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Discrete Mathematics 244 (2002) 223–230

DISCRETE
MATHEMATICS

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On subgraphs of Cartesian product graphs

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Received 2 July 1999; revised 15 April 2000; accepted 21 December 2000

Dedicated to Gert Sabidussi on the occasion of his seventieth birthday

Abstract

Graphs which can be represented as nontrivial subgraphs of Cartesian product graphs are characterized. As a corollary it is shown that any bipartite, $K_{2,3}$ -free graph of radius 2 has such a representation. An infinite family of graphs which have no such representation and contain no proper representable subgraph is also constructed. Only a finite number of such graphs have been previously known. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Graph products; Cartesian product; S -prime graph; Path coloring

1. Introduction

Different kinds of subgraphs of Cartesian product graphs have already been considered. Retracts and isometric subgraphs of Cartesian product graphs were studied, for instance, in [1,4,10,12,14]; see also [5] for a survey on related algorithmic results. A lot of attention has been given to the problem of when a given graph can be considered subgraph (or induced subgraph) of a hypercube, which is the simplest Cartesian product graph. A sample of this research is [2,3,9,13]. In this note we consider the problem which graphs are subgraphs of Cartesian product graphs.

Let $*$ be a graph product. A graph G is *prime* with respect to $*$ if G cannot be represented as the product of two nontrivial graphs, i.e., if the identity $G = G_1 * G_2$ implies that G_1 or G_2 is the one-vertex graph K_1 . A graph G is called *S -prime* (S stands for “subgraph”) with respect to $*$ if G cannot be represented as a nontrivial subgraph of a $*$ -product graph. Here a subgraph G of $G_1 * G_2$ is called *nontrivial* if the projections

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¹ Supported by the Ministry of Science and Technology of Slovenia under the Grant 0101-P-504.

² Supported by the Ministry of Science and Technology of Slovenia under the Grant 0101-P-511.

of G on both G_1 and G_2 contain at least two vertices. Graphs which are not S -prime will be called S -composite. Note that an S -prime graph with respect to $*$ is always prime with respect to $*$. For this reason we renamed the notion of quasiprimeness, introduced for this concept by Lamprey and Barnes [7], as S -primeness.

Sabidussi [11] showed that the only S -prime graphs with respect to the direct product are complete graphs or complete graphs minus an edge. Lamprey and Barnes [7] observed that the only S -prime graphs with respect to the strong product and the lexicographic product are K_1 , $K_1 \cup K_1$ and K_2 . Thus, among the four standard graph products, only the Cartesian product offers interesting problems. Since we will exclusively consider this product in the rest of the paper, an S -prime (S -composite) graph will mean an S -prime (S -composite) graph with respect to the Cartesian product.

We consider finite undirected graphs without loops or multiple edges. The set of neighbors of a vertex u is denoted by $N(u)$. The Cartesian product $G \square H$ of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is the graph with vertex set $V(G) \times V(H)$ where vertex (a, x) is adjacent to vertex (b, y) whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. For a fixed vertex a of G , the vertices $\{(a, x) \mid x \in V(H)\}$ induce a subgraph of $G \square H$ isomorphic to H . We call it an H -layer and denote it by H^a . Analogously we define G -layers. With this terminology a subgraph of $G \square H$ is nontrivial if it intersects at least two G -layers and at least two H -layers. We refer to [6] for more information on graph products.

In Section 2, a characterization of S -prime graphs is presented and some corollaries are deduced. In Section 3 we first simplify the definition of basic S -prime graphs due to Lamprey and Barnes [8]. This result enables us to construct an infinite family of graphs which cannot be represented as nontrivial subgraphs of Cartesian product graphs and in addition contain no proper representable subgraph. Only a finite number of such graphs have been previously known.

2. A characterization

Note that every graph which contains a vertex of degree one or a cut vertex is S -composite. In particular, this holds for any tree on at least three vertices. Examples of S -prime graphs are provided by complete graphs and complete bipartite graphs on at least five vertices. Before presenting a characterization of S -prime graphs, we briefly recall two previously known characterizations.

A *plotting* of a graph G is a drawing of G in the Euclidean plane in which every pair of adjacent vertices has either the same abscissa or the same ordinate. A plotting is *trivial* if all vertices are plotted on the same horizontal or vertical line. Then S -prime graphs can be (trivially) characterized as graphs with only trivial plottings. Another characterization is provided in [8] by the construction of every S -prime graph from basic S -prime graphs.

A surjective mapping $c: V(G) \rightarrow \{1, 2, \dots, k\}$ is called a k -coloring of G . (Note that it need not be a coloring in the usual sense, i.e., adjacent vertices may receive

the same color.) Let c be a k -coloring of G and let P be a path of G on at least two vertices. Then we say that P is *well-colored*, if for any two consecutive vertices u and v of P we have $c(u) \neq c(v)$. We call a k -coloring c of G a *path k -coloring*, if for any well-colored u, v -path of G we have $c(u) \neq c(v)$.

Theorem 1. *Let G be a connected graph on at least three vertices. Then G is S -composite if and only if there exists a path k -coloring of G with $2 \leq k \leq |V(G)| - 1$.*

Proof. Assume first that G is S -composite and let G be a subgraph of $G_1 \square G_2$ which intersects at least two G_1 -layers and at least two G_2 -layers. Let $V(G_1) = \{a_1, a_2, \dots, a_k\}$ and let $V(G_2) = \{x_1, x_2, \dots, x_\ell\}$. Define a k -coloring of G by $c(a_i, x_j) = i$. Let P be an arbitrary well-colored path between vertices (a_i, x_j) and $(a_{i'}, x_{j'})$ of G . Then, by the definition of the Cartesian product it follows that $j = j'$. Therefore (using the definition of the Cartesian product again) we infer that $i \neq i'$. We conclude that $c(a_i, x_j) \neq c(a_{i'}, x_{j'})$ which shows that c is a k -path coloring. Clearly, $2 \leq k \leq |V(G)| - 1$.

Conversely, let c be a path k -coloring of G with $2 \leq k \leq |V(G)| - 1$. Let G' be a spanning subgraph of G which is obtained by removing all edges of G whose endpoints receive the same color. Let C_1, C_2, \dots, C_t be the connected components of G' . Note first that $t \geq 2$. Indeed, since $k \leq |V(G)| - 1$, there are vertices u and v of G with $c(u) = c(v)$. Then u and v belong to different connected components of G' for otherwise we would have a well-colored u, v -path so that c would not be a path k -coloring.

Now we show that G is a subgraph of $K_k \square K_t$. Let $V(K_k) = \{1, 2, \dots, k\}$ and let $g: V(G) \rightarrow V(K_t)$ be defined by $g(u) = i$, if $u \in C_i$. Let $f: V(G) \rightarrow V(K_k \square K_t)$ be defined by

$$f(u) = (c(u), g(u)).$$

We need to show that f is injective and that it maps edges to edges. Let u and v be different vertices of G . If $c(u) \neq c(v)$ then clearly $f(u) \neq f(v)$. And if $c(u) = c(v)$, then by the above argument we have $g(u) \neq g(v)$. Thus f is injective. Let $uv \in E(G)$. If $c(u) = c(v)$ then $K_t^{c(u)} = K_t^{c(v)}$ and so $f(u)f(v) \in E(K_k \square K_t)$. Finally, if $c(u) \neq c(v)$ then uv is an edge of G' and thus u and v belong to the same component C_i of G' . Therefore, $K_k^{g(u)} = K_k^{g(v)}$ and thus also in this case we conclude that $f(u)f(v) \in E(K_k \square K_t)$.

Since the image of c consists of at least two elements, G lies in at least two K_t -layers and as the same holds for the image of g , G lies in at least two K_k -layers. We conclude that G is S -composite. \square

From Theorem 1 we immediately get the following corollary from [7]:

Corollary 2. *Let G be a connected graph. If there exists a partition $\{A, B\}$ of $V(G)$ such that $A, B \neq \emptyset$ and the set of edges with one endpoint in A and the other in B match, then G is S -composite.*

Proof. Set $k = 2$. \square

Another consequence of Theorem 1 is the following:

Corollary 3. *Let G be a bipartite graph with radius 2 and suppose that G contains no subgraph isomorphic to $K_{2,3}$. Then G is S -composite.*

Proof. Let a be a vertex of G such that all vertices of G are at a distance, at most, of two from it. Let w be a neighbor of a and set

$$W = (\{w\} \cup N(w)) \setminus \{a\}.$$

Let a 2-coloring c of graph G be defined by $c(u) = 1$ if $u \in W$ and $c(u) = 2$ otherwise.

We claim that c is a path 2-coloring. Suppose this is not the case. Then there exists a well-colored path xyz of G for which $c(x) = c(z) \neq c(y)$.

We distinguish two cases. First, let $c(x) = 1$. Suppose $x = w$. Then $y = a$ and since a has only one neighbor of color 1, we cannot find z on P . It follows that $x \neq w$. As G is bipartite, y is adjacent to a and z is adjacent to w . So w, y, a, x, z induce a $K_{2,3}$. Assume now that $c(x) = 2$. By the definition of c , x is not adjacent to w . Since G is bipartite it follows that x is adjacent to a , y is adjacent to w and z is adjacent to a . Hence a, y, w, x, z induce a $K_{2,3}$. \square

3. Basic S -prime graphs

In [8] Lamprey and Barnes defined *basic S -prime graphs* recursively as S -prime graphs on at least three vertices which contain no proper basic S -prime subgraphs. They used this concept to show that every S -prime graph on at least three vertices is either a basic S -prime graph or can be constructed from such graphs by two special rules. We show first how this definition can be simplified.

For a set P of graphs, denote by $\min P$ the set of its minimal elements with respect to the subgraph relation.

Theorem 4. *Let P be any set of graphs. The equation*

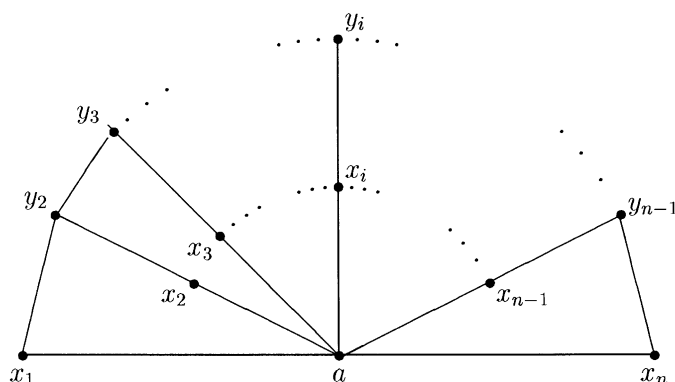
$$X = \{G \in P; \text{no proper subgraph of } G \text{ belongs to } X\} \quad (1)$$

has $X = \min P$ as its unique solution.

Proof. First we show that $X = \min P$ satisfies (1). If $G \in \min P$ then no proper subgraph of G belongs to P , much less to $\min P$. Conversely, if no proper subgraph of $G \in P$ belongs to $\min P$ then $G \in \min P$, by well-foundedness of the subgraph relation.

To prove uniqueness, assume that $X \subseteq P$ satisfies (1). If $G \in \min P$ then no proper subgraph of G belongs to P , much less to X . By (1), $G \in X$, so $\min P \subseteq X$.

Conversely, if $G \in X$ then, by (1), no proper subgraph of G belongs to X . It follows from the preceding paragraph, that no proper subgraph of G belongs to

Fig. 1. Graphs A_n .

$\min P$. Then $G \in \min P$, by well-foundedness of the subgraph relation. It follows that $X = \min P$. \square

Corollary 5. *An S -prime graph is a basic S -prime graph if and only if it has at least three vertices and contains no proper S -prime subgraph on at least three vertices.*

Proof. Let P be the set of all S -prime graphs on at least three vertices and X the set of all basic S -prime graphs. Then, by definition, X satisfies (1). From Theorem 4 it follows that basic S -prime graphs are exactly the minimal S -prime graphs on at least three vertices. \square

The only basic S -prime graphs with six or fewer vertices are K_3 and $K_{2,3}$. In addition, only very few basic S -prime graphs have been known. We are now going to construct an infinite family of basic S -prime graphs.

For $n \geq 3$ let A_n be the graph as shown on Fig. 1. Note that $\deg(a) = n$, $\deg(x_i) = 2$, $\deg(y_i) = 3$, $|V(A_n)| = 2n - 1$ and $|E(A_n)| = 3n - 3$. Note also that $A_3 = K_{2,3}$.

Theorem 6. A_n is a basic S -prime graph for any $n \geq 3$.

Proof. Let $X = \{x_1, \dots, x_n\}$ and $Y = \{y_2, \dots, y_{n-1}\}$. Suppose that A_n is not S -prime and let c be a path k -coloring of A_n with $2 \leq k \leq |V(A_n)| - 1$. We may assume that $c(a) = 1$.

Suppose first that $c(x_i) = 1$ for $i = 1, \dots, n$. As c is a path k -coloring and $c(x_1) = c(x_2) = 1$ we get $c(y_2) = 1$ and similarly $c(y_i) = 1$ for $i = 3, \dots, n - 1$. As $k \geq 2$ this case is thus not possible.

It follows that there exists a vertex x_{i_1} with $c(x_{i_1}) = 2$. Then no other vertex x_j of X receives color 2, for otherwise $x_{i_1}ax_j$ would violate our assumption that c is a path k -coloring. It follows that two different vertices of X cannot be colored with the same color different from 1.

Claim. *The vertices of X are colored with n different colors.*

We have seen above that exactly one vertex of X is colored with 2, $c(x_{i_1}) = 2$. Suppose that all the elements of $X \setminus \{x_{i_1}\}$ receive color 1. Then all vertices y_i , $i \neq i_1$, receive color 1. Now, if $c(y_{i_1}) = 1$ the well-colored path $ax_{i_1}y_{i_1}$ has ends of the same color and if $c(y_{i_1}) \neq 1$ the path $y_{i_1-1}y_{i_1}y_{i_1+1}$ has the same property. Therefore, there must be a vertex $x_{i_2} \in X \setminus \{x_{i_1}\}$ with $c(x_{i_2}) = 3$.

Assume now that $c(x_{i_j}) = j + 1$ for $j = 1, \dots, \ell$, where $2 \leq \ell \leq n - 1$. We wish to show that there exists a vertex $x_{i_{\ell+1}} \in X \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_\ell}\}$ with $c(x_{i_{\ell+1}}) = \ell + 2$.

We have observed above that none of the vertices from $X \setminus \{x_{i_1}, x_{i_2}, \dots, x_{i_\ell}\}$ can be colored with $2, 3, \dots, \ell + 1$. Thus we must show that it is not possible for all vertices in this set to receive color 1. We distinguish three cases.

Case 1. $c(x_1) = c(x_n) = 1$. Let x_p be the first vertex of X (i.e., the vertex with the smallest index) with $c(x_p) \neq 1$ and let x_q be the last such vertex of X . We may, without loss of generality, assume that $c(x_p) = 2$ and $c(x_q) = \ell + 1$. Note first that $c(y_j) = 1$ for $j = 1, \dots, p - 1$ and for $j = q + 1, \dots, n - 1$. Therefore, $c(y_p) = c(x_p)$ and $c(y_q) = c(x_q)$. In addition, it is not difficult to see that for the vertices y_{p+1}, \dots, y_{q-1} we have:

$$c(y_j) = \begin{cases} c(y_{j-1}), & c(x_j) = 1 \\ c(x_j), & \text{otherwise.} \end{cases}$$

Consider now the following two possibilities.

- (1) $c(x_{q-1}) = 1$. In this case $c(y_{q-1}) \neq \ell + 1, 1$. Thus the path $y_{q+1}y_qy_{q-1}x_{q-1}$ is well-colored and yet $c(y_{q+1}) = c(x_{q-1})$.
- (2) $c(x_{q-1}) \neq 1$. Let r be the largest index smaller than q such that $c(x_r) = 1$. Clearly, $r \geq 1$. If $r > p - 1$ then the path $x_r y_r y_{r+1} \dots y_{q-1} y_q y_{q+1}$ provides a contradiction. And if $r = p - 1$ then we consider the path $y_{p-1} y_p \dots y_{q-1} y_q y_{q+1}$. It is a well-colored path by the definition of q and we have $c(y_{p-1}) = c(y_{q+1})$.

We conclude that Case 1 is impossible.

Case 2. $c(x_1) \neq 1$, $c(x_n) = 1$. Suppose first that $c(x_j) \neq 1$ for $j = 1, \dots, n - 1$. Then all these colors are pairwise different and moreover, $c(y_j) = c(x_j)$. Then the path $ax_1y_2 \dots y_{n-1}x_n$ gives a contradiction. Otherwise, we have a vertex x_j , $j < n$, with $c(x_j) = 1$. Then we argue similarly as in Case 1.

Case 3. $c(x_1) \neq 1$, $c(x_n) \neq 1$. This case is treated similarly and is left to the reader.

This proves the claim, i.e., the vertices of X are colored with n different colors, say $c(x_i) = i + 1$ for $i = 1, \dots, n$. Clearly, no vertex of Y can be colored 1. It is also easy to see that no two vertices of Y receive the same color. Suppose that $c(y_j) \neq j + 1$. Set $s = c(y_j)$, $2 \leq s \leq l + 1$ and observe that there is a path from y_j to x_{s-1} which gives us a contradiction. Therefore, $c(y_i) = i + 1$ for $i = 2, \dots, n - 1$. But in this case the path $x_2ax_ny_{n-1} \dots y_2$ gives another contradiction.

By Corollary 5 the proof will be complete by showing that A_n contains no proper S -prime subgraph on $n \geq 3$ vertices. Let H be a proper subgraph of A_n on at least

three vertices. Assume $a \notin V(H)$. Then, if any of the vertices of X belongs to H , it is of degree at most 1 and thus H is S -composite. And if no vertex of X belongs to H then at least one of the vertices from Y is of degree at most 1. Thus we may assume that $a \in H$. We consider two cases.

Case 1. All vertices of X belong to H . Then for $i = 1, \dots, n$ the edges ax_i belong to H as well, for otherwise we would have a vertex of degree 1 in H . Moreover, also the edges x_1y_2, x_ny_{n-1} and x_iy_i for $i = 2, \dots, n-1$ belong to H . Finally, also the edges y_iy_{i+1} , $i = 2, \dots, n-2$ belong to H because otherwise a would be a cut-vertex. We conclude that $H = A_n$.

Case 2. Some vertex of X is not in H . Since $a \in V(H)$, it must be connected to some x_i in H . We may assume that $x_{i-1} \notin H$. Note that $y_i \in V(H)$ and y_ix_i also belongs to H , for otherwise x_i would be of degree 1. In addition, y_i must be adjacent to y_{i-1} and y_{i+1} in H . For otherwise x_i and y_i are adjacent vertices of degree at most two and then it is clear from Corollary 2 that H is S -composite. As $x_{i-1} \notin V(H)$, we observe that y_{i-1} must be adjacent to y_{i-2} in H . By the above arguments we see that $x_{i-2} \in V(H)$ and also that the edges $y_{i-2}x_{i-2}$ and $x_{i-2}a$ are in H . We now define a 3-coloring of H as follows:

$$c(u) = \begin{cases} 1, & u = y_{i-2}, x_{i-2} \\ 2, & u = y_{i-1}, y_i, x_i \\ 3, & \text{otherwise.} \end{cases}$$

It is not difficult to see that c is a path 3-coloring of H . Thus, by Theorem 1 H is S -composite. \square

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