathrm{diam}(T)$. $G$ is said to be *locally $p$-median* if its median index is $p$.

**Proposition 4** *Let $G$ be a graph, $\pi$ a profile on $G$ with $\mathrm{diam}(\pi) \leq d$, and $v$ an element of $\pi$ with $\mathrm{mx}_G(v) \geq 3d$. Let $x$ be a vertex of $G$ with $d(x, \pi) \leq d$. Then the majority strategy started from $x$ produces $M(\pi)$.*

**Proof.** Note first that $d + r \leq 3d$, where $r$ is defined as in Theorem 1. Clearly, all the vertices of $\pi$ belong to $B_d(v)$. Since $d(x, \pi) \leq d$, we have $x \in B_{2d}(v)$. As $B_{2d}(v)$ is an induced subgraph of $B_{3d}(v)$, Theorem 2 implies that $M(\pi)$ is contained in $B_{2d}(v)$. Since, in addition, $B_{3d}(v)$ is a median graph, the majority strategy started from $x$ produces $M(\pi)$ by the above-mentioned theorem of Mulder. □

Hence it is desirable to have graphs with large median index. To construct large graphs with this property we introduce a graph operation called the $d$-join of graphs.

By a *$d$-distance sequence* in a graph $G$ we mean a finite sequence $S$ of distinct vertices of $G$ such that for any two vertices $u, v$ of $S$, $d(u, v) \geq d$. Clearly any permutation of a $d$-distance sequence is also an $d$-distance sequence. Let $G_1$ and $G_2$ be graphs and let $S_1$ and $S_2$ be $d$-distance sequences of equal length in $G_1$ and $G_2$ respectively. Then the *$d$-join* of $G_1$ and $G_2$ with respect to $S_1$ and $S_2$ is the graph obtained from the disjoint union of $G_1$ and $G_2$ by joining the corresponding vertices in $S_1$ and $S_2$ by edges.

The $d$-join construction is illustrated in Fig. 1. The left graph is a 3-join of $C_{12}$ with itself, the right graph is a 4-join of $P_5 \,\square\, P_5$ with itself. (Recall that the *Cartesian product* $G \,\square\, H$ of two graphs has the vertex set $V(G) \times V(H)$ where the vertex $(g, h)$ is adjacent to $(g', h')$ whenever $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$, see [6].)

To show that the $d$-join operation preserves large median index we first recall the following—part of the folklore—result.

**Lemma 5** *A connected graph $G$ is a median graph if and only if every block of $G$ is median.*

**Theorem 6** *Let $G_1$ and $G_2$ be graphs with $\mathrm{mx}(G_1) \geq d$ and $\mathrm{mx}(G_2) \geq d$, and let $G_3$ be a $d$-join of $G_1$ and $G_2$. Then $\mathrm{mx}(G_3) \geq d$.*

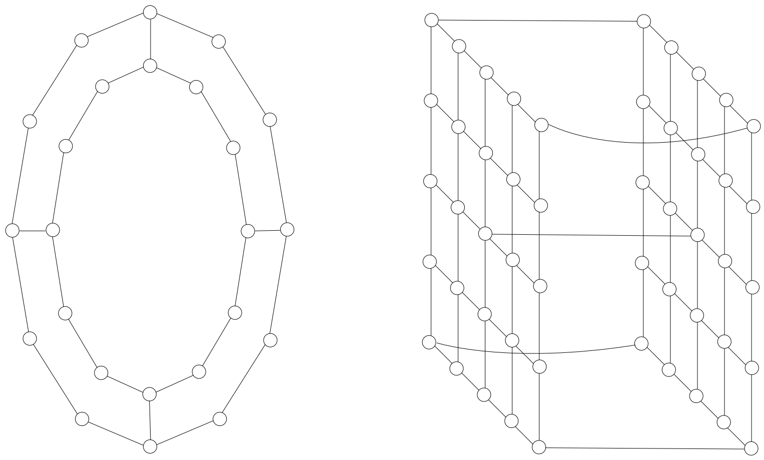

Figure 1: A 3-join and a 4-join

**Proof.** Let $G_3$ be the $d$-join of $G_1$ and $G_2$ with respect to the $d$-distance sequences $S = (s_1, \ldots, s_k)$ and $T=(t_1, \ldots, t_k)$. Consider an arbitrary vertex $x \in V(G_3)$. We need to show that $B_i(x)$ is a median graph for $1 \le i \le d$.

We first show that $B_d(x)$ is a median graph. Assume without loss of generality that $x \in G_1$ and set $a = \min_{1\le i\le k}\{d(x, s_i)\}$. We may in addition assume that $d(x, s_1) = a$.

Suppose first that $a \ge d$. In this case $B_d(x)$ is contained in $G_1$ and as $\mathrm{mx}(G_1) \ge d$, we infer that $B_d(x)$ is a median graph as required.

Suppose that $a < d$. Note first that for $s_i$, $i \ne 1$, we have

$$d \le d(s_1, s_i) \le d(s_1, x) + d(x, s_i) = a + d(x, s_i)$$

and hence $d(x, s_i) \ge d - a$.

Set $T_i(x) = B_d(x) \cap G_2$, $1 \le i \le k$. We claim that $T_i(x) \cap T_j(x) = \emptyset$ for any $i \ne j$. Suppose on the contrary that there exists a vertex $y \in T_i(x) \cap T_j(x)$ where $i \ne j$. Then $t_i$ is on a $y, x$-geodesic and $t_j$ is on another $y, x$-geodesic. Consequently, $d(y, t_i) \le a - 1$ and $d(y, t_j) \le a - 1$ for $i, j \ne 1$, while $d(y, t_1) \le d - a - 1$. Now, if $i = 1$ then

$$d \le d(t_1, t_j) \le d(t_1, y) + d(y, t_j) \le (d - a - 1) + (a - 1) = d - 2\,,$$

a contradiction. And if $i \ne 1$, $j \ne 1$, then

$$d \le d(t_i, t_j) \le d(t_i, y) + d(y, t_j) \le (a - 1) + (a - 1) = 2a - 2\,.$$

By the definition of $a$ we have $a \le d - a$ and so $d \ge 2a$, another contradiction.

Let $H = B_d(x) \cap G_1$ and let $B_d(x) \cap T = \{t_1\} \cup \{t_{i_1}, \ldots, t_{i_r}\}$. Then $H$ is a median graph since $\mathrm{mx}(G_1) \geq d$. Similarly, since $T_{i_j}(x) = B_r(x) \cap G_2$, where $r < d$, $T_{i_j}(x)$ is a median graph. Hence $G$ is obtained from the disjoint union of $H, T_1, T_{i_1}, \ldots, T_{i_r}$ by adding edges between the corresponding $s_i$'s and $t_i$'s. Hence, $H$, $T_1, T_{i_1}, \ldots, T_{i_r}$, and the edges in between are the blocks of $B_d(x)$. Since they are all median, Lemma 5 implies that $B_d(x)$ is median as well.

Finally note that the structure of $B_i(x)$, where $1 \leq i \leq d$, is the same as the structure of $B_d(x)$, hence all the balls $B_i(x)$ are median and so $\mathrm{mx}(G_3) \geq d$. $\square$

Let $G_{kn}$ be the $n$-join of two copies of the cycle $C_{kn}$, where $2 < k \leq n$. That is, the two cycles are connected with $k$ edges such that an $n$-join is constructed. The case $k = 3$ and $n = 4$ is shown in Fig. 1. Since $\mathrm{mx}(C_{kn}) = \lfloor kn/2 \rfloor - 1 \geq n$, Theorem 6 implies that $\mathrm{mx}(G_{kn}) = n$.

For another example consider the Cartesian product $G_n = P_{2n+1} \,\square\, P_{2n+1}$, where $n \geq 1$. Select the four vertices of degree 2 and the central vertex of $G_n$ for a $2n$-*distance sequence* of $G_n$ and let $H_n$ be the $2n$-join of two copies of $G_n$. ($H_2$ is shown in Fig. 1.) Then $\mathrm{mx}(H_n) = 2n$ by Theorem 6.

## 4 Consensus strategies

Different strategies can be formulated that search for the median set of a profile in an arbitrary graph. In [11] Mulder proposed the majority strategy that can be stated in the following slightly more general form.

1. Start at an initial vertex $v$.
2. If we are in $v$ and $w$ is a neighbor of $v$ for which the consensus criteria is fulfilled then we move to $w$.
3. We move only to a vertex already visited if there is no alternative.
4. We stop when (i) we are stuck at a vertex $v$ (ii) we have visited vertices at least twice, and, for each vertex $v$ visited at least twice and each neighbor $w$ of $v$, either $w$ is also visited at least twice or consensus criteria is not fulfilled.
5. We park at the vertices where we get stuck or at each vertex visited twice and erect a traffic sign (that is, the set of vertices visited twice is selected as the result of the strategy).

The above strategy is known as the *majority strategy* if the consensus criteria is to move from $v$ to $w$ provided that

$$|\pi_{wv}| \geq \frac{1}{2}|\pi|,$$

where $\pi_{wv}$ is the subprofile of $\pi$ consisting of the elements of $\pi$ closer to $w$ than to $v$. Other well-known strategies are:

- *Condorcet*: $|\pi_{vw}| \leq \frac{1}{2}|\pi|$.
- *Plurality*: $|\pi_{vw}| \leq |\pi_{wv}|$.

It is easy to observe that in the case of bipartite graphs the majority strategy, the condorcet strategy, and the plurality strategy coincide. This observation together with Theorem 1 yields the following result for graphs that are bipartite in a vicinity of a profile.

**Proposition 7** *If $\pi$ is any profile on $G$ with* $\text{diam}(\pi) \leq d$ *and if $v$ is any vertex in $\pi$ such that the induced subgraph $B_{3d}(v)$ is bipartite, then the majority strategy, the condorcet strategy, and the plurality strategy coincide on $B_d(v)$.*

Conversely, if the majority strategy works optimally when starting from vertices of a profile, the graph must be bipartite in a vicinity of the profile. More precisely:

**Proposition 8** *If for each profile $\pi$ with* $\text{diam}(\pi) \leq d$ *, the majority strategy produces $M(\pi)$, starting from a vertex in $\pi$, then $G$ does not contain any odd cycle of length less than $2d+3$.*

**Proof.** The lemma is obvious for $k = 0$ since the profile contains a single vertex, and every single vertex graph is trivially bipartite.

Now assume that $k \geq 1$. We first prove that $G$ is triangle-free. Assume that $G$ contains a triangle $u$, $v$, $w$. Consider the profile $\pi = u, v, w$. Then $D(x, \pi) = 2$ for $x$ in $\pi$ and $D(x, \pi) \geq 3$ for $x$ outside $\pi$. So $M(\pi) = \{u, v, w\}$. If we apply majority strategy starting at $u$, we find that we are stuck at $u$ and we do not get all of $M(\pi)$. Hence $G$ has to be triangle free.

Assume that $G$ contains an odd cycle of length less than $2k + 3$. Let $C$ be a minimal odd cycle in $G$ of length $t < 2k+3$. Then $C$ is an isometric cycle in $G$. Take any vertex $u$ of $C$ and let $v$ and $w$ be vertices on $C$ at a distance $t$ from $u$. Now we have $D(v, \pi) = D(w, \pi) = t+1$. Take any vertex $x$ distinct from $v$ and $w$. Since $G$ is triangle free, $x$ cannot be adjacent to both $v$ and $w$, say $d(x, w) \geq 2$. Because of the triangle inequality, we have $d(x, u) + d(x, v) \geq t$. Hence $D(x, \pi) \geq t + 2$, therefore $M(\pi) = \{v, w\}$. We apply the majority strategy from initial position $v$ with respect to $\pi$. Let $x$ be any neighbor of $v$. If $x = w$, only $x$ is nearer to $x$ than $v$. If $x \neq w$, then only $u$ could be nearer to $x$ than $v$. Hence we do not move to $x$, so that we are stuck at $v$ . Again we do not get all of $M(\pi)$. □

To conclude the paper we propose two additional consensus strategies, the idea for them coming from artificial intelligence [12]. In these strategies the consensus criteria is to move from $v$ to $w$ provided that:

- *Hill Climbing*: $D(w, \pi) \leq D(v, \pi)$.

- *Steepest Ascent Hill Climbing*: $D(w, \pi) \leq D(v, \pi)$ and $D(w, \pi)$ is minimum among all neighbors of $v$.

It seems that these strategies offer new insights into the consensus theory and will be studied elsewhere [1].

## References


[1] K. Balakrishnan, M. Changat and H. M. Mulder, Consensus strategies on graphs, in preparation.

[2] H.-J. Bandelt and J.-P. Barthélemy, Medians in median graphs, Discrete Appl. Math. 8 (1984) 131–142.

[3] H.-J. Bandelt and V. Chepoi, Graphs with connected medians, SIAM J. Discrete. Math. 15 (2002) 268-282.

[4] J. P. Barthélemy and M. F. Janowitz, A formal theory of consensus, SIAM J. Discrete Math. 4 (1991) 305–322.

[5] J. P. Barthélemy, B. Leclerc, and B. Monjardet, On the use of ordered sets in problems of comparison and consensus of classifications, J. Classification 3 (1986) 187–224.

[6] W. Imrich and S. Klavžar, *Product Graphs: Structure and Recognition*, John Wiley & Sons, New York, 2000.

[7] S. Klavžar and H. M. Mulder, Median graphs: characterizations, location theory and related structures, J. Combin. Math. Combin. Comp. 30 (1999) 103–127.

[8] B. Leclerc, The median procedure in the semilattice of orders, Discrete Appl. Math. 127 (2003) 285–302.

[9] F. R. McMorris, H. M. Mulder, and F. S. Roberts, The median procedure on median graphs, Discrete Appl. Math. 84 (1998) 165–181.

[10] F. R. McMorris, H. M. Mulder, and R. C. Powers, The median function on median graphs and semilattices, Discrete Appl. Math. 101 (2000) 221–230.

[11] H. M. Mulder, The majority strategy on graphs, Discrete Appl. Math. 80 (1997) 97–105.

[12] P. H. Winston, *Artificial Intelligence, 3rd Edition*, Addison-Wesley Publishing Co., Reading, MA, 1992.

[13] H.-G. Yeh and G. J. Chang, Centers and medians of distance-hereditary graphs, Discrete Math. 265 (2003) 297–310.