Maker-Breaker domination game played on strong products of graphs

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Abstract

The Maker-Breaker domination game is played on a graph G by Dominator and Staller, who alternately select unplayed vertices; Dominator aims to occupy a dominating set of G, while Staller tries to prevent this. The possible outcomes of the game played on the strong product based on the outcomes on the factors is determined. The outcomes for the strong product of odd paths, the strong product of stars, the strong product of bistars, and the strong product of stars and bistars serve this purpose. Lower and upper bounds on the Maker-Breaker domination number and the Staller-start Maker-Breaker domination number of strong products are also proved.

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1 Introduction

The Maker-Breaker game was introduced by Erdős and Selfridge back in 1973 [16]. The game is played on an arbitrary hypergraph by two players, Maker and Breaker, who alternately claim previously unselected vertices. Maker aims to select all vertices of some hyperedge, while Breaker attempts to prevent this from happening. The Maker-Breaker game has been widely studied in both, general setting and specific instance; see the book [22], the recent works [1, 14, 15, 23, 27], and the references therein.

In this paper, we focus on the Maker-Breaker domination game, introduced in 2020 by Duchêne, Gledel, Parreau, and Renault [12]. The *Maker-Breaker domination game* (briefly, the *MBD game*) is played on a graph $G = (V(G), E(G))$ by two players called Dominator and Staller. These names were chosen according to the well-studied domination game [3, 4]. As in the Maker-Breaker game, the players alternately choose previously unselected vertices of G. Dominator aims to occupy all vertices of a dominating set of G, whereas Staller aims to select at least one vertex from every dominating set of G. There are two versions of the game, depending on which player starts the game. If Dominator is the first player, the game is called the *D-game*; if Staller is the first player, it is referred to as the *S-game*.

Several graph invariants are naturally associated with the MBD game [5, 20]. The *Maker-Breaker domination number*, denoted by $\gamma_{\rm MB}(G)$, is defined as the minimum number of vertices selected by Dominator to win the D-game on G when both players play optimally. Analogously, $\gamma'_{\rm MB}(G)$ is the minimum number of vertices selected by Dominator to win the S-game on G when both players play optimally. If Dominator has no winning strategy in the D-game (S-game), then we set $\gamma_{\rm MB}(G) = \infty$ ($\gamma'_{\rm MB}(G) = \infty$). The *Staller-Maker-Breaker domination number*, denoted by $\gamma_{\rm SMB}(G)$, is the minimum number of vertices selected by Staller to win the D-game on G under optimal play. If Staller has no winning strategy, we define $\gamma_{\rm SMB}(G) = \infty$. Similarly, $\gamma'_{\rm SMB}(G)$ is defined for the S-game. Collectively, the parameters $\gamma_{\rm MB}(G)$, $\gamma'_{\rm MB}(G)$, $\gamma_{\rm SMB}(G)$, and $\gamma'_{\rm SMB}(G)$ are called the *MBD numbers* of G.

In the seminal paper [12], it was shown, among other results, that determining the winner of the MBD game can be done efficiently for trees, whereas the problem is PSPACE-complete even for bipartite and split graphs. In [13], the authors

presented a linear-time algorithm for the Maker-Maker version of the game. The paper [5] mainly investigates the parameters $\gamma_{\rm SMB}(G)$ and $\gamma'_{\rm SMB}(G)$ and, among other contributions, establishes an exact formula for $\gamma'_{\rm SMB}(G)$ when G is a path. In [6], trees T satisfying $\gamma_{\rm SMB}(T) = k$ were characterized for each positive integer k, and exact formulas for $\gamma_{\rm SMB}(G)$ and $\gamma'_{\rm SMB}(G)$ were obtained for caterpillars. Furthermore, in the main result of [18], the parameters $\gamma_{\rm MB}$ and $\gamma'_{\rm MB}$ were determined for Cartesian products of K_2 and a path. The MBD game on Cartesian products of paths, stars, and complete bipartite graphs was further studied in [10]. Criticality aspects of the MBD game were investigated in [2, 7]. The total version of the MBD game was introduced in [19] and subsequently investigated further in [17].

The strong product of graphs has been the subject of intensive research for decades; for a comprehensive overview up to 2011, see the book [21]. However, this product is still the subject of intensive research from many different angles; see, for example, [8, 11, 24, 26]. In this paper, we investigate the Maker-Breaker domination game (MBD game) played on the strong product of graphs. The main results are collected in Section 3 where we consider the outcome of the Maker-Breaker domination game played on strong products of graphs. In particular we focus on the outcome for the strong product of odd paths, the strong product of stars, the strong product of bistars, and the strong product of stars and bistars.

2 Preliminaries

Graphs considered in this paper are connected, unless stated otherwise. Let $G = (V(G), E(G))$ be a graph. The order of G is denoted by $n(G)$. For a vertex $v \in V(G)$, the *open neighborhood* of v is denoted by $N(v)$, while its *closed neighborhood* is denoted by $N[v]$. The minimum and maximum degrees of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A vertex of degree 1 is called a *leaf or a pendant vertex* . For $k \in \mathbb{N}$ we will use the notation $[k] = \{1, \ldots, k\}$.

A set $D \subseteq V(G)$ is called a *dominating set* of G if every vertex in $V(G) \setminus D$ has a neighbour in D. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set of cardinality $\gamma(G)$ is called a *γ-set* of G. A vertex of degree $n(G) - 1$ is called a *dominating vertex*. The domination number of a path is $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$. A set $S \subseteq V(G)$ is called a *packing set* if the distance between any two distinct vertices in S is at least 3. The *packing number* of G, denoted by $\rho(G)$, is the maximum cardinality of such a set. A packing set of cardinality $\rho(G)$ is called a *ρ-set* of G. For instance, the packing number of a path is $\rho(P_n) = \left\lceil \frac{n}{3} \right\rceil$. In fact, Meir and Moon [25] proved that the domination number coincides with the packing number for every tree.

The *outcome* $o(G)$ of the MBD game played on G is either $\mathcal{D}$, or $\mathcal{S}$, or $\mathcal{N}$. Here, $o(G) = \mathcal{D}$ means that Dominator has a winning strategy no matter who starts the game; $o(G) = \mathcal{S}$ means that Staller has a winning strategy no matter who starts the game; and $o(G) = \mathcal{N}$ means that the first player has a winning strategy. Theoretically there exists the fourth possible option, but it was proved in [12] that it never occurs.

The natural strategy in the general Maker-Breaker game is a pairing strategy [22]. For the Maker-Breaker domination game, the corresponding concept was introduced in [12] and is defined as follows. A subset of pairs of vertices $\{\{u_1, v_1\}, \ldots, \{u_k, v_k\}\}$ of V is a *pairing dominating set* if all the vertices are distinct and if the intersection of the closed neighborhoods of each pair covers the vertices of the graph:

$$V = \bigcup_{i=1}^{k} N[u_i] \cap N[v_i].$$

Note that the key property of a pairing dominating set is that if a player chooses a vertex in $\{u_j, v_j\}$ for each $j \in [k]$, then a dominating set of G is created.

The strong product of two graphs G and H is the graph $G \boxtimes H$ such that $V(G \boxtimes H) = V(G) \times V(H)$, and two vertices $(g, h), (g', h') \in V(G \boxtimes H)$ are adjacent if either $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$, or $gg' \in E(G)$ and $hh' \in E(H)$. A G-layer of $G \boxtimes H$ is defined as $G^h = \{(g, h) : g \in V(G)\}$ and an H-layer of $G \boxtimes H$ is defined as ${}^gH = \{(g, h) : h \in V(H)\}$.

In the rest of the preliminaries, we collect known results that we use later.

Lemma 2.1 (No-Skip Lemma for Dominator [20]) *In an optimal strategy of Dominator to achieve $\gamma_{\rm MB}(G)$ or $\gamma'_{\rm MB}(G)$ it is never an advantage for him to skip a move. Moreover, if Staller skips a move it can never disadvantage Dominator.*

As observed in [20, Theorem 3.1], No-Skip Lemma for Dominator implies that for a graph G we have

$$\gamma(G) \le \gamma_{\rm MB}(G) \le \gamma'_{\rm MB}(G). \tag{1}$$

Lemma 2.2 (No-Skip Lemma for Staller [5]) *In an optimal strategy of Staller to achieve $\gamma_{\rm SMB}(G)$ or $\gamma'_{\rm SMB}(G)$ it is never an advantage for her to skip a move. Moreover, if Dominator skips a move it can never disadvantage Staller.*

For $X \subseteq V(G)$, let $G|X$ denote the graph G in which vertices from X are considered as being already dominated.

Theorem 2.3 (Continuation Principle [20]) *Let G be a graph with $A, B \subseteq V(G)$. If $B \subseteq A$ then $\gamma_{\rm MB}(G|A) \le \gamma_{\rm MB}(G|B)$ and $\gamma'_{\rm MB}(G|A) \le \gamma'_{\rm MB}(G|B)$.*

Theorem 2.4 (Pairing Strategy [12]) *If G admits a pairing dominating set, then Dominator has a winning strategy on G in the D-game as well as in the S-game.*

Theorem 2.5 [5] *Let G be a graph.*

1. *If $\gamma'_{\rm SMB}(G) < \infty$, then $\gamma'_{\rm SMB}(G) \leq \left\lceil \frac{n(G)}{2} \right\rceil$.*
2. *If $\gamma_{\rm SMB}(G) < \infty$, then $\gamma_{\rm SMB}(G) \leq \left\lfloor \frac{n(G)}{2} \right\rfloor$.*

Moreover, both bounds are sharp.

Proposition 2.6 [5] *If G is a graph, then $\delta(G) + 1 \leq \gamma'_{\rm SMB}(G) \leq \gamma_{\rm SMB}(G)$.*

3 Main results

In this section we consider the outcome of the Maker-Breaker domination game played on strong products of graphs. We will divide our research into three subsections that reflect the possible outcomes of the factor graphs.

3.1 At least one factor has outcome $\mathcal{D}$

The main result of this subsection reads as follows.

Theorem 3.1 *If G and H are graphs, where $o(G) = \mathcal{D}$, then $o(G \boxtimes H) = \mathcal{D}$. Furthermore, the following inequalities hold:*

$$\gamma(G)\rho(H) \leq \gamma_{\rm MB}(G \boxtimes H) \leq \gamma_{\rm MB}(G) + \gamma'_{\rm MB}(G)(\gamma(H) - 1)\,,$$
$$\gamma(G)\rho(H) \leq \gamma'_{\rm MB}(G \boxtimes H) \leq \gamma'_{\rm MB}(G)\gamma(H)\,.$$

Proof. We first prove the upper bounds. For this sake, let D_H be a γ-set of H.

Consider first the S-game played on $G \boxtimes H$ and the following strategy of Dominator. He will only select vertices in layers G^h, where $h \in D_H$. In each of these layers he will follow his optimal strategy in either D-game or S-game. More precisely, if Staller is the first to select a vertex of some layer G^h, $h \in D_H$, then Dominator replies optimally with his strategy in the S-game played on G. After that, as long as Staller plays in G^h, Dominator replies accordingly, unless he has already constructed a dominating set of G^h. In that case, assuming that Staller played some vertex not in the layers G^h, where $h \in D_H$, Dominator starts or continues his strategy in some of the layers G^h, $h \in D_H$, in which the game is not over yet.

Using the above described strategy, in each of the layers G^h, $h \in D_H$, Dominator always follows Staller, or plays some consecutive moves, so that the game restricted to such a layer is a game in which Staller has skipped some moves. Hence, by the two No-Skip Lemmas and by (1), Dominator plays at most $\gamma'_{\rm MB}(G)$ vertices in each of the layers G^h, $h \in D_H$. Moreover, as soon as in each of these layers Dominator finishes the game, the facts that he has selected a dominating set in each of them and that D_H is a γ-set of H, imply that he has constructed a dominating set of $G \boxtimes H$. Consequently, $\gamma'_{\rm MB}(G \boxtimes H) \le \gamma'_{\rm MB}(G)\gamma(H)$.

Assume next that the D-game is played on $G \boxtimes H$. Then Dominator uses the same strategy as above, the only change is that he is able to emulate his optimal D-game on at least one of the layers G^h, where $h \in D_H$. In this particular layer Dominator will thus play at most $\gamma_{\rm MB}(G)$ vertices. In the remaining $\gamma(H)-1$ layers G^h, $h \in D_H$, it may happen that everywhere an S-game will be played. This then implies the first upper bound of the statement.

The above proved upper bounds also imply that $o(G \boxtimes H) = \mathcal{D}$.

It remains to prove the lower bounds. For this sake, let P_H be a ρ-set of H. If $h \in P_H$, then the vertices from the layer G^h can be dominated only from the set $V(G) \times N_H[h]$. This in turn implies that in both games, Dominator needs to play at least $\gamma(G)$ vertices from $V(G) \times N_H[h]$ to create a dominating set of $G \boxtimes H$. As this holds for every layer G^h, $h \in P_H$, we can conclude that Dominator must play at least $\rho(H)\gamma(G)$ vertices to win the game. □

We add that the lower bound of Theorem 3.3 cannot be improved from $\gamma(G)\rho(H)$ to $\gamma_{\rm MB}(G)\rho(H)$. For instance, consider $X = P_6 \boxtimes K_2$. Then we can infer that $\gamma_{\rm MB}(X) = \gamma'_{\rm MB}(X) = 2$, while $\gamma_{\rm MB}(P_6)\rho(K_2) = 3$.

To demonstrate that all the bounds of Theorem 3.1 are sharp, consider the following special case of the theorem.

Corollary 3.2 *If $o(G) = \mathcal{D}$, $\gamma(G) = \gamma'_{\rm MB}(G)$, and $\gamma(H) = \rho(H)$, then*

$$\gamma_{\rm MB}(G \boxtimes H) = \gamma'_{\rm MB}(G \boxtimes H) = \gamma(G)\gamma(H)\,.$$

Proof. Note first that the assumption $\gamma(G) = \gamma'_{\rm MB}(G)$ implies that also the equality $\gamma(G) = \gamma_{\rm MB}(G)$ holds. The assertion now follows directly from Theorem 3.1. □

As part of [20, Theorem 3.1] it was proved that for any integers r, s, t, where $2 \le r \le s \le t$, there exists a graph G such that $\gamma(G) = r$, $\gamma_{\rm MB}(G) = s$, and $\gamma'_{\rm MB}(G) = t$. Such graphs G for which also $o(G) = \mathcal{D}$ holds, together with an arbitrary tree T as the second factor, yield specific examples for Corollary 3.2. In particular, every graph G with at least two dominating vertices can be used for this purpose.

By the commutativity of the strong product, we can exchange the roles of G and H to reach the parallel conclusions for H. In particular, we directly get the following consequence.

Corollary 3.3 *If $o(G) = o(H) = \mathcal{D}$, then*

$$\begin{aligned}
&\gamma_{\rm MB}(G \boxtimes H) \geq \max\{\rho(G)\gamma(H), \rho(H)\gamma(G)\}\,,\\
&\gamma_{\rm MB}(G \boxtimes H) \leq \min\{\gamma_{\rm MB}(G) + \gamma'_{\rm MB}(G)(\gamma(H) - 1),\\
&\qquad\qquad\qquad \gamma_{\rm MB}(H) + \gamma'_{\rm MB}(H)(\gamma(G) - 1)\}\,,\\
&\gamma'_{\rm MB}(G \boxtimes H) \geq \max\{\rho(G)\gamma(H), \rho(H)\gamma(G)\}\,,\\
&\gamma'_{\rm MB}(G \boxtimes H) \leq \min\{\gamma'_{\rm MB}(G)\gamma(H), \gamma'_{\rm MB}(H)\gamma(G)\}\,.
\end{aligned}$$

3.2 Both factors have outcome $\mathcal{N}$

In this subsection we turn our attention to the case when $o(G) = o(H) = \mathcal{N}$. As we will see, in this case $o(G \boxtimes H)$ is not uniquely determined.

We first consider the strong product of two odd paths, for which we recall from [20] that $o(P_{2k+1}) = \mathcal{N}$ for every $k \geq 1$.

Theorem 3.4 *If $n, m \geq 1$, then $o(P_{2n+1} \boxtimes P_{2m+1}) = \mathcal{D}$. Moreover,*

$$\gamma_{\rm MB}(P_{2n+1} \boxtimes P_{2m+1}) \leq \gamma'_{\rm MB}(P_{2n+1} \boxtimes P_{2m+1}) \leq 2n + 1 + (m - 1)\left\lceil \frac{2n+1}{3} \right\rceil.$$

Proof. Let $n, m \geq 1$ be fixed, and set $G = P_{2n+1} \boxtimes P_{2m+1}$ for the rest of the proof. Set also $V(G) = [2n + 1] \times [2m + 1]$ with the natural adjacency in P_{2n+1} and in P_{2m+1}, and consequently in G.

Let D be a γ-set of P_{2n+1} and set

$$\begin{aligned}
M_1 &= \{\{(d, 2i - 1), (d, 2i)\} :\ d \in D, i \in [m]\},\\
M_2 &= \{\{(v, 2m), (v, 2m + 1)\} :\ v \in [2n + 1] \setminus D\}.
\end{aligned}$$

Clearly, $M_1 \cap M_2 = \emptyset$. We claim that $M_1 \cup M_2$ is a pairing dominating set. A vertex of the form $(k, 2i - 1)$ or $(k, 2i)$, where $k \in [2n + 1]$ and $i \in [m]$, is dominated by both $(d, 2i - 1)$ and $(d, 2i)$ for some $d \in D$. Since D is a γ-set of P_{2n+1}, every vertex i, $i \in D$, is adjacent to some vertex $v \in [2n + 1] \setminus D$. So any vertex of the form $(k, 2m+1)$, where $k \in [2n+1]$, is dominated by both $(v, 2m)$ and $(v, 2m+1)$ for some $v \in [2n+1] \setminus D$. This proves the claim, which implies that $o(G) = \mathcal{D}$. In view of (1) we henceforth get $\gamma_{\rm MB}(G) \leq \gamma'_{\rm MB}(G) \leq |M_1 \cup M_2| = m\lceil \frac{2n+1}{3} \rceil + (2n + 1 - \lceil \frac{2n+1}{3} \rceil) = 2n + 1 + (m - 1)\lceil \frac{2n+1}{3} \rceil$. □

Note now that $o(K_{1,n}) = \mathcal{N}$ for all $n \geq 2$. This leads to the situation in which the outcome of the product is $\mathcal{N}$ when both factors also have outcome $\mathcal{N}$.

Theorem 3.5 *If $n \geq 1$, then*

$$o(K_{1,n} \boxtimes K_{1,n}) = \begin{cases} \mathcal{D}; & n \in \{1,2,3\}\,, \\ \mathcal{N}; & n \geq 4\,. \end{cases}$$

Moreover $\gamma_{\rm MB}(K_{1,n} \boxtimes K_{1,n}) = 1$ for $n \geq 1$ and $\gamma'_{\rm SMB}(K_{1,n} \boxtimes K_{1,n}) = 5$ for $n \geq 4$.

Proof. Since $K_{1,1} \boxtimes K_{1,1} \cong K_4$, we have $o(K_4) = \mathcal{D}$, and because $K_{1,2} \cong P_3$, the conclusion for $n = 2$ follows by Theorem 3.4. Now consider the case when $n = 3$. Let $V(K_{1,3}) = \{w, u_1, u_2, u_3\}$, where w is the center of the star $K_{1,3}$. Since (w, w) is a dominating vertex of $K_{1,3} \boxtimes K_{1,3}$, we have $\gamma_{\rm MB}(K_{1,3} \boxtimes K_{1,3}) = 1$.

Consider the S-game played on $K_{1,3} \boxtimes K_{1,3}$. Staller selects the vertex (w, w) as her first move; otherwise, Dominator can win the game by selecting the vertex (w, w). Clearly $\{(w, u_1), (w, u_2), (w, u_3)\}$ is a dominating set of $K_{1,3} \boxtimes K_{1,3}$. Let Dominator select (w, u_1) as his first move. If Staller selects (w, u_2) or (w, u_3), then Dominator selects the other one. Let Dominator select the vertices (w, u_1) and (w, u_2) in his moves and Staller select the vertices (w, w) and (w, u_3) in her moves. The only remaining undominated vertices are (w, u_3) and (u_i, u_3), $i \in [3]$. Clearly, $\{(u_i, w), (u_i, u_3)\}$ for $i \in [3]$ is a pairing dominating set for the remaining undominated vertices. So, Dominator can win this game by selecting one vertex from the above pairing dominating set for each $i \in [3]$. Again, if Staller selects a vertex of the form (u_i, w), say (u_1, w), as her second move. Now, Dominator selects (w, u_2) as his second move. So Staller must select (w, u_3); otherwise, Dominator can win the game by selecting (w, u_3) as his next move. Since all neighbors of (u_1, u_3) are selected by Staller, Dominator must select (u_1, u_3) as his next move. Now, Dominator can win the game by selecting one vertex from the set $\{(u_i, u_3), (u_i, w)\}$ for $i \in \{2, 3\}$. Thus we conclude that $o(K_{1,3} \boxtimes K_{1,3}) = \mathcal{D}$.

Hence, assume in the rest that $n \geq 4$. Let $V(K_{1,n}) = \{w, u_1, u_2, \ldots, u_n\}$, where w is the center of the star $K_{1,n}$. Since (w, w) is a dominating vertex of $K_{1,n} \boxtimes K_{1,n}$, we have $\gamma_{\rm MB}(K_{1,n} \boxtimes K_{1,n}) = 1$.

Consider now the S-game on $K_{1,n} \boxtimes K_{1,n}$ and the following strategy of Staller. She first selects the vertex (w, w). By the Continuation Principle, Dominator first selects vertices of the form (w, u_i) or (u_i, w). By the symmetry and the commutativity of the strong product, we may also without loss of generality assume that the first vertex selected by Dominator is (w, u_1). Staller then selects the vertex (w, u_2) as her second move. If Dominator now selects his second move, then Staller selects an unplayed vertex (w, u_l) where all vertices of the form (u_i, u_l) for $i \in [n]$ are unplayed. Let

the vertices selected by Staller up to this stage be $\{(w,w),(w,u_2),(w,u_l)\}$. Now Dominator responds with his third optimal move. Since $n \geq 4$, Staller can also select her fourth move as a vertex (u_j,w) in which all vertices (u_j,u_k), $k \in [n]$ are unplayed. Clearly, all vertices in $N((u_j,u_2)) = \{(w,w),(u_j,w),(w,u_2)\}$ and $N((u_j,u_l)) = \{(w,w),(u_j,w),(w,u_l)\}$ are played by Staller. So Staller can win the game by selecting her fifth move as either the vertex (u_j,u_2) or the vertex (u_j,u_l) depending on the fourth move of Dominator. So Staller can win the game by selecting these five vertices. Hence $o(K_{1,n} \boxtimes K_{1,n}) = \mathcal{N}$ for $n \geq 4$ and therefore $\gamma'_{\rm SMB}(K_{1,n} \boxtimes K_{1,n}) \leq 5$.

Now we show that $\gamma'_{\rm SMB}(K_{1,n} \boxtimes K_{1,n}) \geq 5$ for $n \geq 4$. Clearly, the closed neighborhood of any vertex of the form $(w,w),(u_i,w)$ and (w,u_i) consists of at least five vertices. Suppose that Staller can win the game in four moves. This is possible when Staller selects all the vertices in $N[(u_i,u_j)] = \{(w,w),(w,u_j),(u_i,w),(u_i,u_j)\}$ for some vertex of the form (u_i,u_j). Now we show that Dominator has a strategy to prevent this from happening. Clearly, Staller first selects the vertex (w,w); otherwise, Dominator can win the game by selecting the vertex (w,w). To win within four moves, Staller's second to fourth moves must form one triple $\{(u_i,w),(w,u_j),(u_i,u_j)\}$, but these can be played in any order. If Staller plays two vertices in the triple, then Dominator selects the remaining one in the triple after Staller's third move. Therefore, Staller cannot win the game with four moves. Thus $\gamma'_{\rm SMB}(K_{1,n} \boxtimes K_{1,n}) \geq 5$ and we can conclude that $\gamma'_{\rm SMB}(K_{1,n} \boxtimes K_{1,n}) = 5$ for $n \geq 4$. □

3.3 Both factors have outcome $\mathcal{S}$

In this subsection, we turn our attention to the case when $o(G) = o(H) = \mathcal{S}$ and demonstrate that also in this case $o(G \boxtimes H)$ is not uniquely determined.

A bistar graph $B_{n,n}$, $n \geq 1$, is constructed by taking two star graphs $K_{1,n}$ and joining their central vertices with an edge. It can be easily seen that $o(B_{n,n}) = \mathcal{S}$ for $n \geq 2$.

Theorem 3.6 *If $n \geq 1$ then*

$$o(B_{n,n} \boxtimes B_{n,n}) = \begin{cases} \mathcal{D}; & n \in \{1,2,3\}\,, \\ \mathcal{S}; & n \geq 4\,. \end{cases}$$

Moreover $\gamma_{\rm SMB}(B_{n,n} \boxtimes B_{n,n}) = \gamma'_{\rm SMB}(B_{n,n} \boxtimes B_{n,n}) = 5$ for $n \geq 4$.

Proof. Let u and v be the central vertices of $B_{n,n}$ and let $u_1,\ldots,u_n$ be the pendant vertices adjacent to u and $v_1,\ldots,v_n$ be the pendant vertices adjacent to v. Clearly, $B_{1,1} \cong P_4$, hence by Theorem 3.1 we get $o(B_{1,1} \boxtimes B_{1,1}) = o(P_4 \boxtimes P_4) = \mathcal{D}$.

Consider the graph $B_{2,2} \boxtimes B_{2,2}$ and the set of pairs of vertices

$$\begin{aligned} S = \Big\{ & \{(u,u),(u,u_1)\}, \{(u_1,u),(u_1,u_2)\}, \{(u_2,u),(u_2,u_2)\}, \\ & \{(v,u),(v,u_1)\}, \{(v_1,u),(v_1,u_2)\}, \{(v_2,u),(v_2,u_2)\}, \\ & \{(u,v),(u,v_1)\}, \{(u_1,v),(u_1,v_2)\}, \{(u_2,v),(u_2,v_2)\}, \\ & \{(v,v),(v,v_1)\}, \{(v_1,v),(v_1,v_2)\}, \{(v_2,v),(v_2,v_2)\} \Big\}. \end{aligned}$$

In Fig. 1, the pairs of vertices from S are the endpoints of thickened, dashed edges. One can easily verify that this is a pairing dominating set for $B_{2,2} \boxtimes B_{2,2}$. Hence $o(B_{2,2} \boxtimes B_{2,2}) = \mathcal{D}$.

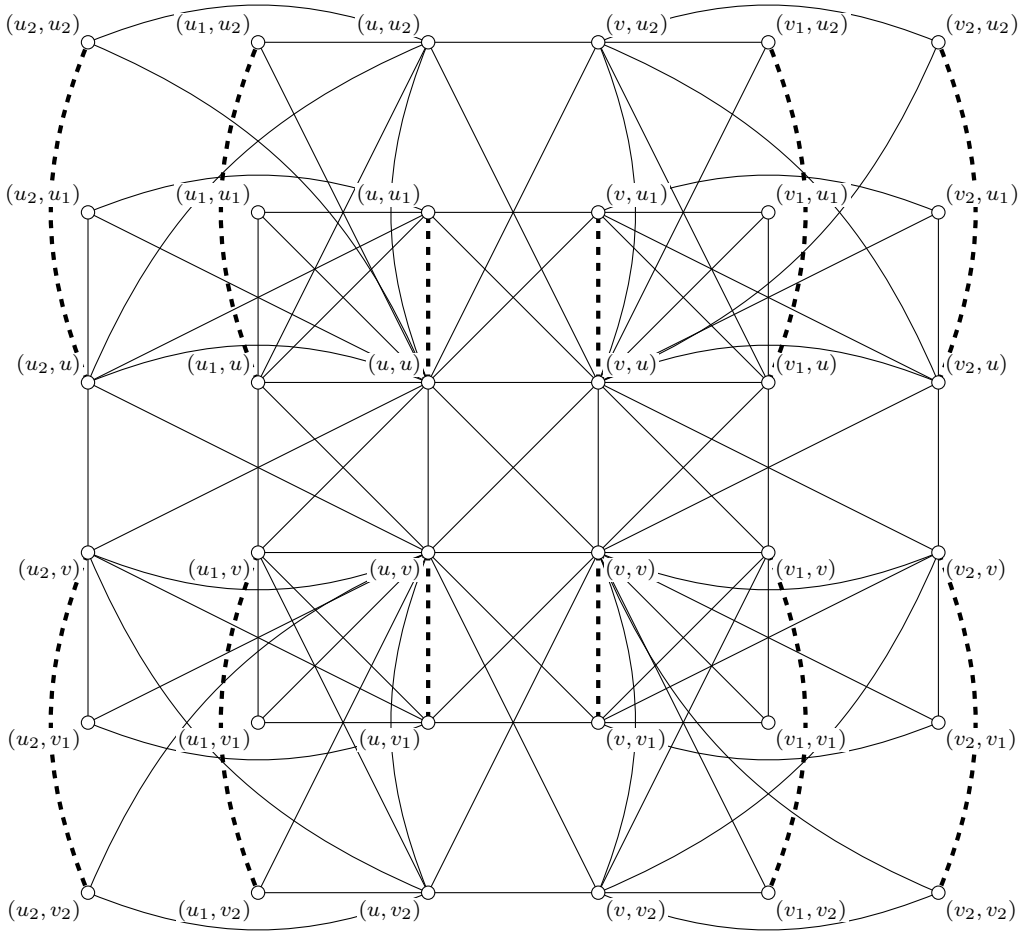


Figure 1: $B_{2,2} \boxtimes B_{2,2}$

Next, we show that $o(B_{3,3} \boxtimes B_{3,3}) = \mathcal{D}$. We can observe that $B_{3,3} \boxtimes B_{3,3}$ can be split into four disjoint copies of $K_{1,3} \boxtimes K_{1,3}$ plus some extra edges. (Use Fig. 1 to infer such a decomposition of $B_{2,2} \boxtimes B_{2,2}$ into four disjoint copies of $K_{1,2} \boxtimes K_{1,2}$ plus some extra edges.) The extra edges only help Dominator. It is known from Theorem 3.5 that $o(K_{1,3} \boxtimes K_{1,3}) = \mathcal{D}$. Hence, Dominator wins the S-game on $B_{3,3} \boxtimes B_{3,3}$ by always replying, according to his winning strategy on $K_{1,3} \boxtimes K_{1,3}$, in the copy in which Staller has just played. Thus $o(B_{3,3} \boxtimes B_{3,3}) = \mathcal{D}$.

Finally, we show that $o(B_{n,n} \boxtimes B_{n,n}) = \mathcal{S}$ for $n \geq 4$. Let $S_u = \{u, u_1, u_2, \ldots, u_n\}$ and $S_v = \{v, v_1, v_2, \ldots, v_n\}$. Consider a D-game on $B_{n,n} \boxtimes B_{n,n}$. Assume without loss of generality that Dominator first selects a vertex in $S_v \times S_v$. Then Staller selects the vertex (u, u) as her first move. Now Dominator plays his optimal response. Since $n \geq 4$, Staller can select two vertices of the form (u_i, u) and (u_j, u) where all vertices of the form (u_i, u_k) and (u_j, u_k), $k \in [n]$ are unplayed. Again, Staller can also select a vertex (u, u_l) where all vertices of the form (u_r, u_l), $r \in [n]$ are unplayed. Therefore, Staller can win the game in her fifth move by selecting either the vertex (u_i, u_l) or the vertex (u_j, u_l), whatever Dominator's strategy. Hence, Staller wins the game on $B_{n,n} \boxtimes B_{n,n}$ as a second player and $\gamma_{\mathrm{SMB}}(B_{n,n} \boxtimes B_{n,n}) \leq 5$. Therefore, she can win the game as the first player, so that $o(B_{n,n} \boxtimes B_{n,n}) = \mathcal{S}$ for $n \geq 4$.

Now we show that $\gamma'_{\mathrm{SMB}}(B_{n,n} \boxtimes B_{n,n}) \geq 5$ for $n \geq 4$. Suppose that Staller can win the game in four moves. By the Continuation Principle, we can assume that Staller first selects a vertex from $\{(u, u), (u, v), (v, u), (v, v)\}$, let it be (u, u). To win in four moves, Staller's second to fourth moves must come from a triple $\{(u_i, u), (u, u_j), (u_i, u_j)\}$, but these can be played in any order. If Staller plays two vertices in the triple, then Dominator selects the remaining one in the triple after Staller's third move. Therefore, Staller cannot win the game with four moves. Thus $\gamma'_{\mathrm{SMB}}(B_{n,n} \boxtimes B_{n,n}) \geq 5$ and get $5 \leq \gamma'_{\mathrm{SMB}}(B_{n,n} \boxtimes B_{n,n}) \leq \gamma_{\mathrm{SMB}}(B_{n,n} \boxtimes B_{n,n}) \leq 5$ for $n \geq 4$. Hence $\gamma'_{\mathrm{SMB}}(B_{n,n} \boxtimes B_{n,n}) = \gamma_{\mathrm{SMB}}(B_{n,n} \boxtimes B_{n,n}) = 5$. □

Theorem 3.7 *If $n \geq 1$ then*

$$o(K_{1,n} \boxtimes B_{n,n}) = \begin{cases} \mathcal{D}; & n \in \{1, 2, 3\}, \\ \mathcal{S}; & n \geq 4. \end{cases}$$

Moreover $\gamma_{\mathrm{SMB}}(K_{1,n} \boxtimes B_{n,n}) = \gamma'_{\mathrm{SMB}}(K_{1,n} \boxtimes B_{n,n}) = 5$ for $n \geq 4$.

Proof. Let u be the central vertex, and let $u_1, u_2, \ldots, u_n$ be the pendant vertices of $K_{1,n}$. Let v and w be the central vertices of $B_{n,n}$. The pendant vertices adjacent to v in $B_{n,n}$ are $v_1, v_2, \ldots, v_n$ and the pendant vertices adjacent to w in $B_{n,n}$

are $w_1, w_2, \ldots, w_n$. Since $B_{1,1} \cong P_4$, we have $K_{1,1} \boxtimes B_{1,1} \cong P_2 \boxtimes P_4$. Hence, by Theorem 3.1 we get $o(K_{1,1} \boxtimes B_{1,1}) = \mathcal{D}$.

Now we show that $o(K_{1,2} \boxtimes B_{2,2}) = \mathcal{D}$. Consider the set of pairs of vertices

$$P = \Big\{ \{(u,v),(u,v_1)\}, \{(u,w),(u,w_1)\}, \{(u_1,v),(u_1,v_2)\}, \\ \{(u_1,w),(u_1,w_2)\}, \{(u_2,v),(u_2,v_2)\}, \{(u_2,w),(u_2,w_2)\} \Big\}.$$

In Fig. 2, the pairs of vertices from P are the endpoints of thickened, dashed edges. Then we observe that P forms a pairing dominating set of $K_{1,2} \boxtimes B_{2,2}$. Hence $o(K_{1,2} \boxtimes B_{2,2}) = \mathcal{D}$.

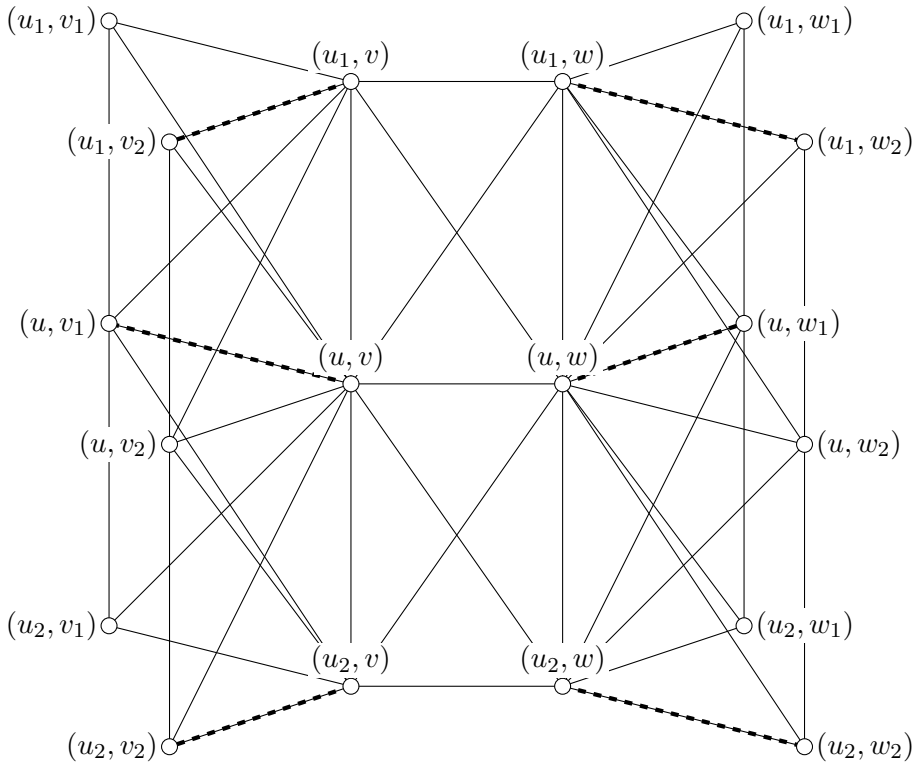


Figure 2: $K_{1,2} \boxtimes B_{2,2}$

Next we show that $o(K_{1,3} \boxtimes B_{3,3}) = \mathcal{D}$. Let $S_u = \{u, u_1, u_2, u_3\}$ be the vertex set of $K_{1,3}$. Let $S_v = \{v, v_1, v_2, v_3\}$ and $S_w = \{w, w_1, w_2, w_3\}$ be the vertices of $B_{3,3}$. Then, $S_u \times S_v$ and $S_u \times S_w$ induce a split of the graph $K_{1,3} \boxtimes B_{3,3}$ into two copies of $K_{1,3} \boxtimes K_{1,3}$ plus some extra edges. The extra edges only help Dominator. By Theorem 3.5 we know that $o(K_{1,3} \boxtimes K_{1,3}) = \mathcal{D}$. Hence, Dominator wins the S-game

on $K_{1,3} \boxtimes B_{3,3}$ by always replying, according to his winning strategy on $K_{1,3} \boxtimes K_{1,3}$, in the copy in which Staller has just played. Thus $o(K_{1,3} \boxtimes B_{3,3}) = \mathcal{D}$.

We next claim that $o(K_{1,n} \boxtimes B_{n,n}) = \mathcal{S}$ for $n \geq 4$. Consider the D-game on $K_{1,n} \boxtimes B_{n,n}$, and assume without loss of generality that Dominator first selects a vertex in $S_u \times S_w$. Then Staller selects the vertex (u, v) as her next move. Now Dominator plays his optimal response. Since $n \geq 4$, Staller can select two vertices of the form (u_i, v) and (u_j, v), where all vertices of the form (u_i, v_k) and (u_j, v_k), $k \in [n]$, are unplayed. Again, Staller can also select a vertex (u, v_l) where all vertices of the form (u_r, v_l), $r \in [n]$ are unplayed. Therefore, Staller can win the game in her fifth move by selecting either the vertex (u_i, v_l) or the vertex (u_j, v_l), whatever Dominator's strategy. Hence, Staller wins the game on $K_{1,n} \boxtimes B_{n,n}$ as a second player and $\gamma_{\mathrm{SMB}}(K_{1,n} \boxtimes B_{n,n}) \leq 5$. Therefore, she can win the game as the first player, that is $o(K_{1,n} \boxtimes B_{n,n}) = \mathcal{S}$ for $n \geq 4$.

Now we show that $\gamma'_{\mathrm{SMB}}(K_{1,n} \boxtimes B_{n,n}) \geq 5$ for $n \geq 4$. Suppose that Staller can win the game in four moves. By the Continuation Principle, we can assume that Staller first selects a vertex from $\{(u, v), (u, w)\}$, let it be (u, v). To win in four moves, Staller's second to fourth moves must come from a triple $\{(u_i, v), (u, v_j), (u_i, v_j)\}$, but these can be played in any order. If Staller plays two vertices in the triple, then Dominator selects the remaining one in the triple after Staller's third move. Therefore, Staller cannot win the game with four moves. Thus $\gamma'_{\mathrm{SMB}}(K_{1,n} \boxtimes B_{n,n}) \geq 5$ and get $5 \leq \gamma'_{\mathrm{SMB}}(K_{1,n} \boxtimes B_{n,n}) \leq \gamma_{\mathrm{SMB}}(K_{1,n} \boxtimes B_{n,n}) \leq 5$ for $n \geq 4$. Hence $\gamma'_{\mathrm{SMB}}(K_{1,n} \boxtimes B_{n,n}) = \gamma_{\mathrm{SMB}}(K_{1,n} \boxtimes B_{n,n}) = 5$.

□

4 Concluding remarks

In this paper, we studied the Maker-Breaker domination game on the strong product of graphs. If G and H are connected graphs, then by Theorem 3.1, $o(G \boxtimes H) = \mathcal{D}$ whenever $o(G) = \mathcal{D}$ or $o(H) = \mathcal{D}$. On the other hand, when neither factor has outcome $\mathcal{D}$, the outcome of the game on $G \boxtimes H$ is no longer determined by $o(G)$ and $o(H)$ alone. Indeed, each of the remaining cells admits more than one possibility. We summarise the situation in Table 1, where a bold entry is the only possible outcome; the other entries list the outcomes realised here.

In Table 1, in cases when neither factor has outcome $\mathcal{D}$, it remains unclear whether a third outcome could occur. We summarise this in the following two problems.

Open Problem 4.1 *Are there connected graphs G and H with $o(G) = o(H) = \mathcal{N}$ for which $o(G \boxtimes H) = \mathcal{S}$?*

G \ H	$\mathcal{D}$	$\mathcal{N}$	$\mathcal{S}$
$\mathcal{D}$	$\boldsymbol{\mathcal{D}}$	$\boldsymbol{\mathcal{D}}$	$\boldsymbol{\mathcal{D}}$
$\mathcal{N}$	$\boldsymbol{\mathcal{D}}$	$\mathcal{D}/\mathcal{N}$	$\mathcal{D}/\mathcal{S}$
$\mathcal{S}$	$\boldsymbol{\mathcal{D}}$	$\mathcal{D}/\mathcal{S}$	$\mathcal{D}/\mathcal{S}$

Table 1: Outcome of $G \boxtimes H$ in terms of $o(G)$ and $o(H)$.

Open Problem 4.2 *Are there connected graphs G and H, where $o(G) = \mathcal{S}$, such that $o(G \boxtimes H) = \mathcal{N}$?*

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Declaration of interests

The authors declare that they have no conflict of interest.

Data availability

Our manuscript has no associated data.

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