

# Median graphs: characterizations, location theory and related structures

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## Abstract

Median graphs are surveyed from the point of view of their characterizations, their role in location theory and their connections with median structures. The median structures we present include ternary algebras, betweenness, interval structures, semilattices, hypergraphs, join geometries and conflict models. In addition, some new characterizations of median graphs as meshed graphs are presented and a new characterization in terms of location theory is given.

## Contents

1. Introduction
2. New characterizations of median graphs
3. A review of median graphs characterizations
4. Median graphs and location theory
5. Related structures
6. Concluding remarks

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# 1 Introduction

Median graphs form by now a well-studied class of graphs. They can be viewed as the natural common generalization of trees and hypercubes. Grid graphs and the covering graphs of distributive lattices are median graphs as well. Characterizations abound, and can be found dispersed in the literature. Some of these have been rediscovered either exactly in the same form, or in a slight disguise. Here we present (almost) all characterizations from the literature in a unified and structured way as well as some new ones.

Median graphs have various origins: they were obtained from algebraic structures as well as hypergraphs. We collect the known structures related to median graphs from diverse fields of (discrete) mathematics.

An important application of median graphs is within location theory. From this point of view there are also several characterizations.

The paper is organized as follows. First we give the notions and notations used throughout the paper. In Section 2 we present new characterizations of median graphs as meshed graphs. We use these to group together the known characterizations in Section 3 in a more or less systematic way. Some short proofs are also included. In Section 4 we discuss the role of median graphs in location theory by collecting characterizations involving various locational notions. We include also a new characterization here. In Section 5, we review structures that are equivalent to median graphs, e.g. from the areas of ternary algebras, of semilattices, of set functions, of hypergraphs, of convexities, of geometries, and of conflict models. Relevant references to the literature are given throughout the paper. This overview of characterizing properties of median graphs can also serve to gain insight in the rich structure theory developed by now on median graphs.

All graphs considered in this paper are finite, connected, undirected graphs, without loops or multiple edges. We wish to point out that most of the characterizations we are going to present for median graphs extend to infinite graphs as well.

The **distance**  $d_G(u, v)$  between vertices  $u$  and  $v$  of a graph  $G$  will be the usual shortest path distance. Whenever the graph  $G$  will be clear from the context, we will shortly write  $d(u, v)$ . A shortest path between two vertices is also called a **geodesic**. A subgraph  $H$  of a graph  $G$  is an **isometric** subgraph if  $d_H(u, v) = d_G(u, v)$ , for all  $u, v \in V(H)$ . The distance  $d(u, H)$  between a vertex  $u$  of a graph  $G$  and a subgraph  $H$  of  $G$  is the minimum distance  $d(u, x)$  over all vertices  $x$  of  $H$ . For an edge  $ab$  of a graph  $G$ , let

$$\begin{aligned} W_{ab} &:= \{u \in V(G) \mid d(a, u) < d(b, u)\}, \\ U_{ab} &:= \{u \in W_{ab} \mid u \text{ is adjacent to a vertex in } W_{ba}\}, \end{aligned}$$

i.e.,  $W_{ab}$  consists of vertices of  $G$  closer to  $a$  than to  $b$  and  $U_{ab}$  are those

vertices from  $W_{ab}$  which have a neighbor in  $W_{ba}$ . Note that  $V = W_{ab} \cup W_{ba}$  for bipartite graphs.

The **interval**  $I(u, v)$  between vertices  $u$  and  $v$  consists of all vertices on shortest paths between  $u$  and  $v$ . A subset  $W$  of vertices of  $G$  is **convex** if  $I(u, v) \subseteq W$  for any  $u, v \in W$ . Observe that the intersection of two convex sets is again convex. A **convex subgraph** is a subgraph induced by a convex set. In the sequel we will not distinguish between a subset  $W$  of vertices in a graph  $G$  and the subgraph of  $G$  induced by  $W$ . A subgraph  $H$  of  $G$  is **2-convex**, if for any  $u, v \in V(H)$  with  $d(u, v) = 2$  all common neighbors of  $u$  and  $v$  belong to  $H$ .

A family  $\mathcal{F}$  of subsets of a set  $X$  has the **Helly property** if any finite family of pairwise non-disjoint sets of  $\mathcal{F}$  has a non-empty intersection. The family  $\mathcal{F}$  has the **separation property**  $S_2$  if any two disjoint points are in complementary sets from  $\mathcal{F}$ .

A **median** of three vertices  $u, v$  and  $w$  is a vertex that lies in  $I(u, v) \cap I(u, w) \cap I(v, w)$ . A connected graph  $G$  is a **median graph** if every triple of its vertices has a unique median. For an example of a median graph see Fig. 1, where also the sets  $W_{ab}$  and  $W_{ba}$  corresponding to an edge  $ab$  are shown. An important feature is that for any edge between  $W_{ab}$  and  $W_{ba}$ , say  $uv$  with  $u$  in  $W_{ab}$  and  $v$  in  $W_{ba}$ , we have that  $W_{ab} = W_{uv}$  and  $W_{ba} = W_{vu}$ .

A graph is called **meshed** if it satisfies the **quadrangle property**: for any vertices  $u, v, w$  and  $z$  with  $d(u, w) = d(v, w) = k = d(z, w) - 1$  and  $d(u, v) = 2$  with  $z$  a common neighbour of  $u$  and  $v$ , there exists a common neighbour  $x$  of  $u$  and  $v$  with  $d(x, w) = k - 1$ .

Finally, the vertices of the  $n$ -dimensional **hypercube**  $Q_n$  are all words from  $\{0, 1\}^n$  and two vertices are adjacent if they differ in precisely one coordinate.

## 2 New characterizations of median graphs

In this section we present different characterizations of median graphs. All will be given by the use of meshed graphs, i.e. the graphs which fulfil the quadrangle property. The characterizations will thus be given in the form: quadrangle property and an additional condition. The additional conditions will be of several different types and we use them in the next section to collect and group the new and known characterizations of median graphs.

Let  $G$  be a bipartite graph, and let  $u, v$  and  $w$  be vertices of  $G$ . Consider any path  $P : v = v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k = w$  from  $v$  to  $w$ . When we walk along  $P$  from  $v$  to  $w$ , then, because  $G$  is bipartite, in each step we either go one step nearer to  $u$  or one step away from  $u$ . Therefore, we say that the

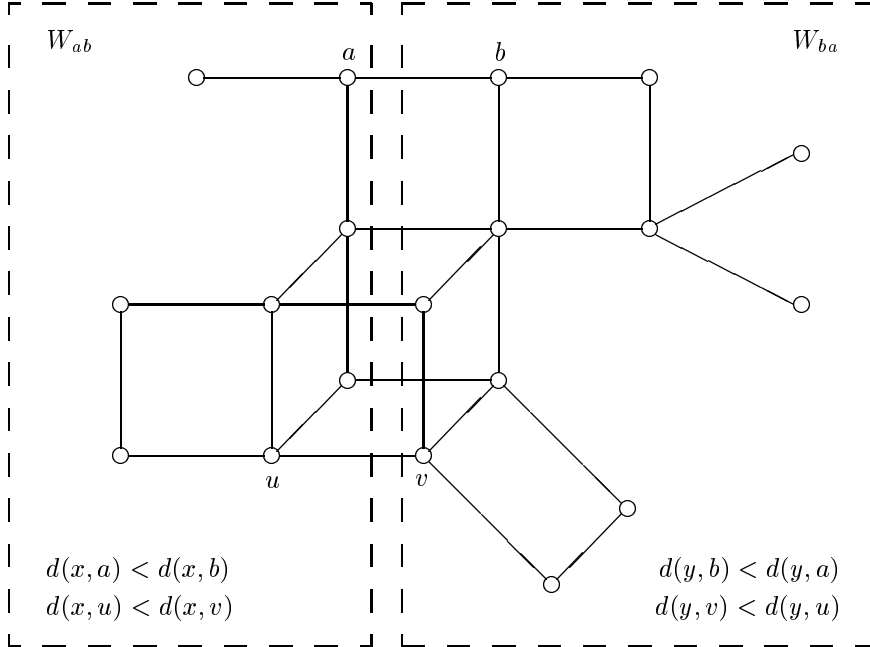


Figure 1: A median graph and the sets  $W_{ab}$  and  $W_{ba}$

edge  $v_i \rightarrow v_{i+1}$  on  $P$  is **downward** with respect to  $u$  if  $d(u, v_i) > d(u, v_{i+1})$ , and **upward** otherwise. We say that  $P$  is **down-up** with respect to  $u$  if there exists  $j$  with  $1 \leq j \leq k$  such that all edges on  $P$  from  $v$  to  $v_j$  are downward and all edges on  $P$  from  $v_j$  to  $w$  are upward.

The following lemma was already observed in the submitted version of Bandelt and Mulder [10], but to reduce the length of the paper it was left out from the published version.

**Lemma 1** *Let  $G$  be a connected, bipartite, meshed graph. Then, for any three vertices  $u$ ,  $v$  and  $w$  of  $G$ , there exists a down-up geodesic from  $v$  to  $w$  with respect to  $u$ .*

**Proof.** Let  $P : v = v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k = w$  be a geodesic from  $v$  to  $w$  such that the distance from  $u$  to  $P$ , viz.  $\sum_{i=1}^k d(u, v_i)$ , is as small as possible. Assume that there is an upward edge  $v_{i-1} \rightarrow v_i$  followed by a downward edge  $v_i \rightarrow v_{i+1}$ . Then we have

$$d(u, v_{i-1}) = d(u, v_{i+1}) = d(u, v_i) - 1.$$

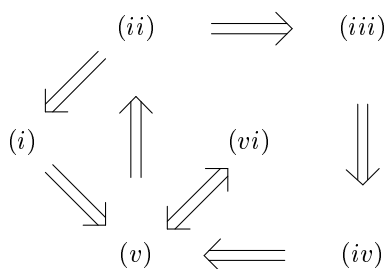
By the quadrangle property, we find a common neighbor  $v'_i$  of  $v_{i-1}$  and  $v_{i+1}$  with  $d(u, v'_i) = d(u, v_i) - 2$ . If we replace  $v_i$  by  $v'_i$  in  $P$ , then we obtain a geodesic from  $v$  to  $w$  with smaller distance to  $u$  than  $P$ . This contradicts the minimality of  $P$ .  $\square$

Note that a down-up geodesic  $P$  from  $v$  to  $w$  with respect to  $u$  is contained in  $I(u, v) \cup I(u, w)$ . The unique vertex on  $P$  nearest to  $u$  is the unique vertex of  $P$  in  $I(u, v) \cap I(u, w)$ . We call this vertex the **bottom vertex** of  $P$  with respect to  $u$ .

**Theorem 2** *Let  $G$  be a connected, bipartite, meshed graph. Then the following conditions are equivalent.*

- (i)  $G$  is an induced subgraph of a hypercube.
- (ii)  $G$  is an isometric subgraph of a hypercube.
- (iii) Every interval of  $G$  is convex.
- (iv) Every interval of  $G$  is 2-convex.
- (v)  $G$  contains no  $K_{2,3}$  as a subgraph.
- (vi) The convex sets of  $G$  have the separation property  $S_2$ .

**Proof.** The proof is organized as follows:



Note first that the implications  $(ii) \Rightarrow (i) \Rightarrow (v)$  as well as  $(iii) \Rightarrow (iv) \Rightarrow (v)$  are trivial.

To see that  $(vi)$  implies  $(v)$  suppose that  $G$  contains a  $K_{2,3}$  as a subgraph. If  $uv$  is any edge of  $K_{2,3}$  then it is easy to see that  $u$  and  $v$  cannot be separated by complementary convex sets.

$(ii) \Rightarrow (iii)$ . Let  $G$  be an isometric subgraph of a hypercube  $Q_n$  and let  $I(u, v)$  be an interval of  $G$ . Recall that the vertices of  $Q_n$  are the words from  $\{0, 1\}^n$ . Without loss of generality we may assume that  $u = 0 \dots 0b_i \dots b_n$  and  $v = 1 \dots 1b_i \dots b_n$ . As  $G$  is an isometric subgraph of  $Q_n$ , the vertices of  $I(u, v)$  are labeled  $c_1 \dots c_{i-1}b_i \dots b_n$  where  $c_j \in \{0, 1\}$ . Moreover, a vertex on a shortest path between two vertices from  $I(u, v)$  must be labeled likewise, hence  $I(u, v)$  is convex.

(v)  $\Rightarrow$  (ii). By a well-known theorem of Djoković [23], it is enough to prove that, for all edges  $ab$  of  $G$ , the sets  $W_{ab}$  and  $W_{ba}$  are convex.

Assume the contrary, and let  $x, y$  be vertices in  $W_{ab}$  such that  $I(x, y) \cap W_{ba} \neq \emptyset$  with  $d(x, y)$  as small as possible. Without loss of generality we may take  $d(x, b) \geq d(y, b)$ . Because of minimality of  $d(x, y)$ , there exists a geodesic  $P : x \rightarrow x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_k \rightarrow y$  of length  $k + 1$  with all its internal vertices in  $W_{ba}$ . We consider two cases.

Case 1.  $k = 1$ .

Then we have  $d(a, y) = d(b, x_1)$  and therefore

$$d(a, x) = d(a, y) = d(b, x_1) = d(a, x_1) - 1 = d(b, x) - 1.$$

By the quadrangle property, there is a common neighbor  $z$  of  $x$  and  $y$  in  $I(a, x) \cap I(a, y) \subseteq W_{ab}$  with  $d(a, z) = d(a, x) - 1 = d(b, x_1) - 1$ . Again, by the quadrangle property, there is a common neighbor  $w$  of  $z$  and  $x_1$  in  $I(b, z) \cap I(b, x_1) \subseteq W_{ba}$ . But now the vertices  $z, x_1, w, x$  and  $y$  induce a  $K_{2,3}$  in  $G$  which is forbidden. This settles Case 1.

Case 2.  $k > 1$ .

Using Lemma 1, we may choose  $P$  to be such that the geodesic  $x_1 \rightarrow \dots \rightarrow x_k$  is down-up with respect to  $b$  (for such a geodesic lies in  $I(b, x_1) \cup I(b, x_k) \subseteq W_{ba}$ ). Hence also  $P$  itself is down-up with respect to  $b$ . Let  $x_j$  be the bottom vertex of  $P$ . Since  $d(x, b) \geq d(y, b)$ , we have  $j > 1$ . It follows that  $d(b, x_2) = d(b, x_1) - 1 = d(b, x) - 2$ , so that

$$d(a, x_2) = d(a, x_1) - 1 = d(a, x).$$

By the quadrangle property, we find a common neighbor  $x'_2$  of  $x$  and  $x_2$  in  $I(a, x) \cap I(a, x_2) \subseteq W_{ab}$  with  $d(a, x'_2) = d(a, x_2) - 1$ . Proceeding in the same way, we find a path  $x \rightarrow x'_2 \rightarrow x'_3 \rightarrow \dots \rightarrow x'_j$  in  $W_{ab}$  such that  $x'_i$  is adjacent to  $x_i$  and  $d(a, x'_i) = d(a, x_i) - 1$ , for  $2 \leq i \leq j$ . If  $j < k$ , then, using the quadrangle property, we also find a path  $y \rightarrow x''_{k-1} \rightarrow \dots \rightarrow x''_j$  in  $W_{ab}$  such that  $x''_i$  is adjacent to  $x_i$  and  $d(a, x''_i) = d(a, x_i) - 1$ , for  $j \leq i \leq k - 1$ . By Case 1, we have  $x'_j = x''_j$ . But now we have constructed a path  $P' : x \rightarrow x'_2 \rightarrow \dots \rightarrow x'_j = x''_j \rightarrow \dots \rightarrow x''_{k-1} \rightarrow y$  from  $x$  to  $y$  of length  $k - 1$  contradicting the fact that  $d(x, y) = k + 1$ . This settles Case 2.

It remains to prove that (v) implies (vi). Let  $u$  and  $v$  be any vertices of  $G$  and let  $P$  be a geodesic between  $u$  and  $v$ . Let  $w$  be the neighbor of  $u$  on  $P$ . Then clearly  $u \in W_{uw}$  and  $v \in W_{vu}$ . Moreover, as we have already proved that (v) implies (ii), we have, using the theorem of Djoković [23] again, that  $W_{uw}$  and  $W_{vu}$  are convex.  $\square$

Bandelt [3] proved that a connected graph is a median graph if and only if it is bipartite, fulfils the quadrangle property and is  $K_{2,3}$ -free. We note

that this characterization also follows immediately from a more general result due to Bandelt, Mulder and Wilkeit [12], which characterizes quasi-median graphs as graphs fulfilling the quadrangle and the so-called triangle property and having no induced  $K_{2,3}$  or  $K_4 - e$  (i.e.  $K_4$  minus an edge). As the triangle property and having no induced  $K_4 - e$  are trivial in the bipartite case, and median graphs are just bipartite quasi-median graphs, the result follows.

Hence Theorem 2 in fact gives characterizations of median graphs.

### 3 A review of median graphs characterizations

In this section we review known characterizations of median graphs together with the characterizations given in Theorem 2. Some short proofs are also included. We have collected the characterizations into the following five types:

- characterizations close to the definition of median graphs,
- characterizations in terms of hypercube subgraphs,
- characterizations involving convex subgraphs,
- characterizations with conditions on intervals,
- characterizations with separation properties.

In some cases, for instance in the case of convex intervals, a characterization could be placed into two different types. We have, however, presented each characterization only once.

We begin with characterizations of median graphs which are closely related to the definition. A **Steiner point** of a set of three vertices  $u$ ,  $v$  and  $w$  of a graph is a vertex  $x$  that minimizes the sum  $d(x, u) + d(x, v) + d(x, w)$ .

**Theorem 3** *For a connected graph  $G$ , the following conditions are equivalent.*

- (i)  $G$  is a median graph.
- (ii) Every triple of  $G$  has a unique Steiner point.
- (iii)  $G$  is triangle-free and any triple  $u, v, w$  of  $G$  with  $d(u, v) = 2$  has a unique median.
- (iv) Every triple of vertices of  $G$  has a median and  $G$  has no  $K_{2,3}$  as a subgraph.

Theorem 3 (ii) is due to Avann [2] and is the first characterization of median graphs. We refer also to Chung, Graham and Saks [20]. That the

uniqueness of medians can be replaced by forbidding  $K_{2,3}$  was implicitly proved by Mulder [38] and explicitly by Bandelt [3], see also Bandelt [6]. In addition, it is an immediate corollary of Theorem 1 from Bandelt and Mulder [10]. The minimality of conditions posed on medians of triples of vertices from Theorem 3 (iii) is due to Mulder [38].

We next collect those characterizations of median graphs, where median graphs are considered as subgraphs of hypercubes. Some definitions are needed first. A subgraph  $H$  of a graph  $G$  is **median closed** if, with any triple of vertices of  $H$ , their median is also in  $H$ . A (necessarily induced) subgraph  $H$  of a graph  $G$  is a **retract** of  $G$ , if there is a map  $r$  from  $V(G)$  to  $V(H)$  that maps each edge of  $G$  either to an edge of  $H$  or to a vertex of  $H$ , and fixes  $H$ , i.e.,  $r(v) = v$  for every  $v \in V(H)$ . If we allow that  $r$  maps edges to edges only, we call  $H$  a **strong retract** of  $G$ .

**Theorem 4** *For a connected graph  $G$ , the following conditions are equivalent.*

- (i)  $G$  is a median graph.
- (ii)  $G$  is a meshed, induced subgraph of a hypercube.
- (iii)  $G$  is a meshed, isometric subgraph of a hypercube.
- (iv)  $G$  is a median closed, induced subgraph of a hypercube.
- (v)  $G$  is a median closed, isometric subgraph of a hypercube.
- (vi)  $G$  is a retract of a hypercube.

Characterizations (ii) and (iii) are given in Theorem 2 (i) and (ii), respectively. Characterizations (iv) and (v) appear most frequently in the literature among all characterizations of median graphs. They are due to Mulder [36, 37, 38], for alternative proofs we refer also to Mulder and Schrijver [42] and Chung, Graham and Saks [20].

Theorem 4 (vi) is due to Bandelt [4]. Moreover, he showed that both retract and strong retract can be used in the statement of the theorem. A proof using the convex expansion theorem (which we present in the next theorem) is given by Mulder in [40]. The results can also be deduced from a generalization to quasi-median graphs which was obtained independently by Chung, Graham and Saks [21] and Wilkeit [54]. We wish to add that an important step to Bandelt's theorem was the following theorem due to Duffus and Rival [24]: *a graph  $G$  is the covering graph of a distributive lattice of length  $n$  if and only if  $G$  is a retract of  $Q_n$  and  $G$  is of diameter  $n$ .*

We next summarize characterizations involving convex subgraphs. In addition, two relevant characterizations will be included later in Theorem 6. A subgraph  $H$  of a graph  $G$  is called **gated** in  $G$ , if, for every  $v \in V(G)$ , there exists a vertex  $x \in V(H)$  such that, for every  $u \in V(H)$ ,  $x$  lies on a shortest path from  $v$  to  $u$ . If such a vertex exists, it must be unique. We



call it the **gate** of  $v$  in  $H$ . We call a subgraph  $H$  of a graph  $G$  **weakly gated** in  $G$ , if, for every  $v \in V(G)$ , there exists a unique vertex  $x \in V(H)$ , which minimizes the distance between  $v$  and the vertices of  $H$ .

**Theorem 5** *For a connected graph  $G$ , the following conditions are equivalent.*

- (i)  $G$  is a median graph.
- (ii)  $I(u, v) \cap I(v, w) = \{v\}$  implies  $d(u, w) = d(u, v) + d(v, w)$ , for any vertices  $u, v$  and  $w$ , and intervals of  $G$  are convex.
- (iii)  $G$  is a bipartite and for every edge  $ab$  of  $G$ , the sets  $U_{ab}$  and  $U_{ba}$  are convex.
- (iv) The convex closure of any isometric cycle in  $G$  is a hypercube.
- (v)  $G$  can be obtained from the one vertex graph by the convex expansion procedure.
- (vi)  $G$  can be obtained from hypercubes by a sequence of convex amalgams.
- (vii) Every convex set of  $G$  is weakly gated and  $G$  contains no  $K_{2,3}$  as a subgraph.

Theorem 5 (ii) is due to Mulder [38] while (iii) and (iv) are from Bandelt [3]. For (iii) we refer also to Bandelt, Mulder and Wilkeit [12] where a generalization to quasi-median graphs is presented.

The **convex expansion theorem**, i.e. Theorem 5 (v), gives the most structural insight into median graphs among all the characterizations of these graphs. The theorem was proved by Mulder [36, 38], we also refer to Mulder [40]. Roughly speaking, the convex expansion of a graph is obtained from the graph by selecting two convex sets with nonempty intersection and without edges between vertices not in the intersection and expanding the intersection by blowing each vertex to an edge. For instance, the median graph of Fig. 2 is expanded to that of Fig. 1. We have already mentioned that Theorem 4 (vi) can also be proved using the convex expansion theorem. In addition, several other characterizations follow from the convex expansion theorem by induction.

Two technical variations of the convex expansion procedure were given by Jha and Slutzky [30], and by Hagauer, Imrich and Klavžar [26], respectively, to give fast recognition algorithms for median graphs. For more information on recognition algorithms we refer to Imrich and Klavžar [27].

Theorem 5 (vi) was proved by Bandelt and van de Vel [13]. It is an immediate corollary of (v), see Mulder [39]. Roughly speaking, a convex amalgamation consists of gluing together two graphs along common convex subgraphs.

The last characterization is from Berrachedi and Mollard [17]. We give an outline of its proof. If  $G$  is a median graph then by Theorem 2 (v) it

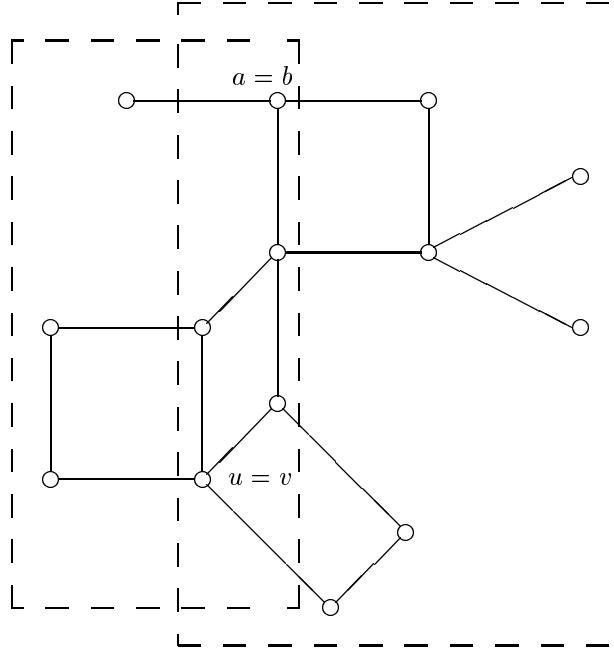


Figure 2: A contraction of the graph from Fig. 1

contains no  $K_{2,3}$ . In addition, using Lemma 1, it follows easily that convex sets of  $G$  are weakly gated. Conversely, assume that every convex set of  $G$  is weakly gated and that  $G$  contains no  $K_{2,3}$  as a subgraph. Then the first condition immediately implies that  $G$  is bipartite and moreover,  $G$  must also be meshed. By Theorem 2 (v) we conclude that  $G$  is a median graph.

We next give different conditions on the intervals of a given graph  $G$  which ensures that  $G$  is a median graph. As we already mentioned, two conditions also involve convexity whereas Theorem 5 (ii) also involves intervals.

**Theorem 6** *For a connected graph  $G$ , the following conditions are equivalent.*

- (i)  $G$  is a median graph.
- (ii)  $G$  is a bipartite, meshed graph with 2-convex intervals.
- (iii)  $G$  is a bipartite, meshed graph with convex intervals.
- (iv)  $G$  is bipartite and every interval induces a median graph.
- (v) Every interval of  $G$  is weakly gated.
- (vi) Every interval of  $G$  is gated.

Theorem 6 (ii) and (iii) is Theorem 2 (iv) and (iii), respectively. Theorem 6 (iv) is due to Bandelt and Mulder [9] and (v) is a recent result due to Berrachedi [16]. In the sequel we will give a short proof of it. Note that (vi) is a slightly stronger condition than the one of (v). However, since gatedness is a well-known concept we have included also this condition. To see that intervals in median graphs are indeed gated, recall that there is a unique nearest vertex in the interval, which, by Theorem 5 (ii) must be the gate in the interval.

We now give a short proof of Theorem 6 (v). Let every interval of  $G$  be weakly gated. Then it easily follows that  $G$  must be bipartite, that it contains no  $K_{2,3}$  and that fulfils the quadrangle property. Thus  $G$  is a median graph. Conversely, suppose on the contrary that there is an interval  $I(u, v)$  of  $G$  and a vertex  $x$  of  $G$  such that  $d(x, I(u, v)) = d(x, x') = d(x, x'') = k \geq 1$ , where  $x' \neq x''$ . Note first that 2-convexity implies  $k > 1$ . Let  $P = x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_n \rightarrow x_{n+1}$  be a path in  $I(u, v)$  between  $x'$  and  $x''$ , where  $n \geq 1$ ,  $x_0 = x'$  and  $x_{n+1} = x''$ . Then clearly  $d(x, x_1) = d(x, x_n) = k + 1$ . Furthermore, as  $G$  is bipartite it follows that for some  $i$ , where  $1 \leq i \leq n - 1$ , we have  $d(x, x_i) = k + i$  and  $d(x, x_{i-1}) = d(x, x_{i+1}) = k + i - 1$ . By the quadrangle property, it then follows that there exists a vertex  $x'_i$  such that  $x'_i$  is adjacent to both  $x_{i-1}$  and  $x_{i+1}$  and that  $d(x, x'_i) = k + i - 2$ . In addition, by 2-convexity,  $x'_i$  belongs to  $I(u, v)$ . Repeating this argument we see that there exists a vertex  $x'_1$  which lies in  $I(u, v)$  with  $d(x, x'_1) = k - 1$ , a contradiction.

The last theorem in this section collects characterizations of median graphs involving separation axioms.

**Theorem 7** *For a connected graph  $G$ , the following conditions are equivalent.*

- (i)  $G$  is a median graph.
- (ii)  $G$  is a bipartite, meshed graph and its convex sets have the separation property  $S_2$ .
- (iii) The intervals of  $G$  have the Helly property.
- (iv) The convex sets of  $G$  have the Helly property and  $G$  contains no  $K_{2,3}$  as a subgraph.
- (v) The convex sets of  $G$  have the Helly property and the separation property  $S_2$ .
- (vi) The gated sets of  $G$  have the Helly property and the separation property  $S_2$ .

Theorem 7 (ii) is Theorem 2 (vi), (iii) is due to Tardif [51], (iv) was proved by Soltan and Chepoi [50]. Characterizations (v) and (vi) are due to Bandelt [5], where one more similar characterization is given.

## 4 Median graphs and location theory

Let  $G$  be a (connected) graph. A **profile** of **length**  $p$  on  $G$  is a finite sequence  $\pi = v_1, v_2, \dots, v_p$  of vertices of  $G$ . We denote  $p = |\pi|$ . A **median** of  $\pi$  is a vertex  $x$  minimizing  $D(x, \pi) := \sum_{i=1}^p d(x, v_i)$ , and the **median set**  $M(\pi)$  of  $\pi$  consists of all medians of  $\pi$ . In location theory the **Median Problem** is then: given  $\pi$ , find  $M(\pi)$ . In this terminology the median graphs are precisely those graphs in which  $|M(\pi)| = 1$ , for all profiles of length 3. Let  $V^*$  be the set of all nonempty profiles of  $G$ . Then  $M$  can be viewed as a function  $M : V^* \rightarrow \mathcal{P}(V) \setminus \{\emptyset\}$ . For median graphs, the function  $M$  is by now well studied, see e.g. Bandelt and Barthélemy [7] and McMorris, Mulder and Roberts [35]. Some simple properties of the function  $M$  for arbitrary (connected) graphs are:

**Anonymity (A)** : for any profile  $\pi = v_1, v_2, \dots, v_p$  and any permutation  $\sigma$  of  $\{1, 2, \dots, p\}$ , we have

$$M(\pi) = M(\pi^\sigma),$$

where  $\pi^\sigma = v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(p)}$ .

**Faithfulness (F)** :  $M(v) = \{v\}$ , for all  $v \in V(G)$ .

**Betweenness (B)** :  $M(u, v) = I(u, v)$ , for all  $u, v \in V(G)$ .

**Consistency (C)** : if  $M(\pi) \cap M(\sigma) \neq \emptyset$  for profiles  $\pi$  and  $\sigma$ , then  $M(\pi, \sigma) = M(\pi) \cap M(\sigma)$ .

For a profile  $\pi = v_1, v_2, \dots, v_p$ , we denote by  $\pi - v_i$  the vertex-deleted profile obtained from  $\pi$  by deleting  $v_i$ . McMorris, Mulder and Roberts [35] proved that in a median graph  $M$  also satisfies:

**Convexity (K)** : let  $\pi = v_1, v_2, \dots, v_p$  be a profile with  $p \geq 2$  in  $G$ ; if  $\bigcap_{i=1}^p M(\pi - v_i) = \emptyset$ , then  $M(\pi) = \text{Con}[\bigcup_{i=1}^p M(\pi - v_i)]$ .

Here  $\text{Con}(H)$  denotes the **convex hull** of a subgraph  $H$  in  $G$ , i.e., the smallest convex subgraph of  $G$  containing  $H$ .

An interesting problem in location theory is the following. Let  $G$  be a (connected) graph, and let  $L : V^* \rightarrow \mathcal{P}(V) \setminus \{\emptyset\}$  be some function on the profiles of  $G$ . What axioms should  $L$  satisfy to force that  $L = M$  on  $G$ . Of course,  $L$  should satisfy some or all of the necessary axioms (A), (B), (C) and (F) above, plus some extra axioms depending on the structure of  $G$ . In McMorris, Mulder and Roberts [35] it is proved that, *if  $G$  is a median graph, then  $L = M$  if and only if  $L$  satisfies (A), (C), (F) and (K)*. This result provides us with a new characterization of median graphs in terms of location theory.

**Theorem 8** *Let  $G$  be a connected graph. Then  $G$  is a median graph if and only if  $G$  contains no induced  $K_3$  or  $K_{2,3}$  and  $M$  satisfies (K) on  $G$ .*

**Proof.** For a proof that  $M$  satisfies (K) on a median graph we refer the reader to McMorris, Mulder and Roberts [35].

So let  $G$  be a triangle-free graph without  $K_{2,3}$  and with  $M$  satisfying (K) on  $G$ . First we prove that  $G$  is bipartite. Assume the contrary, and let  $C$  be a cycle of length  $2k + 1$  with  $k$  as small as possible. Because  $G$  is triangle-free, we have  $k > 1$ . Note that  $C$  is isometric in  $G$ . Take any edge  $vw$  on  $C$  and the vertex  $u$  on  $C$  with  $d(u, v) = d(u, w) = k$ . Because of minimality of  $k$ , we have

$$I(u, v) \cap I(u, w) = \{u\},$$

so that  $M(u, v) \cap M(u, w) \cap M(v, w) = \{u\} \cap \{v, w\} = \emptyset$ . Set  $\pi = u, v, w$ . By (K), we infer that  $u$  is in  $M(\pi)$ . Clearly, we have  $D(v, \pi) = D(w, \pi) = k + 1$ , and  $D(u, \pi) = 2k \geq k + 2$ , whence  $u$  is not in  $M(\pi)$ . This contradiction tells us that  $G$  is bipartite.

Now we prove that  $G$  is meshed. Assume the contrary, and let  $u, v, w$  and  $z$  be vertices such that  $z$  is a common neighbor of  $v$  and  $w$ , and  $k = d(u, v) = d(u, w) = d(u, z) - 1$ , and that there is no common neighbor of  $v$  and  $w$  at distance  $k - 1$  to  $u$ , with  $k$  as small as possible. Note that we have  $k \geq 2$ , and

$$\begin{aligned} I(u, v) \cap I(u, w) &= \{u\}, \\ I(u, v) \cap I(u, w) \cap I(v, w) &= \emptyset. \end{aligned}$$

Set  $\pi = u, v, w$ . By (K), we infer that  $z$  is in  $M(\pi)$ . On the other hand, we have  $D(v, \pi) = k + 2 = D(w, \pi)$  and  $D(z, \pi) = k + 1 + 1 + 1 = k + 3$ , whence  $z$  is not in  $M(\pi)$ . This contradiction tells us that  $G$  is meshed, and, by Theorem 2 (v),  $G$  is a median graph.  $\square$

Several other characterizations of median graphs in terms of location theory are known. Before we present them three more concepts should be introduced.

Let  $G$  be a connected graph and let  $\pi$  be a profile of length at least two. The **centroid** of  $G$  with respect to  $\pi$  consists of those vertices  $u$  of  $G$  for which the number of vertices of  $\pi$  that belong to a maximal convex set not containing  $u$  is minimum.

The second concept is seemingly completely different than the concepts introduced by now, but it will give us a very interesting characterization of median graphs. Chung, Graham and Saks [20, 21] defined a new graph invariant called windex. We roughly present it here, for more details see Chung, Graham and Saks [21]. We have a system which is represented by

a connected graph  $G$ . A sequence of requests, say  $r_1, r_2, \dots, r_n$ , where  $r_i$  are vertices of  $G$ , is coming to the system dynamically. Our goal is to find, from an initial vertex  $s_0$ , a state sequence, say  $s_0, s_1, s_2, \dots, s_n$ , where  $s_i$  are vertices of  $G$ , such that

$$\sum_{i=1}^n (d(s_{i-1}, s_i) + d(s_i, r_i))$$

is as small as possible. Then the **windex** of a graph  $G$  is the smallest  $k$ , such that there exists an optimal algorithm for the above problem if, at each stage  $i$ , when we want to determine  $s_{i+1}$ , we know  $k$  requests in advance, i.e.  $r_{i+1}, \dots, r_{i+k}$ .

In a tree, it is easily seen that one can find  $M(\pi)$ , for a profile  $\pi$ , by starting in an arbitrary vertex and moving in the tree to the majority of  $\pi$ . This idea is generalized by Mulder [41] as follows. For an edge  $uv$  of a connected graph  $G$ , and a profile  $\pi$  of  $G$ , we denote by  $\pi_{uv}$  the subprofile of  $\pi$  consisting of all elements of  $\pi$  nearer to  $u$  than to  $v$ . The **majority strategy** for  $\pi$  reads as follows: start at an initial vertex  $u$  of  $G$ ; if  $v$  is a neighbor of  $u$  with  $|\pi_{uv}| \geq |\pi|/2$ , then move to  $v$ ; move to a vertex already visited twice only if there is no other choice; stop when either we are stuck at a vertex  $v$  (i.e.  $|\pi_{wv}| < |\pi|/2$ , for all neighbors  $w$  of  $v$ ) or we have visited vertices at least twice, and, for each vertex  $v$  visited at least twice and each neighbor  $w$  of  $v$ , either  $w$  is also visited twice or  $|\pi_{wv}| < |\pi|/2$ . We say that the majority strategy produces from initial vertex  $u$ , for  $\pi$ , the set consisting of the single vertex where we get stuck or of all vertices visited at least twice. If the majority strategy produces for  $\pi$  the same set  $W$  from any initial vertex, then we just say that it **produces**  $W$  for  $\pi$ .

**Theorem 9** *Let  $G$  be a connected graph. Then the following conditions are equivalent.*

- (i)  $G$  is a median graph.
- (ii)  $G$  contains no  $K_3$  or  $K_{2,3}$  and  $M$  satisfies (K).
- (iii)  $G$  is triangle-free and  $M(\pi)$  is connected for all  $\pi$ .
- (iv)  $G$  has windex 2.
- (v)  $M(\pi)$  consists of all vertices  $x$  such that  $W_{yx}$  contains at most half of  $\pi$  for any neighbor  $y$  of  $x$ .
- (vi) The majority strategy produces  $M(\pi)$ , for all  $\pi$ .
- (vii) The majority strategy produces  $M(\pi)$ , for all  $\pi$  of length 3.
- (viii) The majority strategy produces the same set from each initial position, for all  $\pi$ .
- (ix) The majority strategy produces the same set from each initial position, for all  $\pi$  of length 3.
- (x)  $|M(\pi)| = 1$ , for all  $\pi$  of odd length.

- (xi) *There exists an odd integer  $p = 2k + 1 \geq 3$  such that  $|M(\pi)| = 1$ , for all  $\pi$  of length  $p$ .*
- (xii)  *$G$  is triangle-free, and the centroid of  $\pi$  coincides with  $M(\pi)$ , for all  $\pi$ .*
- (xiii)  *$M(\pi)$  is the intersection of all convex sets containing at least half of  $\pi$ , for all  $\pi$ .*

Theorem 9 (ii) is Theorem 8. The characterization (iii) is due to Bandelt [3], see also Bandelt and Mulder [10]. Theorem 9 (iv) was proved by Chung, Graham and Saks [20], see also Chung, Graham and Saks [21] and Mulder [40]. A short direct proof that the windex of a median graph is equal two is given in Klavžar [31]. Theorem 9 (v) is from Bandelt and Barthélemy [7], see also McMorris, Mulder and Roberts [35] while theorems (vi) – (ix) are due to Mulder [41]. Theorems 9 (x) and (xi) are due to Bandelt and Barthélemy [7]. The last two characterizations from Theorem 9 are due to Bandelt [6], where one can also find some other similar characterizations.

## 5 Related structures

Median structures can be found in many different guises. Here we present the main ones: in various algebraic terms, in terms of set functions, in terms of hypergraphs, in terms of convexities, in terms of geometries, and in terms of conflict models. Pertinent references to the literature are given. In the case of trees and hypercubes appropriate axioms can be found in the literature, we do not review these here.

### 5.1 Ternary algebras

In this subsection, we present the independent discoveries of median algebras.

A **ternary algebra**  $(V, M)$  consists of a set  $V$  and a ternary operator

$$M : V \times V \times V \rightarrow V.$$

The graph  $G_M = (V, E_M)$  of  $(V, M)$  is defined by

$$uv \in E_M \iff u \neq v \text{ and } M(u, x, v) \in \{u, v\} \text{ for all } x \in V.$$

A ternary algebra  $(V, M)$  is a **median algebra** if it satisfies the axioms:

- (a1)  $M(u, u, v) = u$ ;
- (a2)  $M(u, v, w)$  is invariant under all six permutations;
- (a3)  $M(u, M(v, w, x), y) = M(M(u, v, y), w, M(u, x, y))$ .

Then  $(V, M)$  is a median algebra if and only if its graph  $G_M$  is a median graph. Conversely, let  $G = (V, E)$  be a median graph, and define  $M(u, v, w)$  to be the median of the triple  $u, v, w$ . Then  $(V, M)$  is a median algebra.

Median algebras in this sense were introduced by Avann [1] as ternary distributive semi-lattices. In [1] he announced and in [2] he established the relationship with median graphs, which he called unique ternary distance graphs.

The second discovery of median algebras was by Sholander [47]. He defined a median algebra (named by him median semi-lattice) to be a ternary algebra  $(V, M)$  satisfying

$$\begin{aligned} \text{(m1)} \quad & M(u, u, v) = u; \\ \text{(m2)} \quad & M(M(u, v, w), M(u, v, x), y) = M(M(w, x, y), u, v). \end{aligned}$$

In [48, 49] Sholander proved the equivalence of these median algebras with the structures given below in Sections 5.2 - 5.4.

The third independent discovery of median algebras was by Nebeský in [43] under the name of normal graphic algebra (or simple graphic algebra in [44]). He defined a median algebra to be a ternary algebra  $(V, M)$  satisfying

$$\begin{aligned} \text{(n1)} \quad & M(u, u, v) = u; \\ \text{(n2)} \quad & M(u, v, w) = M(w, v, u) = M(v, u, w); \\ \text{(n3)} \quad & M(M(u, v, w), w, x) = M(u, M(v, w, x), w). \end{aligned}$$

He proved the relation between median algebras and median graphs in [44]. The term median algebra seems to be independently introduced by Evans [25], Isbell [28] and Mulder [38]. Many other axiom systems for median algebras can be found in the literature, e.g. in Kolibiar and Marcisová [34], Isbell [28], Bandelt and Hedlíková [8].

## 5.2 Betweenness

A **betweenness structure**  $(V, B)$  consists of a set  $V$  and a **betweenness relation**

$$B \subseteq V \times V \times V$$

satisfying at least the conditions  $(u, u, v) \in B$  and  $(u, v, w) \in B$  if and only if  $(w, v, u) \in B$ , for all  $u, v, w \in V$ . If  $(u, v, w) \in B$ , then we say that “ $v$  is between  $u$  and  $w$ ”. The graph  $G_B = (V, E_B)$  of  $(V, B)$  is given by

$$uv \in E_B \iff u \neq v \text{ and } (u, x, v) \in B \text{ only if } x \in \{u, v\}.$$

A betweenness structure  $(V, B)$  is a **median betweenness structure** if it satisfies the axioms:



- (b1) for all  $u, v, w \in V$ , there exists  $x$  such that  $(u, x, v)$ ,  $(v, x, w)$ ,  $(w, x, u) \in B$ ;
- (b2) if  $(u, v, u) \in B$ , then  $v = u$ ;
- (b3) if  $(u, v, w)$ ,  $(u, v, x)$ ,  $(w, y, x) \in B$ , then  $(y, v, u) \in B$ .

Then  $(V, B)$  is a median betweenness structure if and only if its graph  $G_B$  is a median graph. Conversely, let  $G = (V, E)$  be a median graph with interval function  $I$ , and let  $B$  be the betweenness relation on  $V$  defined by  $(u, v, w) \in B$  if  $v \in I(u, w)$ . Then  $(V, B)$  is a median betweenness structure. This gives a one-to-one correspondence between the median betweenness structures  $(V, B)$  and the median graphs with vertex set  $V$ .

The median betweenness structures were introduced by Sholander [48] and proven to be equivalent with his median algebras from Section 5.1.

### 5.3 Interval structures

An **interval structure**  $(V, I)$  consists of a set  $V$  and an **interval function**

$$I : V \times V \rightarrow \mathcal{P}(V)$$

satisfying at least the conditions  $u \in I(u, v)$  and  $I(u, v) = I(v, u)$ , for all  $u, v \in V$ . The graph  $G_I = (V, E_I)$  of the interval structure is defined by

$$uv \in E_I \iff u \neq v \text{ and } I(u, v) = \{u, v\}.$$

An interval structure  $(V, I)$  is a **median interval structure** if it satisfies the axioms:

- (s1) there exists  $z$  such that  $I(u, v) \cap I(u, w) = I(u, z)$ , for all  $u, v, w \in V$ ;
- (s2) if  $I(u, v) \cap I(u, w) = I(u, v)$ , then  $I(x, u) \cap I(x, w) \subseteq I(v, x)$ , for any  $x \in V$ ;
- (s3) if  $I(u, v) \cap I(u, w) = I(u, u)$ , then  $I(u, u) \cap I(v, w) = \{u\}$ .

Then  $(V, I)$  is a median interval structure if and only if its graph  $G_I$  is a median graph. Conversely, let  $G = (V, E)$  be a median graph with interval function  $I$ . Then  $(V, I)$  is a median interval structure. This gives a one-to-one correspondence between the median interval structures  $(V, I)$  and the median graphs with vertex set  $V$ .

Median interval structure  $(V, I)$  with the axioms (s1), (s2) and (s3) were introduced by Sholander [48], which he called median segments, and proven to be equivalent to his median algebras from Section 5.1. The equivalence with median graphs follows from Avann [2]. In Mulder and Schrijver [42] a different set of axioms and direct proofs of the relation between median interval structures and median graphs were given, cf. Mulder [38]. A median interval structure sensu Mulder and Schrijver is an interval structure  $(V, I)$  satisfying

- (i1) if  $x, y \in I(u, v)$  then  $I(x, y) \subseteq I(u, v)$ ;
- (i2)  $|I(u, v) \cap I(v, w) \cap I(w, u)| = 1$ , for all  $u, v, w \in V$ .

## 5.4 Semilattices

A **semilattice** is a partially ordered set  $(V, \leq)$  in which any two elements  $u, v$  have a **meet** (greatest lower bound)  $u \wedge v$ . If  $u$  and  $v$  have a **join** (least upper bound), then we denote it by  $u \vee v$ . For  $u \leq v$ , we define the **order interval** to be  $[u, v] = \{w \mid u \leq w \leq v\}$ . As usual, the **covering graph**  $G_{\leq} = (V, E_{\leq})$  of  $(V, \leq)$  is given by

$$uv \in E_{\leq} \iff u \neq v \text{ and } u \leq w \leq v \text{ if and only if } w \in \{u, v\}.$$

A semilattice  $(V, \leq)$  is a **median semilattice** if satisfies the following axioms:

- ( $\ell 1$ ) every order interval is a distributive lattice;
- ( $\ell 2$ ) if  $u \vee v, v \vee w$  and  $w \vee u$  exist, then  $u \vee v \vee w$  exists for any  $u, v, w \in V$ .

Then  $(V, \leq)$  is a median semilattice if and only if its covering graph  $G_{\leq}$  is a median graph. Note that two different median semilattices can have the same median graph as covering graph. Conversely, let  $G = (V, E)$  be a median graph with interval function  $I$  and let  $z$  be a fixed vertex in  $G$ . We define the ordering  $\leq_z$  on  $V$  by  $u \leq_z v$  if  $u \in I(z, v)$ . Then  $(V, \leq_z)$  is a median semilattice with universal lower bound  $z$ .

In the finite case this yields a one-to-one correspondence between the median semilattices and the pairs  $(G, z)$ , where  $G$  is a median graph and  $z$  is a vertex of  $G$ .

Median semilattices were introduced by Sholander [49] and proven to be equivalent with his median algebras, median betweenness structures and median interval structures. The equivalence with median graphs follows from the result of Avann [2] on median algebras and median graphs. A direct proof of the above result in the finite case is given in Mulder [38].

## 5.5 Hypergraphs and convexities

A **copair hypergraph**  $(V, \mathcal{E})$  consists of a set  $V$  and a family  $\mathcal{E}$  of subsets of  $V$  such that  $A \in \mathcal{E}$  implies  $V \setminus A \in \mathcal{E}$ . As usual, its graph  $G_{\mathcal{E}} = (V, E_{\mathcal{E}})$  is given by

$$uv \in E_{\mathcal{E}} \iff u \neq v \text{ and } \bigcap \{A \in \mathcal{E} \mid u, v \in A\} = \{u, v\}.$$

A copair hypergraph  $(V, \mathcal{E})$  is a **maximal Helly copair hypergraph** if it satisfies the conditions:

- (h1)  $\mathcal{E}$  has the Helly property;
- (h2) if  $A \notin \mathcal{E}$ , then  $\mathcal{E} \cup \{A, V \setminus A\}$  does not have the Helly property.

Then a copair hypergraph  $(V, \mathcal{E})$  is maximal Helly if and only if its graph  $G_{\mathcal{E}}$  is a median graph. Conversely, let  $G = (V, E)$  be a median graph, and let  $\mathcal{E}$  consists of the sets  $W_{uv}$ , for  $uv \in E$ . Then  $(V, \mathcal{E})$  is a maximal Helly copair hypergraph.

This result was proven by Mulder and Schrijver in [42] using Theorem 5 (v); see also Mulder [38] and Barthélemy [14].

Note that, if we take the closure  $\mathcal{E}^*$  of  $\mathcal{E}$  by taking all intersections, then  $\mathcal{E}^*$  consists precisely of the convex sets of  $G_{\mathcal{E}}$ . Thus we get an alternative formulation of the above result as follows.

A **convexity**  $(V, \mathcal{C})$  consists of a set  $V$  and a convexity  $\mathcal{C}$  being a family of subsets of  $V$  closed under taking intersections. Its graph  $G_{\mathcal{C}} = (V, E_{\mathcal{C}})$  is given by

$$uv \in E_{\mathcal{C}} \iff u \neq v \text{ and } \bigcap \{A \in \mathcal{C} \mid u, v \in A\} = \{u, v\}.$$

The above result reads then: in a convexity  $(V, \mathcal{C})$  the family of convex sets  $\mathcal{C}$  has the Helly property and the separation property  $S_2$  if and only if its graph  $G_{\mathcal{C}}$  is a median graph (cf. Bandelt and Van de Vel [52]).

## 5.6 Join geometries

A **join geometry**  $(V, \circ)$  consists of a set  $V$  and a **join operator**  $\circ : V \times V \rightarrow \mathcal{P}(V)$  satisfying at least  $u \circ u = \{u\}$ ,  $u \in u \circ v$ ,  $u \circ v = v \circ u$ , and  $u \circ (v \circ w) = (u \circ v) \circ w$ , for all  $u, v, w \in V$ . Its graph  $G_{\circ} = (V, E_{\circ})$  is given by

$$uv \in E_{\circ} \iff u \neq v \text{ and } u \circ v = \{u, v\}.$$

For subsets  $U, W$  of  $V$ , we write  $U \circ W$  for the union of all  $u \circ w$  with  $u \in U$  and  $w \in W$ . If  $U = \{u\}$ , we write  $u \circ W$  instead of  $U \circ W$ . A set  $C$  in a join geometry  $(V, \circ)$  is **convex** if  $C \circ C = C$ . Join geometries were introduced and extensively studied by Prenowitz and Jantosciak [46]. A join geometry  $(V, \circ)$  is a **join space** if it satisfies the following conditions:

- (S4) (Kakutani separation property) if  $C, D$  are disjoint convex sets in  $(V, \circ)$ , then there is a convex set  $H \subset V$  such that  $C \subseteq H$  and  $D \subseteq V \setminus H$  and  $V \setminus H$  is also convex;
- (JHC) (Join-hull commutativity) if  $C$  is a convex set then  $Con(\{u\} \cup C) = u \circ C$ , for  $u$  in  $V$ .

Then the convex sets in a join space  $(V, \circ)$  have the Helly property if and only if its graph  $G_{\circ}$  is a median graph. Conversely, let  $G = (V, E)$  be a median graph with interval function  $I$ , and define the join operator  $\circ$  on  $V$

by  $u \circ v = I(u, v)$ . Then  $(V, \circ)$  is a join space with convex sets having the Helly property.

This result is essentially due to Van de Vel [52], see also Bandelt and Mulder [11] and Van de Vel [53]. We also refer to Nieminen [45] who observed that median algebras are join spaces for the natural join operator  $u \circ v = I(u, v)$ .

## 5.7 Conflict models

A **conflict model**  $(X, \leq, A)$  consists of a set  $X$ , a partial ordering  $\leq$  of  $X$  and a set of edges  $A$  such that  $(X, A)$  is a graph. As usual, a subset  $Y$  of  $X$  is an **ideal** whenever  $x \in Y$  and  $y \leq x$  implies  $y \in Y$ , and  $Y$  is **independent** whenever there are no edges in  $(X, A)$  between vertices of  $Y$ . One can construct a graph  $G = (V, E)$  from a conflict model  $(X, \leq, A)$  as follows: the vertex-set  $V$  of  $G$  consists of all independent ideals of  $(X, \leq, A)$ , and we connect two vertices  $Y$  and  $Z$  by an edge whenever they differ in one element (i.e. have symmetric difference of size 1). Using the equivalence of (i) and (vi) in Theorem 5 one can easily deduce that  $G$  is a median graph. This fact was first observed by Barthélemy and Constantin [15].

Now the problem arises how to construct a conflict model  $(X, \leq, A)$  from a median graph  $G = (V, E)$  such that  $G$  can be reconstructed as above from  $(X, \leq, A)$ . Barthélemy and Constantin [15] gave a nice construction, which amounts to the following. Let  $G = (V, E)$  be a median graph, and let  $z$  be a fixed vertex of  $G$ . The set  $X$  consists of the sets  $\{W_{ab}, W_{ba}\}$ , with  $ab \in E$ . Here we do not count multiplicities, i.e. if  $uv$  is an edge between  $W_{ab}$  and  $W_{ba}$ , then it defines the same set as  $ab$ . We say that  $\{W_{uv}, W_{vu}\}$  and  $\{W_{ab}, W_{ba}\}$  **cross** if the four intersections  $W_p \cap W_q$ , where  $p \in \{uv, vu\}$  and  $q \in \{ab, ba\}$  are nonempty. Now let  $x = \{W_{uv}, W_{vu}\}$  and  $y = \{W_{ab}, W_{ba}\}$  be two distinct non-crossing elements of  $X$ . We put

$$\begin{aligned} x \leq y & \text{ if, say, } z \in W_{uv} \cap W_{ab}, \text{ and } W_{uv} \subseteq W_{ab}, \\ xy \in A & \text{ if } x \text{ and } y \text{ are incomparable with respect to } \leq. \end{aligned}$$

Note that crossing elements of  $X$  are incomparable as well as non-adjacent. Barthélemy and Constantin [15] proved that the so obtained conflict model  $(X, \leq, A)$  reproduces  $G$  by the above construction. These conflict models  $(X, \leq, A)$  constructed from median graphs have the following additional property

$$(p1) \text{ if } xy \in A \text{ and } x \leq u, y \leq v, \text{ then } uv \in A.$$

Barthélemy and Constantin named conflict models satisfying (p1) **sites**. In computer science, they are known as **conflict event structures**, cf. [22]. The main result of Barthélemy and Constantin [15] reads then as follows:

there is a one-to-one correspondence between the sites and the pairs  $(G, z)$ , where  $G$  is a median graph and  $z$  a vertex of  $G$ .

## 6 Concluding remarks

Almost all results above also hold when we allow the set  $V$  to be infinite. For the structures in Section 5, we have to postulate discreteness to make sense of the relation to (infinite) median graphs. Furthermore, we have to postulate some other finiteness condition, which amounts, for instance, in the interval structure case that intervals are finite. The only intrinsic finite characterization of median graphs is Theorem 5 (v):  $G$  is a median graph if and only if it can be obtained from the one vertex graph by the expansion procedure. But all structural properties obtained in the proof by Mulder [36, 38] also hold for the infinite case. The theorem should then read as follows: a graph  $G$  is a median graph if and only if it can be obtained from a median graph  $G'$  by convex expansion, unless  $G$  is  $K_1$  (as in Fig. 2 to Fig. 1).

Those characterizations above that are proved in the literature only for the finite case can easily be proved for the infinite case as well using the following fact: *the convex closure of a finite set in a median graph is finite*. This fact is observed e.g. in Chepoi [18]. For example, McMorris, Mulder and Roberts [35] use induction on the number of expansions to prove that a median graph satisfies the convexity axiom (K) for profiles. By restricting oneself to the convex closure  $Con(\pi)$  of the vertices in the profile  $\pi$ , one only has to consider a finite number of expansions, viz. those involved in  $Con(\pi)$ , i.e. pairs  $W_{ab}, W_{ba}$  with  $ab$  an edge in  $Con(\pi)$ .

Of course, we do not pretend that we have exhausted all possible characterizations of median graphs and all related median structures. This paper just presents the state of the art in a unified and structured way.

### Notes added in proof

Since this paper has been written, several new aspects of median graphs were discovered.

Chepoi [19] characterized median graphs within a new framework. Let us just briefly outline his ideas, for details see the cited paper. A *cubical complex* is a set of cubes on any dimensions which is closed under taking subcubes and nonempty intersections. The *underlying graph* of a cubical complex has its 0-cubes as vertices and its 1-cubes as edges. A cubical complex is a *cubing* if the following condition holds: if three  $(k+2)$ -cubes share a common  $k$ -cube, and pairwise share common  $(k+1)$ -cubes then

they are contained in a  $(k + 3)$ -cube. Now, Chepoi proved that the graphs of cubings are exactly the median graphs.

Imrich, Klavžar and Mulder [29] demonstrated that median graphs are closely related to triangle-free graphs. In particular it is shown that the complexities of recognizing median graphs and of recognizing triangle-free graphs are closely related and that intuitively speaking there are as many median graphs as there are triangle-free graphs.

In [32] an Euler-type formula for median graphs is presented which involves the number of vertices, the number of edges, and the number of cutsets in the cutset coloring of a median graph. The formula is an inequality, where equality is attained if and only if the median graph is cube-free.

Finally, in [33] it is shown that Theorem 6 (*v*) can be relaxed as follows: A connected graph is a median graph if and only if every interval of  $G$  of length at most 2 is weakly gated. Here a subgraph  $H$  of a graph  $G$  is *weakly gated* in  $G$ , if, for every  $v \in V(G)$ , there exists a unique vertex  $x \in V(H)$ , which minimizes the distance between  $v$  and the vertices of  $H$ .

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