Maker-Breaker domination number

Valentin Gledel [a]   Vesna Iršič [b,c]   Sandi Klavžar [b,c,d]

March 13, 2019

[a] Univ Lyon, Université Lyon 1, LIRIS UMR CNRS 5205, F-69621, Lyon, France

[b] Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

[c] Faculty of Mathematics and Physics, University of Ljubljana, Slovenia

[d] Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia

Abstract

The Maker-Breaker domination game is played on a graph G by Dominator and Staller. The players alternatively select a vertex of G that was not yet chosen in the course of the game. Dominator wins if at some point the vertices he has chosen form a dominating set. Staller wins if Dominator cannot form a dominating set. In this paper we introduce the Maker-Breaker domination number $\gamma_{\mathrm{MB}}(G)$ of G as the minimum number of moves of Dominator to win the game provided that he has a winning strategy and is the first to play. If Staller plays first, then the corresponding invariant is denoted $\gamma'_{\mathrm{MB}}(G)$. Comparing the two invariants it turns out that they behave much differently than the related game domination numbers. The invariant $\gamma_{\mathrm{MB}}(G)$ is also compared with the domination number. Using the Erdős-Selfridge Criterion a large class of graphs G is found for which $\gamma_{\mathrm{MB}}(G) > \gamma(G)$ holds. Residual graphs are introduced and used to bound/determine $\gamma_{\mathrm{MB}}(G)$ and $\gamma'_{\mathrm{MB}}(G)$. Using residual graphs, $\gamma_{\mathrm{MB}}(T)$ and $\gamma'_{\mathrm{MB}}(T)$ are determined for an arbitrary tree. The invariants are also obtained for cycles and bounded for union of graphs. A list of open problems and directions for further investigations is given.

Key words: Maker-Breaker domination game; Maker-Breaker domination number; domination game; perfect matching; tree; cycle; union of graphs

AMS Subj. Class: 05C57, 05C69, 91A43

1   Introduction

Maker-Breaker games (as well as other *positional games*) have been introduced by Erdős and Selfridge in [13], and since then have been the subject of numerous studies, see [2, 3, 14, 15]. Maker-Breaker games are played on hypergraphs by two players called Maker and Breaker. They take turns and at each turn the current player selects a new vertex. Maker wins if at some point of the game he has selected all vertices from one of the hyperedges, while Breaker wins if she can keep him from doing it. See [1] and [16] for general introductions on this field.

Very recently, the *Maker-Breaker domination game* was introduced in [12]. The game is played on a graph G with two players named Dominator and Staller. These names were selected to emphasize the domination nature of the game and to be consistent with the usual domination game where these two names are standard by now. (The domination game was introduced in [4] and further studied in dozens of papers, cf. [6, 11, 22, 23, 24].) The players alternatively select a vertex of G that was not yet chosen in the course of the game. Dominator wins if at some point, the vertices he has chosen form a dominating set. Staller wins if Dominator cannot form a dominating set. Note that the Maker-Breaker domination game is a Maker-Breaker game. Indeed, if for a graph G we build a hypergraph $\mathcal{F}$ with the same set of vertices as G, and in which the hyperedges are the dominating sets of G, then Dominator wins the Maker-Breaker domination game on G if and only if Maker wins the Maker-Breaker game on $\mathcal{F}$.

In several papers on Maker-Breaker games the authors were interested in the smallest number of moves needed for Maker to win, see [7, 8, 15]. Also, in [12] it was emphasized that when dealing with the Maker-Breaker games, there are two natural questions: (i) which player has a winning strategy and (ii) what is the minimum number of moves if Dominator has a winning strategy. In the seminal paper question (i) is investigated, while in this paper we study (ii). For this sake we say that if G is a graph, then the *Maker-Breaker domination number* $\gamma_{\mathrm{MB}}(G)$ of G is the minimum number of moves of Dominator to win the game provided that he has a winning strategy and is the first to play. Otherwise we set $\gamma_{\mathrm{MB}}(G) = \infty$. Similarly, $\gamma'_{\mathrm{MB}}(G)$ denotes is the minimum number of moves of Dominator in the game in which Staller plays first.

We proceed as follows. In the next section we list additional definitions and several known results needed in this paper, as well as prove some basic results on the Maker-Breaker domination number. In Section 3 we first compare $\gamma_{\mathrm{MB}}(G)$ with $\gamma'_{\mathrm{MB}}(G)$ and find out that they behave totally different than the related game domination invariants. We also compare $\gamma_{\mathrm{MB}}(G)$ with the domination number and using the Erdős-Selfridge Criterion prove that if the number of γ-sets of G is not too big, then $\gamma_{\mathrm{MB}}(G) > \gamma(G)$. In Section 4 we introduce residual graphs, determine

(resp. bound) $\gamma'_{\mathrm{MB}}(G)$ (resp. $\gamma_{\mathrm{MB}}(G)$) in terms of the residual graph, and determine $\gamma_{\mathrm{MB}}(T)$ and $\gamma'_{\mathrm{MB}}(T)$ for an arbitrary tree. In the next two sections we obtain the invariants for cycles and bound them for union of graphs. We conclude with a list of open problems and directions for further investigation.

2 Preliminaries

Let G be a graph. A vertex of G adjacent to a leaf is a *support vertex* of G. A *perfect matching* of G is a set of pairwise independent edges that cover $V(G)$. The order of G will be denoted with $n(G)$. If u is a vertex of G, then $N[u]$ denotes the *closed neighborhood* of u. If v is another vertex then we set $N[u, v] = N[u] \cap N[v]$. A set $D \subseteq V(G)$ is a *dominating set* of G if $\cup_{u \in D} N[u] = V(G)$. The *domination number* $\gamma(G)$ is the size of a smallest dominating set of G. A dominating set of size $\gamma(G)$ is called a γ-*set* of G.

The Maker-Breaker domination game is called a D-game (resp. S-game) if Dominator (resp. Staller) is the first to play a vertex. The sequence of vertices selected in a D-game will be denoted with $d_1, s_1, d_2, s_2, \ldots$, and the sequence of vertices selected in an S-game with $s'_1, d'_1, s'_2, d'_2, \ldots$ Suppose that Dominator wins a D-game. Then the last vertex played is by Dominator, let it be d_k. By the definition of the game, $\{d_1, \ldots, d_k\}$ is a dominating set of G. Similarly, if Dominator wins an S-game and the last vertex played by Dominator is d'_ℓ, then $\{d'_1, \ldots, d'_\ell\}$ is a dominating set of G.

Let G be a graph, $k \geq 1$, and $u_1, \ldots, u_k, v_1, \ldots, v_k$ pairwise different vertices of G. Then we say that $X = \{\{u_1, v_1\}, \ldots, \{u_k, v_k\}\}$ is a *pairing dominating set* if

$$\bigcup_{i=1}^{k} N[u_i, v_i] = V(G) \,.$$

In the rest we will use this concept via the following interpretation proved in [12, Proposition 9]. To be self-contained, we give here an alternative, short proof.

Lemma 2.1 *Let $u_1, \ldots, u_k, v_1, \ldots, v_k$ be pairwise different vertices of a graph G, and let $X = \{\{u_1, v_1\}, \ldots, \{u_k, v_k\}\}$. Then X is a pairing dominating set if and only if every set $\{x_1, \ldots, x_k\}$, where $x_i \in \{u_i, v_i\}$, $i \in [k]$, is a dominating set of G.*

Proof. Suppose first that X is a pairing dominating set, that is, $\cup_{i=1}^{k} N[u_i, v_i] = V(G)$. Let $\{x_1, \ldots, x_k\}$ be an arbitrary set with $x_i \in \{u_i, v_i\}$, $i \in [k]$. Then $V(G) = \cup_{i=1}^{k} N[u_i, v_i] \subseteq \cup_{i=1}^{k} N[x_i]$. So $\{x_1, \ldots, x_k\}$ is a dominating set of G.

Conversely, consider a set $\{x_1, \ldots, x_k\}$, where $x_i \in \{u_i, v_i\}$, $i \in [k]$, and suppose that $\cup_{i=1}^{k} N[u_i, v_i]$ is a proper subset of $V(G)$. Let $w \in V(G) \setminus \cup_{i=1}^{k} N[u_i, v_i]$. Then

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for every $i \in [k]$ we have $w \notin N[u_i, v_i]$. Let $y_i \in \{u_i, v_i\}$ be such that $w \notin N[y_i]$. But then $\cup_{i=1}^k N[y_i]$ is a proper subset of $V(G)$, that is, $\{y_1, \ldots, y_k\}$ is not a dominating set. $\square$

If $X = \{\{u_1, v_1\}, \ldots, \{u_k, v_k\}\}$ is a pairing dominating set such that $u_i v_i \in E(G)$ holds for $i \in [k]$, then we say that X is a *dominating matching*.

Fact 2.2 [12, Proposition 10] *If G admits a pairing dominating set, then Dominator has a winning strategy on G in the D-game as well as in the S-game.*

The converse of Fact 2.2 does not hold in general. For instance, in [12, Figure 4] a chordal graph is presented on which Dominator has a winning strategy in both games but which admits no pairing dominating set. On the other hand, the converse holds in the class of trees because if Dominator has a winning strategy on a tree T, then it was proved in [12] that T has a dominating matching. Moreover, the converse also holds for co-graphs. The next lemma considers the variation of the game where players might skip some moves. This means that the current player selects no vertex and the previous player just makes another move. The skipped moves do not count in $\gamma_{\mathrm{MB}}(G)$ or $\gamma'_{\mathrm{MB}}(G)$.

Lemma 2.3 (No-Skip Lemma) *In an optimal strategy of Dominator to achieve $\gamma_{\mathrm{MB}}(G)$ or $\gamma'_{\mathrm{MB}}(G)$ it is never an advantage for him to skip a move. Moreover, if Staller skips a move it can never disadvantage Dominator.*

Proof. Suppose a D-game or an S-game is played. Let Dominator and Staller play optimally until some point when Staller decides to skip a move. In that case, Dominator imagines an arbitrary move of Staller, say x, and replies optimally to this move. Since Dominator can always, no matter the way Staller selects vertices, finish the game in no more that $\gamma_{\mathrm{MB}}(G)$ (resp. $\gamma'_{\mathrm{MB}}(G)$) moves, this property is preserved after the imagined move x and the reply to it. Then Dominator proceeds until the end of the game with the same strategy. Note that it may happen that in the course of the game Staller selects a vertex which is not a legal move in the game Dominator is imagining. In that case Dominator imagines that yet some other legal move has been played by Staller. In this way the game on G will finish in no more than $\gamma_{\mathrm{MB}}(G)$ (resp. $\gamma'_{\mathrm{MB}}(G)$) moves.

With a strategy of Staller parallel to the above strategy of Dominator we also infer that it is never an advantage for Dominator to skip a move. $\square$

If G is a graph and $S \subseteq V(G)$, then let $G|S$ denote that graph G in which the vertices from S are declared to be already dominated, that is, Dominator is not obliged to dominate them in the rest of the game. Then we have the following Continuation Principle, a proof of which is much simpler that the corresponding principle for the domination game [21].

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Remark 2.4 (Continuation Principle) *Let G be a graph with $A, B \subseteq V(G)$. If $B \subseteq A$, then $\gamma_{\mathrm{MB}}(G|A) \leq \gamma_{\mathrm{MB}}(G|B)$ and $\gamma'_{\mathrm{MB}}(G|A) \leq \gamma'_{\mathrm{MB}}(G|B)$.*

Indeed, the remark follows from the fact that Dominator can apply the same strategy in $G|A$ as in $G|B$.

Suppose that $\gamma_{\mathrm{MB}}(G) < \infty$. Then in any winning strategy of Dominator, he will play at most half of the vertices (because Staller will play the other half) which in turn implies that

$$1 \leq \gamma_{\mathrm{MB}}(G) \leq \left\lceil \frac{n(G)}{2} \right\rceil . \tag{1}$$

The bound is sharp, consider for instance the disjoint union of K_1 and several copies of K_2. It is also easy to see that all the possible values from (1) can be realized by considering the disjoint union of a complete graph and an appropriate number of K_2s. Similarly, for the S-game, assuming that $\gamma'_{\mathrm{MB}}(G) < \infty$, we have

$$1 \leq \gamma'_{\mathrm{MB}}(G) \leq \left\lfloor \frac{n(G)}{2} \right\rfloor , \tag{2}$$

where again all the values can be realized.

Later we will apply the celebrated Erdős-Selfridge Criterion for Maker-Breaker games that reads as follows.

Theorem 2.5 (Erdős-Selfridge Criterion [13]) *If $\mathcal{F}$ is a hypergraph, then*

$$\sum_{A \in \mathcal{F}} 2^{-|A|} < \frac{1}{2} \ \Rightarrow \ \mathcal{F} \text{ is a Breaker's win}.$$

This theorem together with its proof can also be found in the book [16, Theorem 2.3.3].

3 Maker-Breaker domination numbers

In this section we first compare $\gamma_{\mathrm{MB}}(G)$ with $\gamma'_{\mathrm{MB}}(G)$ and construct graphs for all possible values of the invariants. In the second part we compare $\gamma_{\mathrm{MB}}(G)$ with the domination number and using the Erdős-Selfridge Criterion find a large class of graphs G for which $\gamma_{\mathrm{MB}}(G) > \gamma(G)$ holds.

3.1 Realizations of Maker-Breaker domination numbers

One of the fundamental theorems on the domination game proved in [4, 21] asserts that $|\gamma_g(G) - \gamma'_g(G)| \leq 1$ holds for every graph G. The next result reveals that the situation with the Maker-Breaker domination number is dramatically different.

Theorem 3.1 *If G is a graph, then $\gamma(G) \leq \gamma_{\mathrm{MB}}(G) \leq \gamma'_{\mathrm{MB}}(G)$. Moreover, for any integers r, s, t, where $2 \leq r \leq s \leq t$, there exists a graph G such that $\gamma(G) = r$, $\gamma_{\mathrm{MB}}(G) = s$, and $\gamma'_{\mathrm{MB}}(G) = t$.*

Proof. The assertion $\gamma(G) \leq \gamma_{\mathrm{MB}}(G)$ is clear since $\{d_1, d_2, \ldots\}$ is a dominating set of G.

A D-game can be viewed as an S-game in which Staller has skipped her first move. The No-Skip Lemma thus implies that $\gamma_{\mathrm{MB}}(G) \leq \gamma'_{\mathrm{MB}}(G)$.

Let r, s, t be fixed integers where $2 \leq r \leq s \leq t$. Construct a graph $G_{r,s,t}$ as follows. Start with a path of length $r - 1$ on consecutive vertices $x_1, \ldots, x_r$. Attach $t - r + 1$ pendant triangles at x_1 and $s - r + 1$ pendant triangles at x_2. Finally, at each (if any) of the vertices $x_3, \ldots, x_r$ attach a pendant vertex $y_3, \ldots, y_r$, respectively. The construction should be clear with the aid of Fig. 1.
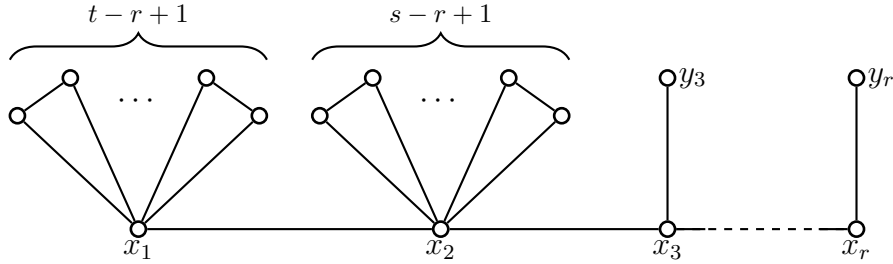

Figure 1: Graph $G_{r,s,t}$

The $t-r+1$ triangle edges opposite to x_1 and the $s-r+1$ triangle edges opposite to x_2 together with the edges $x_3y_3, \ldots, x_ry_r$ form a dominating matching of $G_{r,s,t}$. (If one would like to have a matching, then adding the edge x_1x_2 would do the job.) Hence by Fact 2.2, Dominator has a winning strategy in both games. We claim that $\gamma(G_{r,s,t}) = r$, $\gamma_{\mathrm{MB}}(G_{r,s,t}) = s$, and $\gamma'_{\mathrm{MB}}(G_{r,s,t}) = t$, where the first assertion is clear.

Consider the D-game. We first describe the following strategy of Dominator. He starts the game with the move $d_1 = x_1$. Then no matter how Staller plays, Dominator can proceed such that at most $(r - 2) + (s - r + 1)$ moves of him will be needed to dominate the graph. To do this, whenever Staller plays on x_i or y_i, $i \geq 3$, Dominator replies with y_i or x_i, respectively. Also, Dominator proceeds along the

same lines when Staller plays on a vertex that lies in a triangle attached to x_2. Using this strategy Dominator ensures that he will play at most $1 + (r-2) + (s-r+1) = s$ moves which means that $\gamma_{\mathrm{MB}}(G) \leq s$. To prove the other inequality, consider the following strategy of Staller. If Dominator starts with $d_1 = x_1$, then Staller sets $s_1 = x_2$. Then Dominator will need at least $(r-2) + (s-r+1)$ additional moves to dominate G. On the other hand, if $d_1 \neq x_1$, then Staller sets $s_1 = x_1$, but then Dominator will need at least $(r-2) + (t-r+1) \geq (r-2) + (s-r+1)$ additional moves. In any case, $\gamma_{\mathrm{MB}}(G) \geq 1 + (r-2) + (s-r+1) = s$.

Consider next the S-game. The proof that $\gamma'_{\mathrm{MB}}(G_{r,s,t}) = t$ proceeds similarly as above. To show that $\gamma_{\mathrm{MB}}(G) \geq t$, Staller can apply a strategy to start the S-game with $s'_1 = x_1$. In this way she can guarantee that Dominator will need to play at least $(t-r+1) + 1 + (r-2) = t$ moves. On the other hand, Dominator can play such that no more than t vertices will be selected by him. Whenever Staller plays x_i or y_i, $i \geq 3$, he replies with y_i or x_i, respectively. Moreover, during the game he will be able to play x_1 or x_2. $\square$

Note that if $\gamma(G) = 1$, then also $\gamma_{\mathrm{MB}}(G) = 1$. Hence Theorem 3.1 does not extend to the case $r = 1$. On the other hand, if G_t, $t \geq 1$, is the graph obtained from t disjoint triangles by identifying a vertex from each of the triangles (so that this new vertex is of degree $2t$), then $\gamma(G_t) = 1$, $\gamma_{\mathrm{MB}}(G_t) = 1$, and $\gamma'_{\mathrm{MB}}(G_t) = t$.

Theorem 3.1 extends also to highly connected graphs. To see this, consider the graphs $H_{k,r,s,t}$, $2 \leq r \leq s \leq t$, $k \geq 1$, that are schematically drawn in Fig. 2. Here, each vertex of a K_k clique is adjacent to each vertex of the clique K_{k+r}. Then by arguments similar to those from the proof of Theorem 3.1 one can see that $\gamma(H_{k,r,s,t}) = r$, $\gamma_{\mathrm{MB}}(H_{k,r,s,t}) = s$, and $\gamma'_{\mathrm{MB}}(H_{k,r,s,t}) = t$. Moreover, $H_{k,r,s,t}$ is $(k+1)$-connected.

3.2   Relation with the domination number

As already observed above, $\gamma(G) = 1$ if and only if $\gamma_{\mathrm{MB}}(G) = 1$. In general it would be interesting to characterize the graphs G such that $\gamma_{\mathrm{MB}}(G) = \gamma(G) = k$, where $k \geq 2$ is a fixed integer. For $k = 2$ the answer is simple:

Proposition 3.2 *Let G be a graph with $\gamma(G) = 2$. Then $\gamma_{\mathrm{MB}}(G) = \gamma(G) = 2$ if and only if G has a vertex that lies in at least two γ-sets of G.*

Proof. Suppose $\gamma_{\mathrm{MB}}(G) = \gamma(G) = 2$. After the moves d_1 and s_1 are played, Dominator has a strategy to win the game with the move d_2. Then $\{d_1, d_2\}$ is a γ-set of G. Moreover, Staller has an option to select s_1 such that $\{d_1, s_1\}$ is a γ-set, hence d_1 must lie in at least two γ-sets.
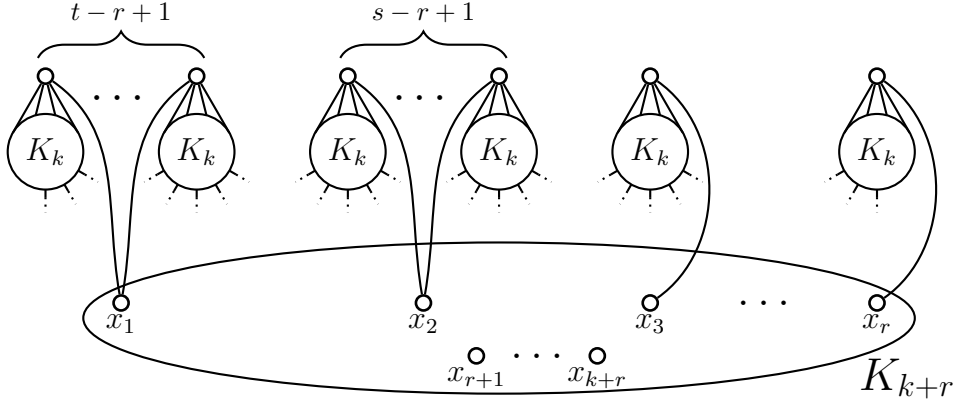
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Figure 2: Graph $H_{k,r,s,t}$

Conversely, let u be a vertex that lies in two γ-sets of G. Then Dominator plays $d_1 = u$, and then no matter which vertex is selected by Staller in her first move, Dominator can finish the game in his second move. $\square$

Proposition 3.2 can be rephrased to hold for larger k also, but this would be more or less just rephrasing the definitions. It would be more interesting to find a structural characterization of the corresponding graphs. This task, however, seems difficult. On the other hand, the Erdős-Selfridge Criterion gives a sufficient condition for $\gamma_{\mathrm{MB}}(G) > \gamma(G)$. Let $X_\gamma(G)$ be the number of γ-sets of a graph G, cf. [9]. Then:

Proposition 3.3 *If G is a graph and $X_\gamma(G) < 2^{\gamma(G)-1}$, then $\gamma_{\mathrm{MB}}(G) > \gamma(G)$.*

Proof. Let $\mathcal{F}$ be the hypergraph with $V(\mathcal{F}) = V(G)$ and whose hyperedges are the γ-sets of G. Then Theorem 2.5 asserts that

$$\sum_{A \in \mathcal{F}} 2^{-|A|} < \frac{1}{2} \;\Rightarrow\; \mathcal{F} \text{ is a Breaker's win}. \tag{3}$$

Since $|E(\mathcal{F})| = X_\gamma(G)$ and each of these hyperedges has size $\gamma(G)$, we can estimate as follows:

$$\sum_{A \in \mathcal{F}} 2^{-|A|} = \sum_{A \in \mathcal{F}} 2^{-\gamma(G)} = X_\gamma(G) \cdot 2^{-\gamma(G)} < 2^{\gamma(G)-1} \cdot 2^{-\gamma(G)} = \frac{1}{2}.$$

Therefore, $\mathcal{F}$ is a Breaker's win by (3). But this means that in G, Dominator is unable to win with $\gamma(G)$ moves, thus $\gamma_{\mathrm{MB}}(G) > \gamma(G)$. $\square$

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Consider the cycles C_{3k-1}, $k \geq 1$. It is known and easy to see that $\gamma(C_{3k-1}) = k$. We now determine the number of γ-sets of C_{3k-1}. Each vertex from a γ-set dominates itself and its two neighbors. As there are k such triplets and $3k - 1$ vertices in the graph, there is only one vertex that is dominated by two vertices from the γ-set, all others are dominated exactly once. Thus if the vertex that is dominated twice is fixed, then the γ-set of the cycle is uniquely determined. As there are $3k - 1$ choices for this vertex, we have $X_\gamma(C_{3k-1}) = 3k - 1$. If $k \geq 5$, then $X_\gamma(C_{3k-1}) = 3k - 1 < 2^{k-1} = 2^{\gamma(C_{3k-1})-1}$, and by Proposition 3.3, we conclude that $\gamma_{\mathrm{MB}}(C_{3k-1}) > k = \gamma(C_{3k-1})$. Actually, $\gamma_{\mathrm{MB}}(C_{3k-1})$ is much bigger than $\gamma_g(C_{3k-1})$ as we will see in Section 5.

The converse of Proposition 3.3 does not hold as the following example shows. If $k \in \{3, 4\}$, then $X_\gamma(C_{3k-1}) = 3k - 1 > 2^{k-1} = 2^{\gamma(C_{3k-1})-1}$, but as we will see in Section 5, $\gamma(C_{3k-1}) = k < k + 1 = \left\lfloor \frac{3k-1}{2} \right\rfloor = \gamma_{\mathrm{MB}}(C_{3k-1})$.

4   Residual graphs

In this section we study the Maker-Breaker domination number on a construction that might be of independent interest and that will be later used to determine the invariant for trees.

If G is a graph, then we say that the *residual graph* $R(G)$ of G is the graph obtained from G by iteratively removing pendant paths P_2 until no such path is present. By a *pendant* P_2 we mean P_2 attached to G with an edge. Hence, when such a pendant P_2 is removed, exactly two vertices and two edges are removed. When $G = P_2$, we can also remove it and obtain the empty graph.

Note that $H = R(G)$ for some graph G if and only if H is the empty graph, $H = K_1$, or each support vertex of H has degree at least 3. This is in particular true if H has no support vertices. We further observe:

Lemma 4.1 *If G is a graph, then $R(G)$ is unique (up to isomorphism).*

Proof. Let $P' = xy$ be an arbitrary pendant P_2 of G, where x is a leaf of G and let z be the other neighbor of y. Then either P' is removed at some point when pendant P_2s are removed from G, or P' is not removed at all. The latter possibility can only happen if after this removal process only the path induced with x, y, z remains, and then the pendant path $P'' = yz$ is removed. In this case, however, we have $R(G) = K_1$. In addition, by induction every pendant P_2 that appears during the removal process will either be eventually removed or will lead to the residual tree K_1. $\qquad \square$

Note that the proof of Lemma 4.1 also reveals that if $R(G) \neq K_1$, then $G \setminus V(R(G))$ is unique. To see that it is not unique in general, consider a path P_{2k+1}, $k \geq 2$, and different sequences of removing pendant P_2s.

Lemma 4.2 *Let G be a graph and $R(G)$ a residual graph of G. Then*

(i) $G \setminus V(R(G))$ is a forest that has a unique perfect matching, and

(ii) G has a perfect matching if and only if $R(G)$ has a perfect matching.

Proof. (i) $G \setminus V(R(G))$ is a forest since it is built from the empty graph by successively attaching to it pendant P_2s. If $x_i y_i$, $i \in I$, are the pendant P_2s that were removed when $R(G)$ was obtained from G, then $\{x_i y_i\}_{i \in I}$ is a unique perfect matching of $G - V(R(G))$.

(ii) If G has a perfect matching, then its restriction to $G \setminus V(R(G))$ must be the unique perfect matching of $G \setminus V(R(G))$, hence $R(G)$ has a perfect matching. Conversely, if $R(G)$ has a perfect matching, then it can be extended to a perfect matching of G by means of (i). $\qquad\square$

For the proof of the main result of this section, we also need the following.

Lemma 4.3 *If T is a tree that admits a perfect matching and $v \in V(T)$, then Staller has a strategy for the S-game such that Dominator has to select at least $\frac{n(T)}{2}$ vertices to dominate T and v is played by Staller in her last move.*

Proof. We prove the claim by induction on $n(T)$. If $T = P_2$ and $v \in V(P_2)$, then Staller can play on v and Dominator has to reply on the other vertex.

Let now $n(T) \geq 4$ and consider T as a BFS-tree rooted at an arbitrary vertex r. Let x be a leaf of this BFS-tree at the largest distance from r and let y be the neighbor of x. Then $\deg(y) = 2$ because T has a perfect matching. Let z be the other neighbor of y. Set $T' = T \setminus \{x, y\}$. As T has a perfect matching, xy belongs to it, hence T' also has a perfect matching. If $v \in V(T')$, then Staller starts on y, Dominator has to reply on x (otherwise Staller would win) and then Staller applies her strategy on T' (by the induction hypothesis). If $v \in \{x, y\}$, then she applies her strategy on T' with her last move on z, and then plays v in her last move. Note that if Dominator plays on v while Staller is playing on T', then Staller wins the game as she can prevent Dominator from playing on one pair of vertices from the matching in T'.

From the above strategy of Staller we conclude that the total number of Dominator's moves was $\frac{n(T')}{2} + 1 = \frac{n(T)}{2}$. $\qquad\square$

Note that by the strategy from the proof of lemma 4.3, unless Staller wants to play on a leaf, she plays on the support vertex, forcing Dominator to reply on its neighboring leaf and separating this P_2 from the rest of the graph.

Theorem 4.4 *Let $R(G)$ be a residual graph of G and let $H = G \setminus V(R(G))$. Then*

(i) $\gamma'_{\mathrm{MB}}(G) = \frac{n(H)}{2} + \gamma'_{\mathrm{MB}}(R(G)),$

(ii) $\frac{n(H)}{2} + \gamma_{\mathrm{MB}}(R(G)) - 1 \leq \gamma_{\mathrm{MB}}(G) \leq \frac{n(H)}{2} + \gamma_{\mathrm{MB}}(R(G)).$

Proof. (i) H has a perfect matching and is a forest by Lemma 4.2(i). Let the S-game be played on G and consider the following strategy of Staller. By Lemma 4.3 she can play on each tree of H and play last on the vertex of this tree adjacent to $R(G)$. Dominator has to reply on the matching (otherwise Staller wins the game). Thus, Dominator makes (at least) $\frac{n(H)}{2}$ moves on H. Moreover, Staller plays on vertices adjacent to $R(G)$, hence no vertex in $R(G)$ will be dominated by the time Staller makes her first move in $R(G)$. Next, Staller is the player to make the first move on $R(G)$ and she follows her optimal strategy there to ensure at least $\gamma'_{\mathrm{MB}}(R(G))$ moves of Dominator.

On the other hand, Dominator's strategy is to then reply wherever Staller plays, H or $R(G)$, with its strategy on this graph. As H has a perfect matching, Dominator makes no more than $\frac{n(H)}{2}$ moves on H. Moreover, he makes at most $\gamma'_{\mathrm{MB}}(R(G))$ moves on $R(G)$. Hence, we have $\gamma'_{\mathrm{MB}}(G) = \frac{n(H)}{2} + \gamma'_{\mathrm{MB}}(R(G))$.

(ii) Suppose now that the D-game is played on G. To prove the upper bound, Dominator's strategy is to start on $R(G)$ and then reply on $R(G)$ or H if Staller plays there. As H has a perfect matching, Dominator makes no more than $\frac{n(H)}{2}$ moves on H. Moreover, he makes at most $\gamma_{\mathrm{MB}}(R(G))$ moves on $R(G)$. Hence we get the upper bound $\gamma_{\mathrm{MB}}(G) \leq \frac{n(H)}{2} + \gamma_{\mathrm{MB}}(R(G))$.

To prove the lower bound, consider the following strategy of Staller depending on the first move of Dominator. We will distinguish two cases, the second with two subcases, which are schematically depicted in Fig. 3.

Case 1: The first move of Dominator is on $R(G)$.

Staller first applies her strategy from Lemma 4.3 on each tree of H, playing the vertex adjacent to $R(G)$ as her last move on each of the trees. With this, she forces Dominator to play (at least) $\frac{n(H)}{2}$ moves on H. After that we have an ordinary D-game played on $R(G)$, so at least $\gamma_{\mathrm{MB}}(R(G))$ moves are made on it by Dominator if Staller follows her strategy there.

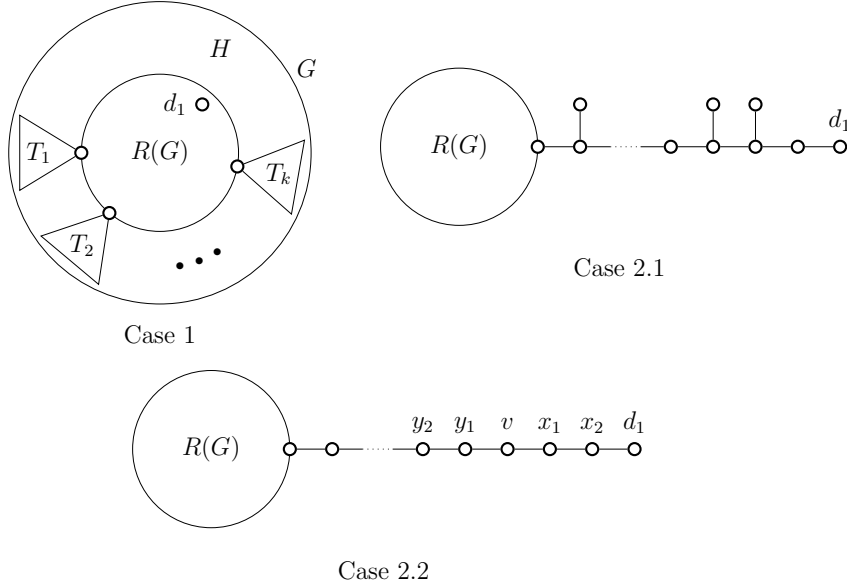*Case 2:* The first move of Dominator is on H.

Figure 3: Representations of the cases from the proof of Theorem 4.4

Let d_1 be the vertex Dominator plays in his first move, let T be the connected component of H containing d_1 (recall that T is a tree), let P be the shortest path between d_1 and $R(G)$ in T, and let M be the unique perfect matching of T (cf. Lemma 4.2(i)).

In this case, Staller first applies her strategy from Lemma 4.3 on all the other trees of H, playing the vertex adjacent to $R(G)$ as her last move on each tree. Next, Staller applies her strategy from Lemma 4.3 on the edges from M, which are not incident with P. Additionally, she plays last on the vertices closest to P. After that, only $R(G)$, P, and maybe some vertices adjacent to P, remain undominated.

  Case 2.1: At least one vertex adjacent to P is still undominated (see Fig. 3).

  Let u be an undominated vertex adjacent to P. Staller plays on its neighbor on P, forcing Dominator to reply on u. Staller does so on each such vertex. After that, the only undominated vertices lie on P, moreover, up to now, at least one move of Dominator was played on each already completely dominated edge from M.

  As long as there are some more undominated edges from M on P, at least one of them, say $e \in M$, is adjacent to a vertex s of P already played by Staller. Her strategy is to play on the vertex of e which is at distance 2 from s. Then Dominator has to reply on the other vertex of e, otherwise

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Staller wins by playing it. Hence, Staller can force Dominator to reply on all remaining edges.

Case 2.2: The only undominated vertices in H lie on P.

Staller's strategy is to play on the vertex v of P at distance 3 from d_1. Dominator has to reply on a neighbor of v, otherwise one of the neighbors of v is not dominated, say u, and Staller can win by playing u and later playing the unplayed vertex from $N[v]$ or $N[u]$. Indeed, in this case, two undominated adjacent vertices are played by Staller, and no matter where Dominator answers, she can play another consecutive vertex and win the game.

Let x_i be a vertex at distance i from v on P in the direction of d_1, and y_i be a vertex at distance i from v on P in the direction of $R(G)$ for all possible $i \geq 1$, see Fig. 3 again.

If Dominator replies on x_1, then Staller's next move is y_2. Now, Dominator has to reply on y_1, otherwise Staller wins. Then Staller repeats this strategy until P is dominated, i.e., she plays on the vertices y_{2k} in the increasing order, and Dominator is forced to reply on y_{2k-1}.

If Dominator replies on y_1, then Staller replies on x_2. After that, Dominator has to play x_1. Next, Staller applies the same strategy as before, taking y_1 as the new d_1.

In both cases, Dominator is forced to play at least one move on each edge of the matching M, hence at least $\frac{n(T)}{2}$ moves are made on T. On $H - T$, at least $\frac{n(H-T)}{2}$ moves are made by Lemma 4.3.

After T is completely dominated, Staller follows her optimal strategy on $R(G)$, but it might happen that one vertex u in $R(G)$ is already dominated (by a move of Dominator in H close to $R(G)$). As Staller's strategy on H forces Dominator to answer on H, Staller will be the first player to play on $R(G)$. But as she can imagine that Dominator's move was u, we have

$$\gamma'_{\mathrm{MB}}(R(G)|u) \geq \gamma'_{\mathrm{MB}}(R(G)|N[u]) \geq \gamma_{\mathrm{MB}}(R(G)) - 1 \,,$$

hence the total number of moves on $R(G)$ is at least $\gamma_{\mathrm{MB}}(R(G)) - 1$.

In either case, Dominator played at least $\frac{n(H)}{2} + \gamma_{\mathrm{MB}}(R(G)) - 1$ moves, which proves the lower bound. $\qquad\square$

Note that in the inequality $\gamma'_{\mathrm{MB}}(G|u) \geq \gamma_{\mathrm{MB}}(G) - 1$ from the above proof, equality can be attained. For example, consider the graph G on Fig. 4. Clearly, $\gamma_{\mathrm{MB}}(G) = 2$ and $\gamma'_{\mathrm{MB}}(G|u) = 1$.
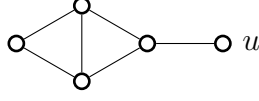
Figure 4: A graph G with the property $\gamma'_{\mathrm{MB}}(G|u) = \gamma_{\mathrm{MB}}(G) - 1$.

Appending to G some trees with perfect matchings, where at least one of them is attached to u, we get graphs that attain the lower bound from Theorem 4.4(ii).

To conclude the section we apply the residual construction to determine the Maker-Breaker domination number of trees. This contrasts the domination game where no such result is known, cf. [5, 17, 18].

Theorem 4.5 *If T a tree, then*

$$\gamma_{\mathrm{MB}}(T) = \begin{cases} \frac{n(T)}{2}; & T \text{ has a perfect matching,} \\ \frac{n(T)-1}{2}; & R(T) \cong K_1, \\ \frac{n(T)-k+1}{2}; & R(T) \cong K_{1,k} \text{ for } k \geq 3, \\ \infty; & \text{otherwise,} \end{cases}$$

and

$$\gamma'_{\mathrm{MB}}(T) = \begin{cases} \frac{n(T)}{2}; & T \text{ has a perfect matching,} \\ \infty; & \text{otherwise.} \end{cases}$$

Proof. Let T be a tree. Note first that by Lemma 4.2 the cases in Theorem 4.5 are disjoint.

If T admits a perfect matching, then $\gamma_{\mathrm{MB}}(T) \leq \frac{n(T)}{2}$ since by playing once on every edge of the matching, Dominator dominates T. By Lemma 4.3, $\gamma_{\mathrm{MB}}(T) \geq \frac{n(T)}{2}$. Hence, $\gamma_{\mathrm{MB}}(T) = \frac{n(T)}{2}$ and by the same reasoning, $\gamma'_{\mathrm{MB}}(T) = \frac{n(T)}{2}$.

If $R(T) \cong K_1$ and v is the remaining vertex, then $T-v$ admits a perfect matching by Lemma 4.3. In this case, Dominator first plays on a neighbor of v (which also belongs to one edge of the matching in $T - v$) and then follows the perfect matching in $T-v$. Thus he makes at most $\frac{n(T)-1}{2}$ moves. By Lemma 4.3, Staller can force him to play on the perfect matching except for his first move, hence $\gamma_{\mathrm{MB}}(T) = \frac{n(T)-1}{2}$.

If $R(T) \cong K_{1,k}$ for some $k \geq 3$, then $T - K_{1,k}$ admits a perfect matching. Dominator can start the game on the center vertex of the star $K_{1,k}$ and then follow the matching in the remaining part. Thus he makes at most $1 + \frac{n(T)-k-1}{2} = \frac{n(T)-k+1}{2}$ moves. On the other hand, Staller can ensure that he cannot win in less moves. If his first move is indeed in the center of a star, then Lemma 4.3 assures that at least $1 + \frac{n(T)-k-1}{2}$ Dominator's moves are needed. If his first move is elsewhere, then Staller

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can ensure that Dominator follows her moves in the perfect matchings in subtrees of $T - K_{1,k}$ while additionally she plays the last move on a vertex closest to $K_{1,k}$ in all subtrees of at least two descendants of the star (again by Lemma 4.3). With this she forces Dominator to play on the center of the star and Dominator makes at least $1 + \frac{n(T)-k-1}{2}$ moves. If Dominator played elsewhere, then Staller would win by playing all vertices in a closed neighborhood of one of those two descendants of the star. This proves that $\gamma_{\text{MB}}(T) = \frac{n(T)-k+1}{2}$.

Otherwise, we know by [12] that Staller wins on T (no matter which player starts the game), hence $\gamma_{\text{MB}}(T) = \gamma'_{\text{MB}}(T) = \infty$. $\qquad\square$

Note that by Theorem 4.5, γ_{MB} and γ'_{MB} of trees can be computed in polynomial time.

5 Cycles

The D-game domination number and the S-game domination number of cycles are given with the following formulas:

$$\gamma_g(C_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil - 1; & n \equiv 3 \mod 4, \\ \left\lceil \frac{n}{2} \right\rceil; & \text{otherwise,} \end{cases} \qquad \gamma'_g(C_n) = \begin{cases} \left\lceil \frac{n-1}{2} \right\rceil - 1; & n \equiv 2 \mod 4, \\ \left\lceil \frac{n-1}{2} \right\rceil; & \text{otherwise.} \end{cases}$$

This fundamental result was first obtained in an unpublished manuscript [20]. The result appeared for the first time in press in the paper [19], where an alternative proof is given. For the total domination game, parallel results were obtain in [10]. The latter paper investigates the total domination game on paths and cycles only. So the (total) game domination number of cycles is far from being straightforward. Here we determine the Maker-Breaker domination number of cycles, a task that turned out to be less involved.

Theorem 5.1 *If $n \geq 3$, then*

$$\gamma_{\text{MB}}(C_n) = \gamma'_{\text{MB}}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor .$$

Proof. We begin by showing that $\gamma_{\text{MB}}(C_n) \leq \left\lfloor \frac{n}{2} \right\rfloor$ and $\gamma'_{\text{MB}}(C_n) \leq \left\lfloor \frac{n}{2} \right\rfloor$. If n is even, C_n has a perfect matching (which is also a dominating matching), thus by Fact 2.2, it holds that $\gamma_{\text{MB}}(C_n) \leq \frac{n}{2} = \left\lfloor \frac{n}{2} \right\rfloor$ and $\gamma'_{\text{MB}}(C_n) \leq \left\lfloor \frac{n}{2} \right\rfloor$. Now consider the case when n is odd. In a D-game, let v be the first vertex played by Dominator. Clearly, among undominated vertices, $V(C_n) - N[v]$, there is a perfect matching. Thus $\gamma_{\text{MB}}(C_n) \leq 1 + \frac{n-3}{2} = \left\lfloor \frac{n}{2} \right\rfloor$. In an S-game, suppose $s'_1 = u$. Then Dominator

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should reply on a neighbor v of the vertex u. Now there is a perfect matching among $V(C_n) - N[v]$, so $\gamma'_{\text{MB}}(C_n) \leq 1 + \frac{n-3}{2} = \left\lfloor \frac{n}{2} \right\rfloor$. This proves the upper bounds.

To find the lower bounds we need to find an appropriate strategy for Staller. Set for the rest of the proof that $V(C_n) = \{x_1, \ldots, x_n\}$, where the adjacencies are natural.

We first show the lower bound for the S-game: $\gamma'_{\text{MB}}(C_n) \geq \left\lfloor \frac{n}{2} \right\rfloor$. Suppose, without loss of generality, that $s'_1 = x_1$. Notice that Dominator has to reply on a neighbor of x_1, for otherwise Staller plays as s'_2 the not yet dominated neighbor of x_1. Then Dominator cannot in one move dominate s'_1 and s'_2. Say he leaves s'_2 undominated. Then Staller can play the other neighbor of s'_2 and win the game as Dominator cannot dominate s'_2. So without loss of generality, Dominator replies with $d'_1 = x_n$. Staller's next move is $s'_2 = x_3$. In order to prevent Staller from winning, Dominator has to play $d'_2 = x_2$. Then Staller continues with the same strategy, forcing Dominator to play on (almost) every second move.

If n is even, the game ends after Staller plays x_{n-1} and Dominator replies on x_{n-2}. So in this case, Dominator plays all even labeled vertices, hence $\gamma'_{\text{MB}}(C_n) \geq \frac{n}{2} = \left\lfloor \frac{n}{2} \right\rfloor$. If n is odd, the game ends after Staller plays x_{n-2} and Dominator replies on x_{n-3} (as x_{n-1} is already dominated by d'_1). So Dominator again plays all even labeled vertices, thus $\gamma'_{\text{MB}}(C_n) \geq \frac{n-1}{2} = \left\lfloor \frac{n}{2} \right\rfloor$.

It remains to prove that $\gamma_{\text{MB}}(C_n) \geq \left\lfloor \frac{n}{2} \right\rfloor$. Assume without loss of generality that $d_1 = x_1$. Staller replies on $s_1 = x_4$ (so at distance 3 from d_1). Again, Dominator has to reply on a neighbor of s_1. If he replies on x_3, then Staller can apply the above strategy by playing x_6 next, and then every second vertex along the cycle to ensure at least $\left\lfloor \frac{n}{2} \right\rfloor$ moves.

If Dominator replies on $d_2 = x_5$, then Staller plays $s_2 = x_8$ (again at distance 3 from d_2 in the same direction as before). But observe that at some point of the game Dominator will have to play on $\{x_2, x_3\}$ to dominate the whole graph.

By repeating this strategy, Staller ensures that among every four consecutive vertices of the cycle, Dominator makes at least two moves (except maybe in the last one, two or three remaining vertices). We now distinguish four different cases.

- If $n \equiv 0 \bmod 4$, then no vertex remains and $\gamma_{\text{MB}}(C_n) \geq \frac{2n}{4} = \left\lfloor \frac{n}{2} \right\rfloor$.

- If $n \equiv 1 \bmod 4$, then only one vertex remains, which is already dominated by d_1, so $\gamma_{\text{MB}}(C_n) \geq \frac{2(n-1)}{4} = \left\lfloor \frac{n}{2} \right\rfloor$.

- If $n \equiv 2 \bmod 4$, then among the remaining two vertices, one is dominated by d_1 but the other is not. So Dominator has to make another move, thus $\gamma_{\text{MB}}(C_n) \geq \frac{2(n-2)}{4} + 1 = \left\lfloor \frac{n}{2} \right\rfloor$.

- If $n \equiv 3 \bmod 4$, then two of the remaining three vertices are not yet dominated,

so Dominator still has to make just one more move. So $\gamma_{\mathrm{MB}}(C_n) \geq \frac{2(n-3)}{4} = \lfloor \frac{n}{2} \rfloor$. $\qquad \square$

6 Union of graphs

The Maker-Breaker domination game was in [12] studied on disjoint unions on graphs, the obtained results were in particular applied to cographs. In this section we complement their investigation with the following result.

Theorem 6.1 *If G and H are graphs, then*

(i) $\gamma_{\mathrm{MB}}(G) + \gamma_{\mathrm{MB}}(H) \leq \gamma_{\mathrm{MB}}(G \cup H) \leq \min\{\gamma'_{\mathrm{MB}}(G) + \gamma_{\mathrm{MB}}(H), \gamma_{\mathrm{MB}}(G) + \gamma'_{\mathrm{MB}}(H)\},$

(ii) $\max\{\gamma'_{\mathrm{MB}}(G) + \gamma_{\mathrm{MB}}(H), \gamma_{\mathrm{MB}}(G) + \gamma'_{\mathrm{MB}}(H)\} \leq \gamma'_{\mathrm{MB}}(G \cup H) \leq \gamma'_{\mathrm{MB}}(G) + \gamma'_{\mathrm{MB}}(H).$

Moreover, all the bounds are sharp.

Proof. (i) It follows directly from the results from [12] that if $\gamma_{\mathrm{MB}}(G) = \infty$ or $\gamma_{\mathrm{MB}}(H) = \infty$, then $\gamma_{\mathrm{MB}}(G \cup H) = \infty$. Suppose then that both $\gamma_{\mathrm{MB}}(G)$ and $\gamma_{\mathrm{MB}}(H)$ are finite. We give a strategy for Staller such that when a D-game is played on $G \cup H$, she can ensure that Dominator will select at least $\gamma_{\mathrm{MB}}(G) + \gamma_{\mathrm{MB}}(H)$ vertices. The strategy of Staller is the following: each time Dominator selects a vertex from G or from H, she answers optimally in G or H (with an optimal strategy restricted to G or H), respectively, as long as this is possible. Suppose without loss of generality that Dominator has first dominated G. Then Staller can reply either in G, provided she has a legal move in G available, or in H. In the first case a usual D-game will be played on H. In the second case we have a game on H in which Dominator has passed one move. By the No-Skip Lemma, Dominator will need to select at least $\gamma_{\mathrm{MB}}(H)$ vertices from H. In any case, $\gamma_{\mathrm{MB}}(G \cup H) \geq \gamma_{\mathrm{MB}}(G) + \gamma_{\mathrm{MB}}(H)$.

To prove the upper bound, suppose first that Dominator starts by playing his optimal move on G and then follows Staller in G or in H whenever she plays in G or in H. In this way (having in mind the No-Skip Lemma) Dominator achieves at most $\gamma_{\mathrm{MB}}(G)$ moves in G and at most $\gamma'_{\mathrm{MB}}(H)$ moves in H, hence at most $\gamma_{\mathrm{MB}}(G) + \gamma'_{\mathrm{MB}}(H)$ vertices for $G \cup H$. If instead he starts by playing his optimal move on H, then he can guarantee to play at most $\gamma'_{\mathrm{MB}}(G) + \gamma_{\mathrm{MB}}(H)$ vertices. By choosing the smaller of the two values, Dominator has a strategy such that he dominates $G \cup H$ with no more than $\min\{\gamma'_{\mathrm{MB}}(G) + \gamma_{\mathrm{MB}}(H), \gamma_{\mathrm{MB}}(G) + \gamma'_{\mathrm{MB}}(H)\}$ moves and so $\gamma_{\mathrm{MB}}(G \cup H) \leq \min\{\gamma'_{\mathrm{MB}}(G) + \gamma_{\mathrm{MB}}(H), \gamma_{\mathrm{MB}}(G) + \gamma'_{\mathrm{MB}}(H)\}$.

(ii) This is done using similar arguments as in (i).

To demonstrate the sharpness of the bounds, consider the graphs $X_{n,m}$, $1 \leq m \leq n$, and Y_k, $k \geq 1$, as depicted in Fig. 5.
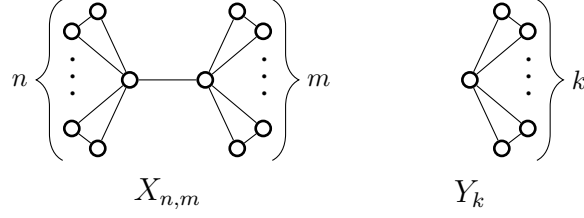
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Figure 5: Representation of graphs $X_{n,m}$ and Y_k

Observe first that $\gamma_{\mathrm{MB}}(X_{n,m}) = m + 1$, $\gamma'_{\mathrm{MB}}(X_{n,m}) = n + 1$, $\gamma_{\mathrm{MB}}(Y_k) = 1$, and $\gamma'_{\mathrm{MB}}(Y_k) = k$. Consider next the union $X_{n,m} \cup Y_k$, where $k \leq m$, and the union $Y_{k_1} \cup Y_{k_2}$, where $k_1 \leq k_2$. Then in the D-game as well in the S-game played in either of the unions, an optimal first move is to play a vertex of highest degree. Moreover, an optimal reply to this move is to play a vertex of second highest degree. Therefore,

- $\gamma_{\mathrm{MB}}(X_{n,m} \cup Y_k) = 1 + m + 1$, reaching the lower bound of (i),

- $\gamma'_{\mathrm{MB}}(X_{n,m} \cup Y_k) = n + 1 + k$, reaching the upper bound of (ii),

- $\gamma_{\mathrm{MB}}(Y_{k_1} \cup Y_{k_2}) = k_1 + 1$, reaching the upper bound of (i), and

- $\gamma'_{\mathrm{MB}}(Y_{k_1} \cup Y_{k_2}) = 1 + k_2$, reaching the lower bound of (ii). $\qquad\square$

7 Concluding remarks

To conclude the paper we list several problems and directions for further investigation of the Maker-Breaker domination number.

1. For the upper bound in (1) we have provided examples of graphs that attain equality. These examples are not connected and it is not difficult to achieve equality with connected graphs of even order. However, we do not know of any connected graph of odd order (different from K_1) for which equality in (1) is achieved. More generally, we ask for a characterization of the extremal graphs with respect to (1) and (2).

2. As we already mentioned, it would be interesting to find a structural characterization of the graphs G for which $\gamma_{\mathrm{MB}}(G) = \gamma(G) = k$ holds, where $k \geq 2$ is a fixed integer.

3. It would also be interesting to investigate $\gamma_{\mathrm{MB}}(G \,\square\, H)$ and $\gamma'_{\mathrm{MB}}(G \,\square\, H)$, where G and H are arbitrary graphs and $G \,\square\, H$ is the Cartesian product of G and H. In particular, it would be interesting to determine $\gamma_{\mathrm{MB}}(P_n \,\square\, P_m)$ (and $\gamma'_{\mathrm{MB}}(P_n \,\square\, P_m)$), as well as $\gamma_{\mathrm{MB}}(G \,\square\, K_2)$ (and $\gamma'_{\mathrm{MB}}(G \,\square\, K_2)$) for an arbitrary graph G.

4. If G is a cograph, then it is not difficult to determine whether Dominator or Staller wins the Maker-Breaker domination game [12]. On the other hand, it does not seem straightforward to determine the Maker-Breaker domination numbers of co-graphs.

5. In this paper we have considered the Maker-Breaker domination number which is an optimization problem from Dominator's point of view. It would likewise be of interest to consider the Staller's point of view, that is, assuming that Staller wins on a graph G, what is the minimum number of moves with which she can achieve the goal?

Acknowledgements

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