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Note

On the fractional chromatic number and the lexicographic product of graphs

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Abstract

For graphs G and H let $G[H]$ be their lexicographic product and let $\chi_f(G) = \inf\{\chi(G[K_n])/n \mid n = 1, 2, \ldots\}$ be the fractional chromatic number of G. For $n \geqslant 1$ set $\mathscr{G}_n = \{G \mid \chi(G[K_n]) = n\chi(G)\}$. Then $\lim_{n\to\infty} \mathscr{G}_n = \{G \mid \chi_f(G) = \chi(G)\}$. Moreover, we prove that for any $n \geqslant 2$ the class $\mathscr{G}_n$ forms a proper subclass of $\mathscr{G}_{n-1}$. As a by-product we show that if G is a χ^*-extremal, vertex transitive graph on $\chi(G)\alpha(G) - 1$ vertices, then for any graph H we have $\chi(G[H]) = \chi(G)\chi(H) - \lfloor \chi(H)/\alpha(G) \rfloor$.

1. Introduction

In this note we consider finite, undirected graphs without loops or multiple edges. As usual, $\chi(G)$ denotes the chromatic number of the graph G and $\alpha(G)$ its independence number. The *lexicographic product* $G[H]$ of graphs G and H has the vertex set $V(G[H]) = V(G) \times V(H)$ and $(a,x)(b,y)$ is an edge of $G[H]$ if $a = b$ and $xy \in E(H)$, or $ab \in E(G)$. The 'product coloring' of $G[H]$ gives immediately $\chi(G[H]) \leqslant \chi(G)\chi(H)$. In the following, we will also (implicitly) use the following result of Geller and Stahl [7]: If $\chi(H) = n$, then for any graph G we have $\chi(G[H]) = \chi(G[K_n])$, where K_n stands for the complete graph on n vertices.

The fractional chromatic number $\chi_f(G)$ of a graph G was introduced by Hilton [10], Rado, Scott [11] and then first studied by Scott [17], Stahl [18], Clarke and Jamison [4], Bollobás and Thomason [2], and others. This graph parameter is also known as the multichromatic number, set-chromatic number, ultimate chromatic number, and can be defined in several ways, cf. [5,11,15,18,21]. For our purposes we introduce

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it as

$$\chi_f(G) = \inf\left\{\frac{\chi(G[K_n])}{n} \mid n = 1, 2, \ldots\right\}.$$

It is interesting to add that $\chi_f(G)$ is also equal to $\lim_{n\to\infty}\sqrt[n]{\chi(G^n)}$, where G^n denotes the nth power of G with respect to the lexicographic product, cf. [9].

Scott [17], see also Stahl [18], showed that $\chi_f(G) = \chi(G[K_k]/k$ for some integer k. (This result is also implicit in [16].) For $n \geqslant 1$ let

$$\mathscr{G}_n = \{G \mid \chi(G[K_n]) = n\chi(G)\}.$$

Observe trivially that $\mathscr{G}_1$ forms the class of all graphs. In this note we are interested in the sequence $(\mathscr{G}_n)_{n=1}^{\infty}$.

Clearly, $\lim_{n\to\infty}\mathscr{G}_n = \{G \mid \chi(G[K_n]) = n\chi(G) \text{ for all } n \geqslant 1\}$. If $G \in \lim_{n\to\infty}\mathscr{G}_n$ then obviously $\chi(G) = \chi_f(G)$. Conversely, if $\chi(G) = \chi_f(G)$ then by the above result of Scott we find that $G \in \lim_{n\to\infty}\mathscr{G}_n$. Hence,

$$\lim_{n\to\infty}\mathscr{G}_n = \{G \mid \chi_f(G) = \chi(G)\}.$$

In the next section we prove that for any $n \geqslant 2$, the class $\mathscr{G}_n$ forms a proper subclass of $\mathscr{G}_{n-1}$. As a byproduct we also show that if G is a χ^*-extremal, vertex transitive graph on $\chi(G)\alpha(G) - 1$ vertices, then for any graph H we have $\chi(G[H]) = \chi(G)\chi(H) - \lfloor\chi(H)/\alpha(G)\rfloor$.

2. Construction using circulant graphs

We will apply a relatively new concept of the star chromatic number of a graph, which was introduced by Vince [19]. Let k and d be two integers with $k \geqslant 2d$. A mapping $c: V(G) \to \{0, 1, \ldots, k-1\}$ is a (k,d)-coloring of G, if for any edge uv of G we have $d \leqslant |c(u) - c(v)| \leqslant k - d$. The star chromatic number of G is then

$$\chi^*(G) = \inf\{k/d \mid G \text{ has a } (k,d)\text{-coloring}\}.$$

For fundamental results on the star chromatic number we refer to [1,3,19,20]. In particular, Zhu [20] proved that if $\chi(H) = n$, then for any graph G with at least one edge $\chi^*(G[H]) = \chi^*(G[K_n])$.

Gao and Zhu [6] called a graph G χ^*-extremal, if $\chi^*(G) = \chi_f(G)$. They proved, among others, the following nice theorem.

Theorem 2.1. *A graph G is a χ^*-extremal graph if and only if for all graphs H we have $\chi^*(G[H]) = \chi^*(G)\chi(H)$.*

The above theorem follows quickly, when we know that the fractional chromatic number is multiplicative on the lexicographic product, i.e., $\chi_f(G[H]) = \chi_f(G)\chi_f(H)$ holds for any graphs G and H. We will use Theorem 2.1 in the following way.

Corollary 2.2. *Let G be a χ^*-extremal, vertex transitive graph on $\chi(G)\alpha(G) - 1$ vertices. Then for any graph H,*

$$\chi(G[H]) = \chi(G)\chi(H) - \left\lfloor \frac{\chi(H)}{\alpha(G)} \right\rfloor .$$

Proof. We have

$$\begin{aligned}
\chi(G[H]) &= \lceil \chi^*(G[H]) \rceil && (1)\\
&= \lceil \chi^*(G)\chi(H) \rceil && (2)\\
&= \lceil \chi_f(G)\chi(H) \rceil && (3)\\
&= \left\lceil \frac{\chi(G)\alpha(G) - 1}{\alpha(G)} \chi(H) \right\rceil && (4)\\
&= \left\lceil \chi(G)\chi(H) - \frac{\chi(H)}{\alpha(G)} \right\rceil \\
&= \chi(G)\chi(H) - \left\lfloor \frac{\chi(H)}{\alpha(G)} \right\rfloor . && (5)
\end{aligned}$$

In the above computation, (1) is well known, cf. [3]. Step (2) is true by Theorem 2.1, while (3) is due to the assumption that G is χ^*-extremal. For Step (4) recall that for a vertex transitive graph G we have $\chi_f(G) = |G|/\alpha(G)$, cf. [6]. Finally, for (5) see [8, Section 3.1]. □

Here we wish to add that the use of the star chromatic number in our construction is not the only way to show that it works. We use it, however, because we believe that the above corollary might be of some independent interest, cf. [13,14]. We also refer to these papers for more results on the chromatic number of graph products, in particular on the lexicographic one.

As already mentioned, we wish to show that $\mathscr{G}_{n-1} \supset \mathscr{G}_n$, for any $n \geqslant 2$. By Corollary 2.2 it suffices to find a χ^*-extremal, vertex transitive graph G on $n\chi(G) - 1$ vertices, where $n = \alpha(G)$. We are going to find such graphs in the class of circulant graphs.

Let N be a set of nonzero elements of $\mathbb{Z}_k$ such that $N = -N$. The circulant graph $G(k,N)$ has vertices $0,1,\dots,k-1$ and i is adjacent to j if and only if $i-j \in N$, where the arithmetic is done mod k. As circulant graphs are Cayley graphs (of Abelian groups) they are clearly vertex transitive. On the other hand, a circulant graph could be, but need not, a χ^*-extremal graph, see. [6].

Let $n \geqslant 3$ and define $G_n = G(3n - 1, \{1,4,\dots,3n - 2\})$.

Lemma 2.3. *For any $n \geqslant 3$ we have $\alpha(G_n) = n$.*

Proof. The vertices $0,3,\dots 3n - 3$ form an independent set of G_n, thus $\alpha(G_n) \geqslant n$.

To prove that $\alpha(G_n) \leqslant n$, let S be a largest independent set of G_n. We may without loss of generality, assume that $0 \in S$. Hence, none of the vertices $1, 4, \ldots, 3n-5, 3n-2$ belongs to S. Moreover, these vertices define a partition of the remaining $2n-2$ vertices of G_n into the following classes:

$$S_1 = \{2,3\}, S_2 = \{5,6\}, \ldots, S_{n-2} = \{3n-7, 3n-6\}, \quad S_{n-1} = \{3n-4, 3n-3\}.$$

Clearly, $|S \cap S_i| \leqslant 1$, for $i = 1, 2, \ldots, n-1$. We conclude that $|S| \leqslant n$. □

Lemma 2.4. *For any $n \geqslant 3$ we have $\chi(G_n) = 3$.*

Proof. Clearly, $\chi(G_n) \geqslant 3$. Define a mapping $c: V(G_n) \to \{1,2,3\}$ as follows:

$$c(i) = i \bmod 3 + 1.$$

By the definition of G_n, if a vertex i is adjacent to a vertex j, then $i \bmod 3 \neq j \bmod 3$. Thus c is a proper 3-coloring of G_n. □

Lemma 2.5. *For any $n \geqslant 3$ we have $\chi_f(G_n) = \chi^*(G_n) = (3n-1)/n$.*

Proof. It is well known, and easy to see, that $\chi_f(G) \leqslant \chi^*(G)$ holds for any graph G, cf. [12,21]. As G_n is vertex transitive, $\chi_f(G_n) = (3n-1)/n$. It remains to show that $\chi^*(G_n) \leqslant (3n-1)/n$. Define

$$c(i) = \left\lfloor \frac{i}{3} \right\rfloor + n \cdot (i \bmod 3).$$

Suppose that i and j are adjacent vertices of G_n. Then $j = (i+1) + 3t$, where $0 \leqslant t \leqslant n-1$. Hence

$$c(j) = \left\lfloor \frac{i+1}{3} \right\rfloor + t + n \cdot ((i+1) \bmod 3).$$

and, therefore, $n \leqslant |c(i) - c(j)| \leqslant 2n$. It follows that c is a $(3n-1, n)$-coloring of G. □

Another way to prove Lemma 2.5 would be to use Theorem 4 of [6]. In this case, however, we would need to introduce one more notion.

Combining Corollary 2.2 with Lemmas 2.3–2.5 we obtain:

Corollary 2.6. *Let $n \geqslant 3$ and let $\chi(H) = m$. Then*

$$\chi(G_n[H]) = 3m - \left\lfloor \frac{m}{n} \right\rfloor.$$

Using Corollary 2.6 we can now prove our main result.

Theorem 2.7. *For any $n \geqslant 2$, the class $\mathcal{G}_n$ forms a proper subclass of $\mathcal{G}_{n-1}$.*

Proof. Suppose that $G \in \mathscr{G}_n$ and let c be an s-coloring of the graph $G[K_{n-1}]$. Then c can be easily extended to an $(s+\chi(G))$-coloring of $G[K_n]$. Hence, $\mathscr{G}_n \subseteq \mathscr{G}_{n-1}$, for any $n \geqslant 2$. To see that the inclusion is strict, for $n \geqslant 3$ use Corollary 2.6 with $H = K_n$ and $H = K_{n-1}$, respectively. Finally, for $n = 2$, the result is well known, see [7], cf. also [14]. (Recall that $\mathscr{G}_1$ is the class of all graphs.) $\square$

To conclude we remark that the mapping

$$c(i, v_j) = i + 1 - 3(j - 1)$$

presents a proper $(3n - 1)$-coloring of the product $G_n[K_n]$, where we have set $V(K_n) = \{v_1, v_2, \ldots, v_n\}$.

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