Two-ended regular median graphs

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Abstract

We show that regular median graphs of linear growth are the Cartesian product of finite hypercubes with the two-way infinite path. Such graphs are Cayley graphs and have only two ends.

For cubic median graphs G the condition of linear growth can be weakened to the condition that G has two ends. For higher degree the relaxation to two-ended graphs is not possible, which we demonstrate by an example of a median graph of degree four that has two ends, but nonlinear growth.

Key words: Median graphs; infinite graphs; Cartesian product of graphs; ends of graphs; growth rate of graphs

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1 Introduction

Median graphs form one of the central classes of graphs in the area of metric graph theory and naturally appear in several other contexts as well [3, 15]. The interest in median graphs was stable for a long time, but recently it increased significantly, as can be seen from the following selection of papers [5, 6, 7, 8, 9, 18, 20].

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The only finite regular median graphs are the k-cubes Q_k, $k \geq 1$. For infinite graphs the situation is quite different—there are abundantly many regular median graphs. For instance, as noted by Bandelt and Mulder [4], already in the cubic case there are $2^{\aleph_0}$ such graphs. For appealing structural results additional conditions to the regularity are needed, such as transitivity or restrictions on the growth rate and the number of ends.

In an earlier paper [13] we considered infinite, vertex-transitive median graphs and proved that there are only finitely many such graphs of given finite degree with finite blocks. We also constructed an infinite family of vertex-transitive median graphs with finite intransitive blocks, and determined all vertex-transitive median graphs of degree four.

The latter result extended work of Bandelt and Mulder [4], who showed hat there exist exactly three cubic vertex transitive median graphs: the 3-cube, the 3-regular tree, and the infinite ladder. By their work infinite ladders are characterized as two-ended vertex transitive cubic median graphs. In this paper we prove that the result also holds without the condition of vertex transitivity; see Theorem 5.

As the infinite ladder is the Cartesian product of a K_2 by the two-way infinite path P_∞, the question arises whether all regular median graphs with two ends have such a structure. In other words, are they the Cartesian product of a hypercube by P_∞? This is not the case, as we show by an example of a median graph G of degree four that has a cut-vertex. This graph G also has cubic growth, whereas Cartesian products of hypercubes by P_∞ have only linear growth.

Thus the question arises how the growth rate is reflected in the structure of regular median graphs. This leads to Theorem 6, our main result. It characterizes two-ended, regular median graphs of linear growth as the class of Cartesian products of finite hypercubes by the two-way infinite path.

Since vertex-transitive graphs of linear growth are regular and have only two ends, this also characterizes vertex-transitive graphs of linear growth as Cartesian products of hypercubes by P_∞.

We do not know whether all regular median graphs of linear growth are also of this form, and pose this as an open problem.

2 Preliminaries

We expect the reader to be familiar with standard concepts, such as the Cartesian product of graphs and the structure of hypercubes. We will write $d_G(u)$ for the degree of the vertex u in a graph G, $d(G)$ for the degree of a regular graph G, Q_d for the d-dimensional hypercube, and $G \,\square\, H$ for the Cartesian product of G by H. By a *ladder* we mean the Cartesian product of a path (finite or infinite) by K_2.

Let us next recall several concepts from metric graph theory that are needed in the sequel. A subgraph H of a graph G is *isometric* if for each pair of vertices u, v

of H there exists a shortest u,v-path in G that lies entirely in H. H is *convex* if all shortest u,v-paths from G lie in H. A graph G is a *median graph* if there exists a unique vertex x to every triple of vertices u, v, w such that x lies simultaneously on a shortest u,v-path, a shortest u,w-path, and a shortest w,v-path.

Median graphs are bipartite and can also be characterized as retracts of hypercubes, both in the finite and the infinite case [2].

For a connected graph G and an edge ab of G we set

$$\begin{aligned}
W_{ab} &= \{w \in V(G) \mid d(a,w) < d(b,w)\},\\
U_{ab} &= \{w \in W_{ab} \mid w \text{ has a neighbor in } W_{ba}\},\\
F_{ab} &= \{e \in E(G) \mid e \text{ is an edge between } W_{ab} \text{ and } W_{ba}\}.
\end{aligned}$$

W_{ab} and U_{ab} are defined as sets of vertices but we will use the same notation also for the subgraphs of G induced by these sets. It should be clear from the context whether we are speaking about a set or a graph. The set, resp. subgraph, U_{ab} is called *peripheral* if $U_{ab} = W_{ab}$.

For bipartite graphs the sets F_{ab} coincide with the equivalence classes of the Djoković-Winkler relation Θ, $G \setminus F_{ab}$ has exactly two connected components, namely W_{ab} and W_{ba}, and these components are convex. For a proof cf. [12, Proposition 2.8].

The sets W_{ab}, U_{ab}, and F_{ab} play a central role in the structure of median graphs, see [12, Section 2.3] and [16]. In particular, in median graphs, the graphs U_{ab} and U_{ba} are isomorphic, in symbols $U_{ab} \cong U_{ba}$, and the edges of F_{ab} define a corresponding isomorphism. Moreover, U_{ab} and U_{ba} are convex subgraphs of G, and thus median graphs by themselves. Also, every U_{ab} is a cut-set.

Infinite graphs are incredibly rich in structure. Many concepts are used for their classification, among them the number of ends and the growth rate, which were mentioned in the introduction.

For the definition of ends[1] we need the concept of one-way infinite paths, which we call *rays*. One says two rays R_1, R_2 of a graph G are *equivalent*, $R_1 \sim R_2$, if there is a third ray R_3 in G that meets both of them infinitely often. It is easily seen that $\sim$ is an equivalence relation. The equivalence classes with respect to this relation are the *ends* of G.

For example, consider P_∞. Clearly it has two ends, just as every $Q_d \,\Box\, P_\infty$. Contrariwise, regular trees T_d of degree $d \geq 2$, that is, trees in which every vertex has the same degree $d \geq 2$, have infinitely many ends for $d > 2$. On the other hand, the infinite star, consisting of three rays that originate from the one and the same vertex, has just three ends.

The growth rate was originally introduced for groups by Adelson-Velsky [1] and its definition is immediately extendable to Cayley graphs. For graphs that are not Cayley graphs it seems to have been used first by Trofimov [19]. Let $B_d(v)$ denote

[1] Our definition follows Halin [11], but the concept goes back to Freudenthal [10].

the number of vertices of distance $\leq d$ from an arbitrary vertex v of a given graph G. If $B_d(v)$ is bounded from above by a polynomial in d, the we say that G has *polynomial growth*. If the degree of that polynomial is linear, we speak of *linear growth*, and it is clear what we mean by *quadratic* or *cubic growth*. As such the definition depends on the choice of v, but it is an easy exercise to show that the growth rate of connected graphs is independent of the choice of v.

Graphs of polynomial growth are just barely infinite and one can expect many properties of finite graphs to hold for them too. In particular this should be true for graphs of linear growth. We also wish to point out that infinite vertex transitive graphs of linear growth have just two ends, see Imrich and Seifter [14].

We continue with several lemmas that we will refer to in the proofs of the main results. A ray that is a shortest path between any two of its vertices will be called a *geodesic ray*.

Lemma 1 *Let F_{ab} be a Θ-class of a k-regular median graph. If U_{ab} is not peripheral, then U_{ba} is not peripheral either and both components of $G \setminus F_{ab}$ contain geodesic rays.*

Proof. If U_{ab} is not peripheral, there must be a vertex in $W_{ab} \setminus U_{ab}$, say c_1. By the connectedness of W_{ab} we can assume that c_1 is adjacent to a vertex in U_{ab}, say a.

Since $d_G(a) = k$ we infer that $d_{U_{ab}}(a) \leq k-2$ and hence, as $U_{ab} \cong U_{ba}$, we conclude that $d_{U_{ba}}(b) \leq k-2$ as well. But then U_{ba} cannot be peripheral either.

Now we consider F_{ac_1}, U_{ac_1} and $U_{c_1 a}$. Since U_{ac_1} does not contain ab and ac_1, it is clear that U_{ac_1} is not peripheral. By the above $U_{c_1 a}$ is also not peripheral.

Proceeding in this manner we arrive at a sequence of vertices $c_1, c_2, \ldots, c_i, c_{i+1}, \ldots$ with the property that no $U_{c_i c_{i+1}}$ and no $U_{c_{i-1} c_i}$ are peripheral. Also, $c_1 c_2 \ldots$ is a shortest path in W_{ab}. Hence W_{ab} contains a geodesic ray. By the same arguments this holds for W_{ba} too. $\square$

Lemma 1 is crucial for our further considerations. Moreover, it gives a new, short proof of the following fundamental fact:

Theorem 2 [17] *Let G be a finite, d-regular median graph ($d \geq 1$). Then $G \cong Q_d$.*

Proof. Let ab be an edge of G. Since G is finite, Lemma 1 implies that U_{ab} and U_{ba} are peripheral. Therefore, $G \cong K_2 \,\square\, U_{ab}$, and induction completes the argument. $\square$

Lemma 3 *Let G be a finite, connected graph in which all vertices are of degree d or $d-1$. If the vertices of degree $d-1$ induce a convex subgraph, then G is not median.*

Proof. Let $v_1, v_2, \ldots, v_k$ be the vertices of degree $d-1$ in G. Take an isomorphic copy φG of G that is disjoint from G and form a new graph H that consists of $G \cup \varphi G$ together with the edges $v_i\, \varphi v_i$, $i = 1, \ldots, k$.

If G is a median graph, then H is a median graph too and the k edges $v_i\, \varphi v_i$, $i = 1, \ldots, k$ form a Θ-class. (This follows from Mulder's Convex Expansion Theorem, cf. [12, Theorem 2.26].) By Theorem 2, G is a hypercube. But then edges in every Θ-class meet all vertices, which clearly is not the case here. $\square$

We conclude the preliminaries with another known fact about median graphs to be used later.

Lemma 4 *If an edge of a median graph is contained in a cycle, then it is contained in a square.*

Proof. Let $e = uv$ be an edge of a median graph G and C a cycle that contains e. Then C contains an edge $f = xy$ different from e that is in the Djoković-Winkler relation Θ to e, cf. [12, Lemma 2.4]. Let $x \in U_{uv}$. Since U_{uv} is connected, there exists a u,x-path in U_{uv}. Consequently, there exists a ladder from e to f. Then e together with the first "rung" after e in the ladder induce the required square. $\square$

3 Main results

We begin with the cubic case and prove the following result that extends the classification of cubic vertex transitive median graphs.

Theorem 5 *Let G be a 2- or 3-regular median graph with two ends. Then G is either the two-way infinite path P_∞ or*

$$G \cong Q_1 \,\square\, P_\infty\,.$$

Proof. Let v be a vertex in a U_{ab}. It is adjacent to a vertex in U_{ba}, thus its degree in U_{ab} can only be 1 or 2. Since the U_{ab}s are connected, they are must be paths or cycles.

If every vertex of a U_{ab} has degree two, then U_{ab} is peripheral. If it is infinite, then G is a ladder, that is $Q_1 \,\square\, P_\infty$, and we are through. If it is finite, then G is a finite regular median graph and has zero ends.

Thus the $U_{ab}s$ are paths and not peripheral. Suppose U_{ab} is a ray R_a with origin a. Then U_{ba} is a ray R_b with origin b and $U_{ab} \cup F_{ab} \cup U_{ba}$ is an infinite ladder L with first rung ab. Since a has degree 1 in U_{ab} it is the origin of an infinite ray P_a in W_{ab}, whereas b is the origin of an infinite ray P_b in W_{ba}. Clearly neither P_a nor P_b can be in the same end as L, because all vertices of L but a and b have degree three, so no ray that is not already in L can meet L infinitely often. If P_a and P_b were in the same end, there would exist a ray R that meets both of them infinitely often. Since P_a and P_b are separated by L this ray would have to meet L infinitely often, which is not possible.

Hence every U_{ab} is a path of finite length. Then $U_{ab} \cup F_{ab} \cup U_{ba}$ is a finite ladder L. We choose longest such ladder. Let ab be the first rung and xy the last, where $x \in U_{ab}$. By Lemma 1 a, b, x and y are origins of infinite rays, say P_a, P_b, P_x and P_y. P_a and P_x are separated from P_b and P_y by the finite sets U_{ab} and U_{ba}. Since we have only two ends, P_a and P_x must be in the same end. Thus there must be third ray R that meets both of them infinitely often. Let P be a subpath of R from P_a to P_x that does not meet U_{ab} and let P'_a and P'_x be the corresponding starting sections of P_a and P_x, respectively. Then

$$U_{ab} \cup P'_a \cup R \cup P'_x$$

is a closed walk in W_{ab}. Clearly U_{ab} is contained in a subcycle of this walk. But then every edge of U_{ab} is in a square Q in W_{ab} by Lemma 4. This is only possible if U_{ab} consists only of one edge. But then $L \cup Q$ is a longer ladder than L, in contradiction to the maximality of L. $\square$

For degree four the situation is quite different, there are median graphs of degree four with two ends which are not the Cartesian product of a square by a two-way infinite path.

To construct such an example we begin with the integer square lattice in the plane. Disengage every edge that comes from the lower half of the plane to $1, 2, 3, \ldots$ from $1, 2, 3\ldots$ Call the new endpoints of these edges $1', 2', \ldots$ Let this graph be G_0 and make infinitely many copies $G_i, i \in \mathbb{Z}$. Now take G_{-1} and glue the vertices $1', 2', \ldots$ of G_{-1} together with $1, 2, 3, \ldots$ of G_0. Do the same for G_{-2} and G_{-1}. Continue by induction, gluing G_{i-1} to G_i for $i = -1, -2, \ldots$.

For G_0 glue $2', 3'$ etc together with $1, 2, \ldots$ of G_1. This leaves one dangling edge. For G_1 continue to glue $1', 2', 3', \ldots$ together with $1, 2, 3, \ldots$ of G_2. Then continue by induction, gluing G_{i+1} to G_i for $i = 2, 3, \ldots$.

The new graph is H, has a dangling edge. It is a median graph with one end. Finally, take another copy of H and glue the dangling edges together. The constructed graph K is shown in Figure 1.

K is a median graph. One way to see this is the following. The only isometric cycles of K are 4-cycles. Moreover, K contains no $K_{2,3}$, hence the convex closure of any C_4 is the C_4 itself. Now apply a theorem of Bandelt (see [15]) asserting that a connected graph is median if and only the convex closure of any isometric cycle is a hypercube.

Hence we have constructed a 4-regular median graph with just two ends. It does not have linear growth, its growth rate is cubic, and it is planar.

Note that the construction of K can be modified such that the obtained 4-regular median graph has an arbitrary number of ends. Instead of matching the one-way

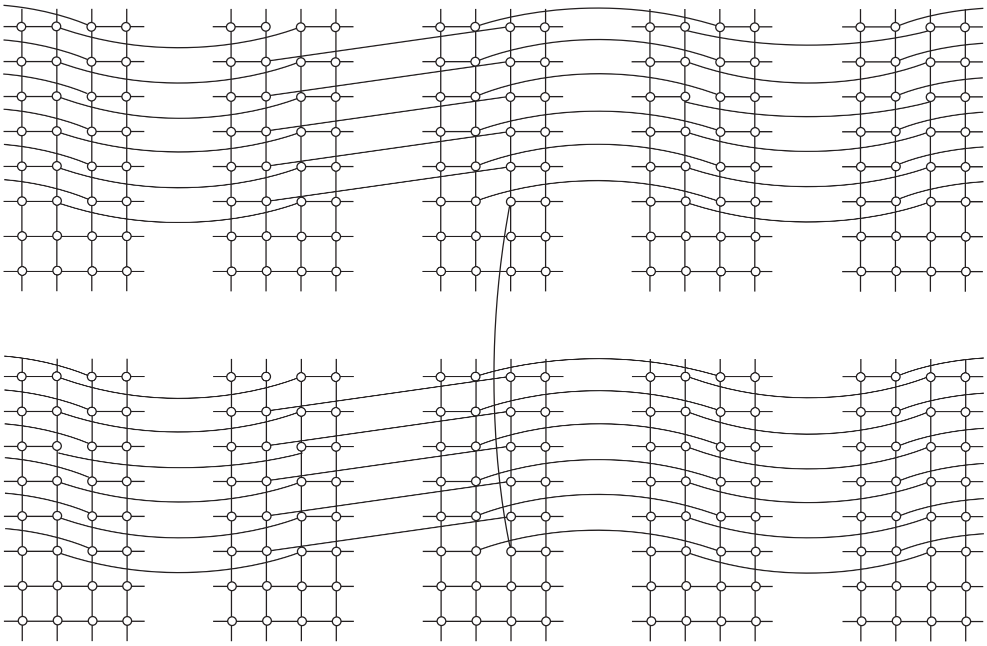

Figure 1: 4-regular two-ended median graph K

infinite path of G_0 with the one of G_1 by a jump of one, one can jump by some fixed distance k which leaves k vertices not yet of degree four and they can be attached by edges between this graphs and k copies of H.

Another easy modification shows that the sets that separate the ends of these examples can be of arbitrary (finite) cardinality.

We are now ready for the main result of the paper.

Theorem 6 *Let G be a two-ended, d-regular, median graph of linear growth. Then*

$$G \cong Q_{d-2} \,\Box\, P_\infty\,.$$

Proof. By Theorem 5 we can assume that $d \geq 4$ and that the theorem is true for all regular two-ended median graphs with linear growth of degree $d-1$.

Case 1: G has an infinite Θ-class.
Let F_{ab} be an infinite Θ-class. If U_{ab} is peripheral, then U_{ba} must also be peripheral by Lemma 1. But then $G = K_2 \,\Box\, U_{ab}$, where U_{ab} is a regular two-ended median graphs with linear growth of degree $d-1$. For such U_{ab} the assertion of the theorem is true, and hence also for

$$G = K_2 \,\Box\, U_{ab} = K_2 \,\Box\, (Q_{d-3} \,\Box\, P_\infty) = (K_2 \,\Box\, Q_{d-3}) \,\Box\, P_\infty = Q_{d-2} \,\Box\, P_\infty.$$

We can therefore assume that U_{ab} is not peripheral. By Lemma 1 we can assume that there is a geodesic ray $P_a = ac_1c_2 \dots c_ic_{i+1} \dots$ in W_{ab}. Since U_{ab} is infinite, it too must contain at least one end.

Since P_a is geodesic the sets $F_{c_{i-1}c_i}$ are mutually disjoint. If they are all infinite, then the number of edges at distance at most n from a is at least $n(n-1)/2$. Since every vertex is incident with just d edges, the number vertices of distance $\leq n$ from a is at least at $n(n-1)/2d$ and G has at least quadratic growth, which is not possible.

Hence, one of the $U_{c_ic_{i+1}}$ is finite. Since it is finite it cannot be peripheral. Thus the set $W_{c_{i+1}c_i}$ contains a ray, which is separated from U_{ab} by a finite cutset. In other words, $ac_1c_2 \cdots c_ic_{i+1} \cdots$ is the representative of an end that is different from the representative of any end in U_{ab}. Similarly there must be a finite $U_{c'_ic'_{i+1}}$ that separates an end from U_{ba}. But then G is not two-ended.

Case 2: All Θ-classes of G are finite.
Let U_{ab} be arbitrarily chosen. Neither U_{ab} nor U_{ba} can be peripheral, otherwise G would be finite. Using Lemma 1 again, there exist two rays P_a and P_b, originating from a, resp. b, that are separated by the finite set U_{ab} and thus represent the two ends of G.

Suppose a has a neighbor a' in U_{ab} that is in turn adjacent to a vertex that is neither in U_{ab} nor in U_{ba}. Then a' is the origin of a ray $P_{a'}$, which must represent the same end as P_a. By Lemma 4, there is a square $aa'c'_1c_1$, where c'_1, c_1 are not in $U_{ab} \cup U_{ba}$.

Now we replace a, a' by c_1, c'_1. Clearly both c_1 and c'_1 have neighbors that are not in $U_{c_1a} \cup U_{ac_1}$. Thus we can find a square $c_1c'_1c'_2c_2$, where c'_2, c_2 are not in $U_{c_2c_1} \cup U_{c_1c_2}$.

Continuing in this manner we arrive at an infinite ladder originating from aa'. We have thus found an infinite Θ-class, hence no neighbor of a is adjacent to a vertex that is neither in U_{ab} nor in U_{ba}.

Let a'' be such a neighbor and consider $F_{aa''}$.

If a vertex $v \neq a$ of P_a meets an edge e of $F_{aa''}$, then there is a ladder L in W_{ab} from aa'' to e. It is possible that aa'' is not the only rung of L that is in U_{ab}. Let xy be the last rung of L in U_{ab}. Thus both vertices x, $y \in U_{ab}$ have neighbors outside of U_{ab} and U_{ba}. But then we can use the arguments we used for aa' for the edge xy to see that L is infinite. Hence a is the only vertex of P_a that meets $F_{aa''}$.

Analogously we show that b is the only vertex of P_b that meets $F_{aa''}$. This means that $U_{aa''}$ separates the two ends P_a and P_b from $W_{a''a}$.

Since $U_{a''a}$ is finite, it is not peripheral. Hence $W_{a''a}$ contains an infinite ray. This ray is separated from the P_a and P_b by a finite set, which implies that G has at least three ends, which is not possible. $\square$

We would like to know whether the condition of two-endedness in Theorem 6 is dispensable and pose this as a problem.

Problem: Is every d-regular median graph G with linear growth of the form

$$Q_{d-2} \,\square\, P_\infty \,?$$

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