Dominating Cartesian products of cycles

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Abstract

Let $\gamma(G)$ be the domination number of a graph G and let $G \,\square\, H$ denote the Cartesian product of graphs G and H. We prove that $\gamma(X) = (\prod_{k=1}^{m} n_k)/(2m + 1)$, where $X = C_1 \,\square\, C_2 \,\square\, \cdots \,\square\, C_m$ and all $n_k = |C_k|$, $1 \leqslant k \leqslant m$, are multiples of $2m + 1$. The methods we use to prove this result immediately lead to an algorithm for finding minimum dominating sets of the considered graphs. Furthermore the domination numbers of products of two cycles are determined exactly if one factor is equal to C_3, C_4 or C_5, respectively.

1. Introduction

A set D of vertices of a simple graph G is called *dominating* if every vertex $w \in V(G) - D$ is adjacent to some vertex $v \in D$. The *domination number* of a graph G, $\gamma(G)$, is the order of a smallest dominating set of G. A dominating set D with $|D| = \gamma(G)$ is called a *minimum dominating set*. The *Cartesian product* $G \,\square\, H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(a, x)(b, y) \in E(G \,\square\, H)$ whenever $x = y$ and $ab \in E(G)$, or $a = b$ and $xy \in E(H)$. For $x \in V(H)$ set $G_x = G \,\square\, \{x\}$ and for $a \in V(G)$ set $H_a = \{a\} \,\square\, H$. We call G_x and H_a a *layer* of G or H, respectively.

There are two basic problems on the domination number of Cartesian products of graphs. The first is a conjecture of Vizing [10], namely that $\gamma(G \,\square\, H) \geqslant \gamma(G)\gamma(H)$. The conjecture is still open. See [4–6, 8, 9] for partial results. The second problem is to determine the domination numbers of particular Cartesian products [8, 9]. Also this problem seems to be a difficult one. It is shown in [3] that even for subgraphs of $P_m \,\square\, P_n$ this problem is NP-complete. Furthermore, the complexity of this problem for $P_m \,\square\, P_n$ itself is still open [7].

In Section 2 we determine the domination number of the Cartesian product of two cycles if one cycle is a triangle. We also prove that $\gamma(C_4 \,\square\, C_n) = n$, $\gamma(C_5 \,\square\, C_{5k}) = 5k$ and $\gamma(C_5 \,\square\, C_n) = n + 1$ for $n \in \{5k + 1, 5k + 2, 5k + 4\}$. In addition, $\gamma(C_5 \,\square\, C_{5k+3})$ $\leqslant 5(k + 1)$. In Section 3 we introduce covering graphs to prove that $\gamma(X) = (\prod_{k=1}^m n_k)/(2m + 1)$, where $X = C_1 \,\square\, C_2 \,\square \cdots \square\, C_m$ and all $n_k = |C_k|$, $1 \leqslant k \leqslant m$, are multiples of $2m + 1$. We also give an algorithm for finding minimum dominating sets of the considered graphs.

2. $C_3 \,\square\, C_n$, $C_4 \,\square\, C_n$ and $C_5 \,\square\, C_n$

We emphasize that throughout this paper the vertices of a path P_n or a cycle C_n are always denoted by $0, 1, \ldots, n - 1$. This notation turned out to be convenient to formulate the proof of Lemma 3.2.

Lemma 2.1. (i) *Let $m \geqslant 2$. Then there exists a minimum dominating set D of $P_m \,\square\, P_n$ such that for every $i \in V(P_n)$, $|(P_m)_i \cap D| \leqslant m - 1$.*

(ii) *Let $m \geqslant 3$. Then there exists a minimum dominating set D of $C_m \,\square\, C_n$ such that for every $i \in V(C_n)$, $|(C_m)_i \cap D| \leqslant m - 1$.*

Proof. (i) Let D be a minimum dominating set of $P_m \,\square\, P_n$. Suppose that $|(P_m)_{i_j} \cap D| = m$ holds for k P_m-layers $(P_m)_{i_j}$, $0 \leqslant k \leqslant n - 1$, $1 \leqslant j \leqslant k$. We now construct a dominating set D' with $|D'| = |D|$ such that only $k - 1$ P_m-layers have m vertices in common with D'.

If $|(P_m)_0 \cap D| = m$ then $(P_m)_1 \cap D = \emptyset$ and $D' = (D \cup \{(0, 1)\}) \setminus \{(0, 0)\}$ has the required properties. If $|(P_m)_{n-1} \cap D| = m$ we construct D' analogously.

Assume now that $|(P_m)_i \cap D| = m$ for some $i \notin \{0, n - 1\}$. If in addition $(P_m)_{i-1} \cap D = (P_m)_{i+1} \cap D = \emptyset$, then we set

$$D' = (D \cup \{(0, i - 1), (1, i + 1)\}) \setminus \{(0, i), (1, i)\} .$$

Assume finally that one of those layers, say $(P_m)_{i+1}$, has nonempty intersection with D and let $(j, i + 1) \in (P_m)_{i+1} \cap D$. Then clearly, $(j, i - 1) \notin (P_m)_{i-1} \cap D$ and furthermore $|(P_m)_{i-1} \cap D| < m - 1$. Then D' is given by

$$D' = (D \cup \{(j, i - 1)\}) \setminus \{(j, i)\} .$$

(ii) The proof is the same as above except that we do not have to consider the cases $|(C_m)_0 \cap D| = m$ and $|(C_m)_{n-1}| \cap D = m$ separately. $\square$

We will use Lemma 2.1 for $m = 2$ in the case of paths and for $m = 3$ and $m = 4$ in the case of cycles. For larger m the lemma seems to be useless because we cannot assume the existence of layers with no vertex from a dominating set.

To show the usefulness of Lemma 2.1 we first reprove a theorem of Jacobson and Kinch [8].

Theorem 2.2. $\gamma(P_2 \,\square\, P_n) = \lceil (n + 1)/2 \rceil$.

Proof. To show that $\gamma(P_2 \,\square\, P_n) \leqslant \lceil (n + 1)/2 \rceil$ we use the construction given in [8].

To show that $\gamma(P_2 \,\square\, P_n) \geqslant \lceil (n + 1)/2 \rceil$ it is clearly sufficient to prove that this bound holds for even n. Let $n = 2k$ and let D be a minimum dominating set satisfying Lemma 2.1 (in the case $m = 2$). Then every second P_2-layer must contain a vertex of D. Hence $\gamma(P_2 \,\square\, P_n) \geqslant k$. If there are exactly k layers without a vertex of D, then either $(P_2)_0$ or $(P_2)_{n-1}$ is not dominated by D. Hence $\gamma(P_2 \,\square\, P_n) \geqslant \lceil (n + 1)/2 \rceil$. $\square$

We mention that also [8, Theorem 8] can be proved using similar arguments. We next use Lemma 2.1 in the case $m = 3$ to prove the following theorem.

Theorem 2.3. $\gamma(C_3 \,\square\, C_n) = n - \lfloor n/4 \rfloor$, $n \geqslant 4$.

Proof. Let D consist of vertices $(1, i)$, $i \equiv 0 \pmod 4$, and vertices $(0, i), (2, i)$, $i \equiv 2 \pmod 4$. If $n \equiv 2 \pmod 4$ then add the vertex $(0, n - 1)$ to the set D. It is straightforward to check that D is a dominating set of $C_3 \,\square\, C_n$ and that $|D| = n - \lfloor n/4 \rfloor$.

Next we show that $\gamma(C_3 \,\square\, C_n) \geqslant n - \lfloor n/4 \rfloor$. Let $n = 4k + t, k \geqslant 1, 3 \geqslant t \geqslant 0$, and let D be a minimum dominating set which satisfies Lemma 2.1. Let s be the number of C_3-layers which contain no vertex of D. Then, since no two empty layers are adjacent,

$$k + 1 \leqslant s \leqslant \left\lfloor \frac{n}{2} \right\rfloor = 2k + \left\lfloor \frac{t}{2} \right\rfloor.$$

As every empty C_3-layer is dominated by exactly two other layers, there are at least $\lceil s/2 \rceil$ C_3-layers with precisely two vertices from D. Hence

$$|D| \geqslant 2\left\lceil \frac{s}{2} \right\rceil + \left(n - \left\lceil \frac{s}{2} \right\rceil - s \right)$$

$$= n - \left(s - \left\lceil \frac{s}{2} \right\rceil \right).$$

Also $s - \lceil s/2 \rceil$ is maximal when $s = 2k + \lfloor t/2 \rfloor$. So

$$|D| \geqslant n - \left(2k + \left\lfloor \frac{t}{2} \right\rfloor - \left\lceil \frac{2k + \lfloor t/2 \rfloor}{2} \right\rceil \right)$$

$$= 3k + t - \left\lfloor \frac{t}{2} \right\rfloor + \left\lceil \frac{\lfloor t/2 \rfloor}{2} \right\rceil$$

$$= 3k + t = n - \left\lfloor \frac{n}{4} \right\rfloor. \quad \square$$

Lemma 2.4. *Let D be a subset of $V(C_4 \square P_{2n+1})$, $n \geqslant 1$, such that*

(i) $|(C_4)_i \cap D| \leqslant 3, 0 \leqslant i \leqslant 2n$,

(ii) $(C_4)_1 \cap D = (C_4)_3 \cap D = \cdots = (C_4)_{2n-1} \cap D = \emptyset$, *and*

(iii) *the layers $(C_4)_1, (C_4)_3, \ldots, (C_4)_{2n-1}$ are dominated by D.*

Then $|D| \geqslant 2n + 1$.

Proof. As $(C_4)_{2i+1} \cap D = \emptyset, 0 \leqslant i \leqslant n-1$, and all $(C_4)_{2i+1}$ are dominated by D it follows that $|(C_4)_{2i} \cap D| \geqslant 1$ and $|((C_4)_{2i} \cup (C_4)_{2(i+1)}) \cap D| \geqslant 4$ for each $0 \leqslant i \leqslant n-1$. Thus,

$$\sum_{i=0}^{n-1} |((C_4)_{2i} \cup (C_4)_{2(i+1)}) \cap D| \geqslant 4n$$

and hence

$$|D| \geqslant \frac{4n + |(C_4)_0 \cap D| + |(C_4)_{2n} \cap D|}{2} \geqslant 2n + 1. \qquad \square$$

Theorem 2.5. $\gamma(C_4 \square C_n) = n, n \geqslant 4$.

Proof. Let D contain the vertices $(0, i), i \equiv 0 \pmod 2$, and vertices $(2, i), i \equiv 1 \pmod 2$, $0 \leqslant i \leqslant n-1$. It is straightforward to verify that D is a dominating set of $C_4 \square C_n$, thus $\gamma(C_4 \square C_n) \leqslant n$.

In the sequel we prove that $\gamma(C_4 \square C_n) \geqslant n$. Let D be a minimum dominating set of $C_4 \square C_n$ and let D satisfy Lemma 2.1. Nothing has to be shown if $|(C_4)_i \cap D| \geqslant 1$ for all i. Suppose therefore that there are s C_4-layers, $s \geqslant 1$, containing no vertex of D. Let $(C_4)_{i_1}, (C_4)_{i_2}, \ldots, (C_4)_{i_s}$ be the empty layers, where $i_1 < i_2 < \cdots < i_s$. By Lemma 2.1 there are no adjacent empty layers. Hence $s \leqslant n/2$. We now consider two cases.

Case 1: $s = n/2$. Clearly $n = 2k$ for some $k \geqslant 1$ in this case. Without loss of generality we assume that $(C_4)_0, (C_4)_2, \ldots, (C_4)_{n-2}$ are the empty layers. Then for each $0 \leqslant i \leqslant k-1$, $|(C_4)_{2i+1} \cap D| \geqslant 2$ and thus it follows that $|D| \geqslant 2k = n$.

Case 2: $s < n/2$. In this case there are indices i_j and i_{j+1} such that $i_{j+1} - i_j \geqslant 3$. (If necessary, all the arithmetic should be done over appropriate module.) This fact allows us to make a partition of $C_4 \square C_n$ in the following way. For every empty layer, say $(C_4)_i$, take a maximal sequence of empty layers including $(C_4)_i$ such that the distance (in C_n) between consecutive empty layers is 2. Add to such a sequence all the layers that dominate the empty layers in the sequence. Such a sequence then forms a part of our partition. Finally, every nonempty layer which is not adjacent to any empty layer forms a part for itself. It is clear that we have a partition of $C_4 \square C_n$. Furthermore, by the definition and by Lemma 2.4 every part of the partition contains at least as many vertices of D as it contains C_4-layers. Hence $|D| \geqslant n$ again holds. $\qquad \square$

We mention that $\gamma(P_4 \square P_n) \geqslant n$, which was first shown in [8, Lemma 10], immediately follows from this result.

We now consider the product $C_5 \square C_n$ and define a set D_1 as follows: D_1 consists of vertices $(0, i)$, $i \equiv 0 \pmod 5$; $(2, i)$, $i \equiv 1 \pmod 5$; $(4, i)$, $i \equiv 2 \pmod 5$; $(1, i)$, $i \equiv 3 \pmod 5$ and $(3, i), i \equiv 4 \pmod 5$. Note that $|D_1| = n$.

Theorem 2.6. *Let $n \geqslant 5$. Then*

$$\gamma(C_5 \square C_n) = \begin{cases} n, & n = 5k, \\ n + 1, & n \in \{5k + 1, 5k + 2, 5k + 4\}. \end{cases}$$

Furthermore, $\gamma(C_5 \square C_{5k+3}) \leqslant 5(k + 1)$.

Proof. If $n = 5k$ then D_1 is a dominating set of $C_5 \square C_n$. As for any graph $G, \gamma(G) \geqslant |V(G)|/(\Delta(G) + 1)$, we conclude that $\gamma(C_5 \square C_n) = n$.

If $n \in \{5k + 1, 5k + 2, 5k + 4\}$ we add the vertex $(3, n - 1)$ to D_1 to obtain a dominating set with $n + 1$ vertices. If $n = 5k + 3$ we finally add the vertices $(1, n - 1)$ and $(3, n - 1)$ to D_1 and thus obtain a dominating set of $C_5 \square C_n$ with $n + 2$ vertices.

Let $n \in \{5k + 1, 5k + 2, 5k + 4\}$ and assume that $\gamma(C_5 \square C_n) = n$. Let D be a corresponding dominating set. We may assume that $(0, 0) \in D$. If $\gamma(C_5 \square C_n) = n$ then D must also be independent and every vertex not in D is dominated by exactly one vertex of D. In particular this implies that $(2, 0) \notin D$ and that it is dominated by exactly one vertex of D. Without loss of generality, we may assume $(2, 1) \in D$. Consider the vertex $(4, 1)$. We have only one possibility to dominate this vertex, thus $(4, 2) \in D$ and so forth $(1, 3) \in D, (3, 4) \in D, (0, 5) \in D, \dots$. We conclude that $D = D_1$, but note that since $n \neq 5k, (3, n - 1) \notin D_1$. Since $(3, 0)$ must be dominated by $(3, n - 1)$, we have a contradiction. $\square$

3. Domination numbers and covering graphs

In this section we use covering graphs to obtain additional results on domination numbers of products of cycles. The following construction of a covering graph of a graph X with respect to a group G can be found in [1] : Each edge $uv \in E(X)$ gives rise to two 1-arcs, $[u, v]$ and $[v, u]$. By $A(X)$ we denote the set of 1-arcs and by $\varphi: A(X) \to G$ we denote a mapping such that $\varphi([u, v]) = (\varphi([v, u]))^{-1}$ for all $[u, v] \in A(X)$. The covering graph $\tilde{X} = \tilde{X}(G, \varphi)$ of X with respect to G is defined on the vertex set $V(\tilde{X}) = G \times V(X)$ and two vertices $(g_1, u), (g_2, v) \in V(\tilde{X})$ are adjacent in $\tilde{X}$ if and only if $[u, v] \in A(X)$ and $g_2 = g_1 \varphi([u, v])$.

As a first example of how to use covering graphs we present the following proposition.

Proposition 3.1. $\gamma(C_{4k} \square C_n) \leqslant kn$.

Proof. Let G be a cyclic group of order k and let a be its generator. Let $Y = C_4 \square C_n$. We fix one edge of the cycle C_4, say e, and define the mapping φ as follows: Each copy

of e in Y is mapped onto a and a^{-1}, respectively, and all other edges of Y are mapped onto the unit element of G. Then the covering graph $\tilde{Y}(G, \varphi)$ is clearly isomorphic to $X = C_{4k} \,\Box\, C_n$. Furthermore, $\tilde{D} = \{(g, d) \,|\, g \in G, d \in D\}$ is a dominating set of X if D is a dominating set of Y. Hence $\gamma(X) \leqslant kn$.   □

Note that the inequality of Proposition 3.1 might be proper. For example, $\gamma(C_{4k} \,\Box\, C_5) \leqslant 4k + 2$ by Theorem 2.6 while the above proposition only claims that $\gamma(C_{4k} \,\Box\, C_5) \leqslant 5k$.

Lemma 3.2. *Let* $X = C_1 \,\Box\, C_2 \,\Box\, \cdots \,\Box\, C_m$ *where* $|C_i| = 2m + 1$ *for each* C_i, $1 \leqslant i \leqslant m$. *Then* $\gamma(X) = (2m + 1)^{m-1}$.

Proof. Since X is regular of degree $2m$, $\gamma(X) \geqslant (2m + 1)^{m-1}$. We now show that a dominating set with $(2m + 1)^{m-1}$ vertices really exists. For convenience we set $n = 2m + 1$. The vertices of G are given by the vectors $(i_1, i_2, \ldots, i_m)$ where each i_j, $1 \leqslant j \leqslant m$, runs through all integers from 0 to $n - 1$.

By $K(i_2, \ldots, i_m)$, $i_j \in \{1, \ldots, n\}$, $2 \leqslant j \leqslant m$, we denote the layers of C_1 in X. We first determine a set D of vertices of X which dominates all vertices of $K(0, \ldots, 0)$. Furthermore, we determine this set in a way such that we can easily extend it to a dominating set of X which contains n^{m-1} vertices. In the sequel all sums concerning entries of the vectors which represent the vertices of X are taken modulo n. We start with $D = \{(0, 0, \ldots, 0)\}$. This vertex dominates $(1, 0 \ldots, 0)$ and $(n - 1, 0, \ldots, 0)$. Then we join all vertices $(2i, i, 0, \ldots, 0)$, $1 \leqslant i \leqslant n - 1$, to D. Clearly the vertex $(2, 1, 0, \ldots, 0)$ dominates $(2, 0, 0, \ldots, 0)$ and $(n - 2, n - 1, 0, \ldots, 0)$ dominates the vertex $(n - 2, 0, 0, \ldots, 0)$. In general, the vertices of D are determined by the following algorithm:

$D := \{(0, \ldots, 0)\}$
for $k = 2$ **to** m **do begin**
    set all entries of *vector* to 0;
    for $i = 1$ **to** $n - 1$ **do begin**
    vector[1] := (*vector*[1] $+ k$) mod n;
    vector[k] := i;
    $D := D \cup$ *vector*
  end
end

Observe first that the above procedure is cyclically closed, i.e. the next vertex after the end of the procedure would be the starting vertex. Of course, the vertices with 1 as their kth entry always dominate the vertices $(k, 0, \ldots, 0)$ and the vertices with $n - 1$ as their kth entry always dominate the vertices $(k \cdot (n - 1), 0, \ldots, 0)$. If $k = m$ then $(m, 0, 0, \ldots, 0)$ is dominated by the vertex with 1 as its mth entry and the vertex $(m + 1, 0, \ldots, 0)$ is dominated by $(m + 1, 0, \ldots, n - 1)$ since $2m^2 \equiv m + 1$ $(\text{mod}(2m + 1))$. It follows that all vertices of $K(0, 0, \ldots, 0)$ are dominated by D.

We now obtain a dominating set D of X as follows: Take any cycle $K(x_2, \ldots, x_n)$ which has nonempty intersection with D but is not dominated by D. Then apply the above given procedure to $K(x_2, \ldots, x_n)$ with that vertex $(x_1, x_2, \ldots, x_n)$ of $K(x_2, \ldots, x_n)$ as starting point which is already contained in D. The set D we thus obtain clearly dominates $K(x_2, \ldots, x_n)$. If it does not dominate the whole graph, then we again choose a cycle $K(y_2, \ldots, y_n)$ which has nonempty intersection with D but is not dominated by D and apply the above algorithm, etc. Hence we finally end with a dominating set D of X which contains exactly one vertex of each cycle $K(i_2, \ldots, i_n)$, $0 \leqslant i_j \leqslant n - 1$, $2 \leqslant j \leqslant m$. For completeness, we finally present the above procedure in its general form:

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D := {(0, …, 0)};
while D does not dominate X do begin
    choose any cycle K(x₂, …, xₙ) not dominated by D
    but has nonempty intersection with D;
    vector := K(x₂, …, xₙ) ∩ D;
        for k = 2 to m do begin
        vector x := vector;
            for i = 1 to n − 1 do begin
                vector[1] := (vector[1] + k) mod n;
                vector[k] := (vector[k] + i) mod n;
                D := D ∪ vector
            end
        end
        vector := vector x
    end
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This completes the proof.   □

Theorem 3.3. *Let* $X = C_1 \square C_2 \square \cdots \square C_m$ *such that all* $n_k = |C_k|$, $1 \leqslant k \leqslant m$, *are multiples of* $2m + 1$. *Then*

$$\gamma(X) = \frac{\prod_{k=1}^{m} n_k}{2m + 1}.$$

Proof. Let $n_k = r_k \cdot (2m + 1)$, $r_k \geqslant 1$, $1 \leqslant k \leqslant m$, and let G be a group defined by $G = \langle a_1 \rangle \times \cdots \times \langle a_m \rangle$, where a_k has order r_k. Hence, if $r_j = 1$ for some $j \in \{1, \ldots, m\}$, we set $a_j = e$.

Let $Y = S_1 \square S_2 \square \cdots \square S_m$, where each S_k, $1 \leqslant k \leqslant m$, is a cycle of length $2m + 1$. We now fix one edge, say e_k, of each cycle S_k, $1 \leqslant k \leqslant m$, and construct a covering graph $\tilde{Y}(G, \varphi)$ where φ is given as follows: Each copy of e_k in Y is mapped onto a_k and a_k^{-1}, respectively. Then the covering graph $\tilde{Y}(G, \varphi)$ is isomorphic to X. Also $\tilde{D} = \{(g, d) \mid g \in G, d \in D\}$ is a dominating set of X if D is a dominating set of Y. Hence, using Lemma 3.2, $\gamma(X) \leqslant (2m + 1)^{(m-1)} \cdot \prod_{k=1}^{m} r_k$, which immediately implies our result.   □

Minor alterations in the algorithm given in the proof of Lemma 3.2 immediately lead to an algorithm to determine minimum dominating sets of the graphs considered in the above theorem. Also, minor modifications of the same algorithm lead to a simple procedure to find small – if not the smallest – dominating sets of products of two cycles.

In [8] Jacobson and Kinch proved that $\lim_{m,n \to \infty} \gamma(P_m \,\square\, P_n)/mn = \frac{1}{5}$. As $\frac{1}{5} \leqslant \gamma(C_m \,\square\, C_n)/mn \leqslant \gamma(P_m \,\square\, P_n)/mn$ we also have the following proposition.

Proposition 3.4. $\lim_{m,n \to \infty} \gamma(C_m \,\square\, C_n)/mn = \frac{1}{5}$.

We finally mention that it is not difficult to show that $\gamma(C_m \,\square\, C_n) \leqslant mn/4$ if $m, n \geqslant 4$. This can be done by applying the ideas of this paper as well as with the help of the results presented in [2].

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