Peripheral Wiener Index of Iterated Line Graphs

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Abstract

The peripheral Wiener index $\mathrm{PW}(G)$ of a graph G is the restriction of the classical Wiener index to the periphery of G. The graph G is called k-PW-line-stable (k-stable for short), if $\mathrm{PW}(L^k(G)) = \mathrm{PW}(G)$, where $L^k(G)$ denotes the k^{th} iterated line graph of G. In this paper, infinite families with the following properties are constructed: three families of 2-stable trees, a family of k-stable trees for every $k \geq 4$, and a family of 2-stable bicyclic graphs. It is also demonstrated that for each $k \geq 3$ and n large enough with respect to k, there exist a k-stable bicyclic graph of order n. Along the way, the research is supported with intensive computer calculations. In particular, all 2-stable trees of order at most 14, all 2-stable unicyclic graphs of order up to 10, and all 2-stable bicyclic graphs of order up to 9 are listed.

Keywords: Wiener index; peripheral Wiener index; iterated line graph

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1 Introduction

If $G = (V(G), E(G))$ is a (connected) graph, then $d_G(x, y)$ denotes the usual shortest path distance between vertices x and y in G. The *eccentricity* $\mathrm{ecc}_G(x)$ of a vertex $x \in V(G)$ is the distance to a furthest vertex from x in G. The *diameter* $\mathrm{diam}(G)$ of G is the maximum eccentricity of G. A vertex x with $\mathrm{ecc}_G(x) = \mathrm{diam}(G)$ is *peripheral*. The set of peripheral vertices of G is the *periphery* of G and denoted by $P(G)$.

The Wiener index is one of the central concepts in metric graph theory as well as in chemical graph theory, see the surveys [9, 11, 13]. For a graph G, it is defined by

$$W(G) = \sum_{\{x,y\} \subseteq V(G)} d_G(x, y)\,.$$

In this paper, we are interested in the variation of this index, the *peripheral Wiener index*, defined by

$$\mathrm{PW}(G) = \sum_{\{x,y\} \subseteq P(G)} d_G(x, y)\,.$$

The peripheral Wiener index was introduced in [16]. It was subsequently studied in [3] on trees, and in [8], where upper and lower bounds were proved and the difference between the Wiener index and the peripheral Wiener index investigated. The peripheral Wiener index has been further studied in various families of graphs; here we refer to [20, 23, 24]. It should be noted that the peripheral hyper-Wiener index has recently also been studied [19].

The *line graph* $L(G)$ of a graph G has vertex set $V(L(G)) = E(G)$, two vertices of $L(G)$ being adjacent if the corresponding edges of G are adjacent. This fundamental graph theory concept has numerous applications in chemistry [6, 7, 17], and remains a subject of research interest [14, 27, 28]. Different topological indices have also been investigated on line graphs, we would like to direct the reader to such investigations of the Steiner Wiener index [10, 21], general Randić indices [12], the first general Zagreb index [4], Kirchoff index [22], the general sum-connectivity index [1, 25], and the harmonic index [26].

One of the main topics of [15] was a study of cycle containing graphs G with the property $\mathrm{PW}(G) = \mathrm{PW}(L(G))$. It was proved that for every graph G there exists a graph H such that G is an induced subgraph of H and $\mathrm{PW}(H) = \mathrm{PW}(L(H))$. Infinite families of graphs G with an arbitrary cyclomatic and with $\mathrm{PW}(G) = \mathrm{PW}(L(G))$ were also constructed. In this article, to continue along this line of inquiry, the following concepts are necessary.

Let k be a positive integer. The k^{th} *iterated line graph* $L^k(G)$ is recursively defined by $L^1(G) = L(G)$ and $L^k(G) = L(L^{k-1}(G))$, $k \geq 2$. See [2, 5] for some studies of this construction. Now, we say that a graph G is *k-PW-line-stable*, if

$$\mathrm{PW}(L^k(G)) = \mathrm{PW}(G)\,.$$

Since only this type of stability is considered, the terminology below will simply refer to G as *k-stable*.

The paper is organized as follows. The next section examines the distance function in line graphs, providing the necessary tools for the subsequent developments. In Section 3, three infinite families of 2-stable trees are constructed. An infinite family of 3-stable trees is also obtained, together with families of k-stable trees for every $k \geq 4$. Section 4 presents an infinite family of 2-stable bicyclic graphs and establishes that, for every $k \geq 3$ and sufficiently large n relative to k, there exists a k-stable bicyclic graph of order n.

Along the way, extensive computer calculations are used to support the results. In particular, all 2-stable trees of order at most 14, all 2-stable unicyclic graphs of order at most 10, and all 2-stable bicyclic graphs of order at most 9 are determined. Furthermore, an exhaustive computer search establishes that no 4-stable tree of order at most 18 exists.

2 On the distance function in line graphs

This section develops the tools required for calculating distances in iterated line graphs, which will be used in the subsequent analysis.

The following definitions, which were introduced in the seminal work [18] on distance functions in iterated line graphs, are recalled first. Let H be a nontrivial subgraph of G. Let $L^1(H,G)$ be the subgraph of $L(G)$ induced by the vertices which are edges of H. Inductively, set $L^k(H,G) = L^{k-1}(L(H,G),L(G))$, that is, $L^k(H,G)$ denotes the subgraph of $L^k(G)$ induced by vertices that are edges of H, $k \geq 1$. Let G be a graph and let v be a vertex in $L^k(G)$. By the *i-butt* $B_i^k(v)$ of the vertex v in $L^k(G)$ we mean the subgraph of $L^{k-i}(G)$ induced by the edges involved into the recursive definition of the vertex v. As observed in [18], a vertex v of $L^k(G)$ lies in $L^k(H,G)$ if and only if $B_k^k(v) \subseteq H$.

Lemma 2.1. [18, Lemma 6] *Let G be a connected graph, and let $L^k(G)$ exist for an integer $k \geq 1$. If u and v are distinct vertices in $L^k(G)$, then the following hold.*

- *If the k-butts of u and v are edge-disjoint, then $d(u,v) = k + d(B_k^k(u), B_k^k(v))$.*
- *If the k-butts of u and v have a common edge, then*

$$d(u,v) = \max\{t :\ t\text{-butts of } u \text{ and } v \text{ are edge-disjoint}\}.$$

We will also need the following consequence:

Corollary 2.2. *Let H be a graph containing pendant paths P and Q of length p and q, respectively, which are attached to a subgraph G in respective vertices u_0 and v_0, see Fig. 1. If $k < p, q$, and u and v are the vertices in which the two paths corresponding to P and Q in $L^k(H)$ are originated, then,*

$$d_{L^k(H)}(u, v) = k + d_H(u_0, v_0)\,.$$

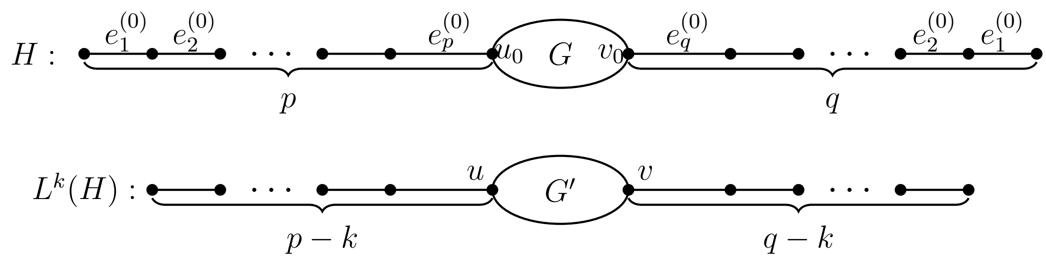


Figure 1: Graph H with pendant paths P and Q and its k^{th} iterated line graph.

Proof. Let $P^{(i)} : u_{i,0}u_{i,1}\dots u_{i,p-i}$, $i \le k$, be the pendant path in $L^i(H)$ that corresponds to P, and let $e_t^{(i)} = u_{i,t-1}u_{i,t}$, $0 \le t \le p-i$. Then by an iterative argument we infer that $B_k^k(u_{kp-k})$ is induced by the the vertex set $\{e_{p-k+1}^{(0)}, e_{p-k+2}^{(0)}, \dots, e_p^{(0)}\}$. Similarly, if $Q^{(i)} : v_{i0}v_{i1}\dots v_{iq-i}$ is a pendant path in $L^i(H)$ which corresponds to Q, then $B_k^k(v_{kq-k})$ is induced by the vertex set $\{\epsilon_{q-k+1}^{(0)}, \epsilon_{q-k+2}^{(0)}, \dots, \epsilon_q^{(0)}\}$, where $\epsilon_t^{(i)} = v_{it-1}v_{it}$. Since $B_k^k(u_{kp-k})$ and $B_k^k(v_{kq-k})$ are edge disjoint, Lemma 2.1 gives

$$\begin{aligned} d_{L^k(H)}(u_{k,p-k}, v_{k,q-k}) &= k + d_H(B_k^k(u_{kp-k}), B_k^k(v_{kq-k})) \\ &= k + d_H(u_{0,p}, v_{0,q})\,, \end{aligned}$$

where $u_{0,p}$ and $v_{0,q}$ are vertices in G in which paths P and Q are respectively attached. □

3 Families of k-stable trees

At the beginning of this section, all 2-stable trees of order at most 14 are presented. Three infinite families of 2-stable trees are then constructed. Subsequently, an infinite family of 3-stable trees is obtained, together with families of k-stable trees for every $k \geq 4$. Interestingly, an exhaustive computer search establishes that no 4-stable tree of order at most 18 exists.

An exhaustive computer search determines all 2-stable trees of order at most 14. The computations require less than one minute on a standard personal computer. The tree shown in Fig. 2 is the unique 2-stable tree of order at most 8, while Table 1 lists all 2-stable trees of orders 9 through 14. Each tree T is labeled as $n_s : t$, where n denotes the order of T, s is a counter for the 2-stable trees of that order, and $t = \mathrm{PW}(T)$.

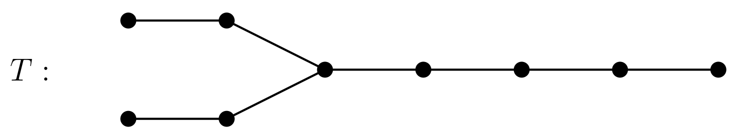
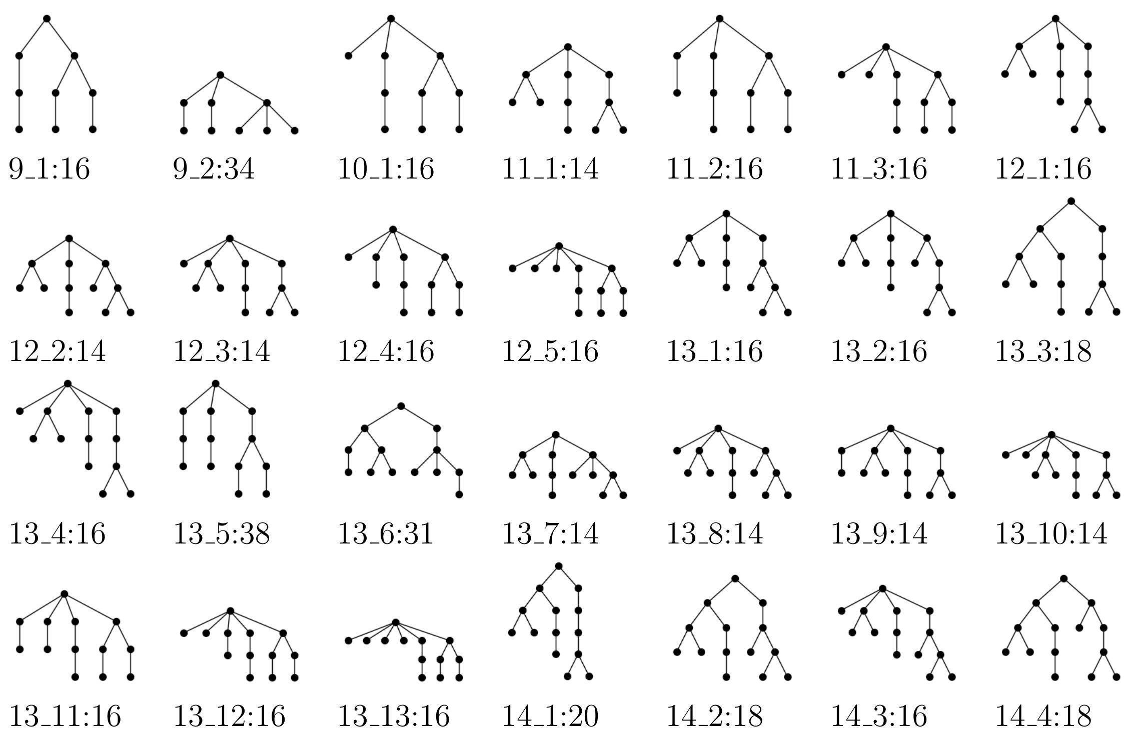
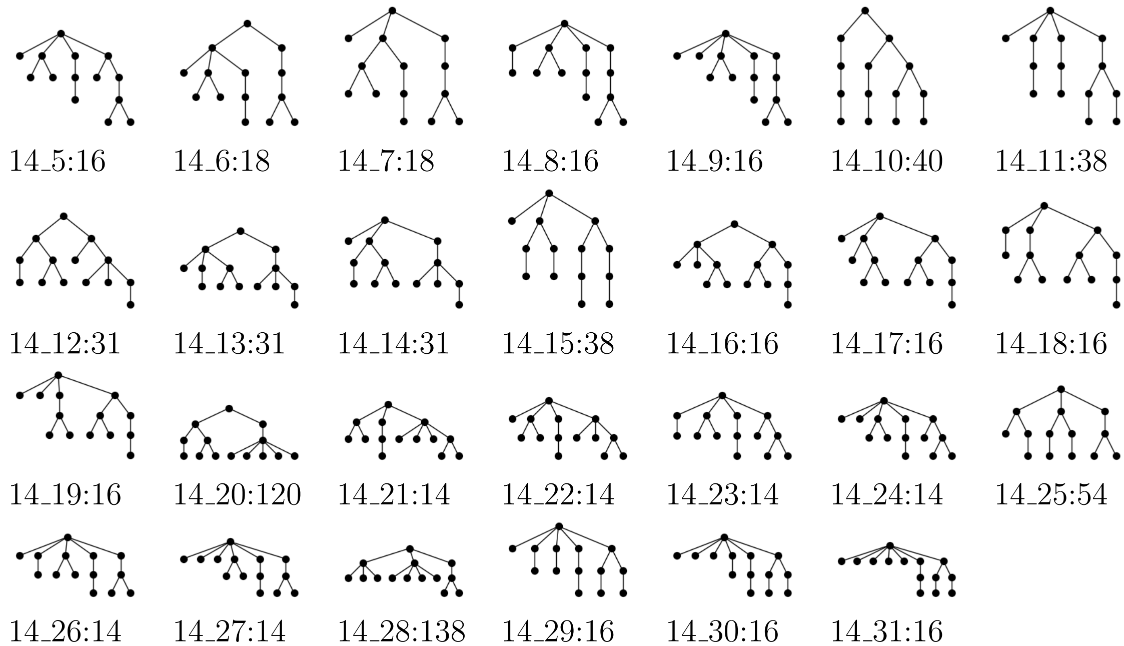


Figure 2: The smallest 2-stable tree.

Table 1: 2-stable trees of order between 9 and 14





With Table 1 in hand, the next goal is to determine infinite families of 2-stable trees. For this sake consider the trees $T_{s,t,d}$, $s, t \geq 0$, $d \geq 5$, as presented in Fig. 3. Note that $T_{2,0,5}$ is the tree T from Fig. 2, $T_{2,1,6}$ is the graph 9_1:16, and $T_{2,2,7}$ is the graph 13_5:38.

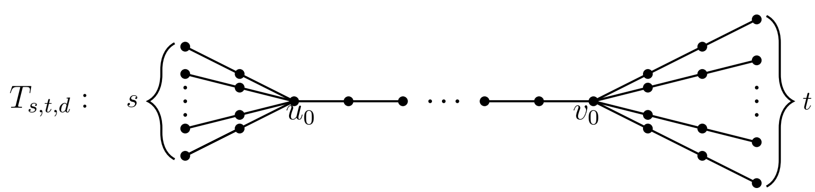


Figure 3: Trees $T_{s,t,d}$.

Theorem 3.1. *Trees $T_{2,d-5,d}$, $d \geq 5$, and $T_{2d-6,2d^3-17d^2+45d-36,d}$, $d \geq 6$, are 2-stable. Moreover, for every $s \geq 3$, the number of 2-stable trees $T_{s,t,d}$ is finite.*

Proof. Let $s, t \geq 1$ and $d \geq 5$. For the rest of the proof set $T = T_{s,t,d}$ and $d = \mathrm{diam}(T)$. Let u_0 be the vertex in which s pendant paths of length 2 are attached, and let v_0 be the vertex in which t pendant paths of length 3 are attached, see Fig. 3.

Clearly, $|P(T)| = s + t$, and

$$\mathrm{PW}(T) = 4\binom{s}{2} + 6\binom{t}{2} + std\,. \tag{1}$$

Edges incident to u_0 and v_0 respectively form complete subgraphs of order $s+1$ and $t+1$, we denote them by $K_{s+1}^{(u_0)}$ and $K_{t+1}^{(v_0)}$. Every vertex in $K_{s+1}^{(u_0)}$ has one pendant vertex except one, say u_1, which is adjacent to a vertex correspond to an edge on the u_0, v_0-path. Similarly, every vertex in $K_{t+1}^{(v_0)}$ has one pendant path of length 2 except one, say v_1, which is adjacent to a vertex correspond to an edge on the u_0, v_0-path.

There are $s + \binom{s+1}{2} = s + \binom{s}{2} + s$ vertices in $L^2(T)$ corresponding to the edges of $K_{s+1}^{(u_0)}$ such that s of them are adjacent to a vertex corresponding to an edge on the u_1, v_1-path, the remaining $s + \binom{s}{2}$ vertices are in $P(L^2(T))$. Let X be the set containing those $s + \binom{s}{2}$ peripheral vertices. We can write $X = \cup_{i=1}^{s} X_i$, where for all X_i, there exist a unique $x_i \in X_i$ such that $x_i \notin X_j$, $i \neq j$, and

$$\begin{aligned}
|X_i \cap X_j| &= 1,\ i \neq j\,,\\
X_i \cap X_j \cap X_k &= \emptyset,\ |\{i,j,k\}| = 3\,,\\
|X_i| &= 1 + s - 1 = s\,.
\end{aligned}$$

Let $\{v_{ij}\} = X_i \cap X_j, 1 \leq i < j \leq s$, and $A = \{x_1, \ldots, x_s\}$.

There are t pendant vertices in $L^2(T)$ which correspond to the t pendant paths of length 2 in $K_{t+1}^{(v_0)}$ and they are in $P(L^2(T))$. Let Y be the set containing those t pendant vertices. Then,

$$P(L^2(T)) = X \cup Y, \quad |X| = s + \binom{s}{2}, \quad |Y| = t\,.$$

Setting $\sigma_P(u)$ to denote the sum of the distance between u and all the vertices from $P = P(L^2(T))$ we have:

$$\begin{aligned}
\sigma_P(x_i) &= \sum_{v \in P} d(x_i, v)\\
&= (s-1) + 2\Big(s + \binom{s}{2} - (s-1) - 1\Big) + t(d-2)\\
&= (s-1)(s+1) + t(d-2), && 1 \leq i \leq s,\\
\sigma_P(v_{ij}) &= 2(s-1) + 2\Big(s + \binom{s}{2} - 2(s-1) - 1\Big) + t(d-2)\\
&= 2\binom{s}{2} + t(d-2), && 1 \leq i < j \leq s,
\end{aligned}$$

$$\sigma_P(u) = 4(t-1) + \big(s + \binom{s}{2}\big)(d-2), \qquad\qquad u \in Y.$$

From here, we can calculate as follows:

$$\begin{aligned}
\mathrm{PW}(L^2(T)) &= \frac{1}{2}\sum_{x\in P(T)} \sigma_P(x) = \frac{1}{2}\Big[\sum_{x\in X}\sigma_P(x) + \sum_{x\in Y}\sigma_P(x)\Big] \\
&= \frac{1}{2}\Big[\sum_{x\in A}\sigma_P(x) + \sum_{x\in X-A}\sigma_P(x) + \sum_{x\in Y}\sigma_P(x)\Big] \\
&= \frac{1}{2}\Big[s[(s-1)(s+1) + t(d-2)] + \binom{s}{2}[2\binom{s}{2} + t(d-2)] \\
&\quad + t[4(t-1) + \big(s + \binom{s}{2}\big)(d-2)]\Big] \\
&= \frac{1}{2}\Big[\binom{s}{2}[2(s+1) + 2\binom{s}{2}] + 8\binom{t}{2} + 2t\Big(s + \binom{s}{2}\Big)(d-2)\Big] \\
&= \binom{s}{2}\Big[(s+1) + \binom{s}{2}\Big] + 4\binom{t}{2} + t\Big(s + \binom{s}{2}\Big)(d-2)\,. \qquad (2)
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathrm{PW}(L^2(T)) - \mathrm{PW}(T) &= \frac{3s}{2} - \frac{7s^2}{4} + \frac{s^4}{4} + t - st - \frac{dst}{2} - s^2t + \frac{ds^2t}{2} - t^2 \\
&= \frac{-1}{4}\big[4t^2 + [2d(s-s^2) + 4(s^2+s) - 4]t - s(s^3 - 7s + 6)\big] \\
&= -[t^2 + [\frac{d}{2}(s-s^2) + (s^2+s) - 1]t - \frac{s}{4}(s-1)(s-2)(s+3)\big]\,.
\end{aligned}$$

If $\mathrm{PW}(L^2(T)) - \mathrm{PW}(T) = 0$, then

$$t = \frac{\frac{d}{2}(s^2-s) - (s^2+s) + 1 \pm \sqrt{\big(\frac{d}{2}(s^2-s) - (s^2+s) + 1\big)^2 + s(s-1)(s-2)(s+3)}}{2}\,. \qquad (3)$$

If $s = 1$, then $t \in \{0, -1\}$, hence (3) has no suitable solution. If $s = 2$, then $t = d-5$, hence $T_{2,d-5,d}$, $d \geq 5$, is 2-stable. Assume in the rest that $s \geq 3$. If (3) has an integer solution, then the square root must be an integer, let it be y^2. Set

$$y^2 = (f(s)d + g(s))^2 + h(s)\,, \qquad (4)$$

where $f(s) = \frac{s}{2}(s-1)$, $g(s) = -s(s+1)+1$, and $h(s) = s(s-1)(s-2)(s+3)$. Then

$$h(s) = y^2 - \big(f(s)d + g(s)\big)^2 = \big(y - (f(s)d + g(s))\big)\big(y + (f(s)d + g(s))\big)\,.$$

Setting

$$a = y - (f(s)d + g(s)) \,,$$
$$b = y + (f(s)d + g(s)) \,,$$

we have $h(s) = ab$. From the two equations above, we can see that

$$d = \frac{\left(\frac{b-a}{2} - g(s)\right)}{f(s)} = \frac{b - a + 2s(s+1) - 2\big)}{s(s-1)} \,.$$

Since $h(s) = s(s-1)(s-2)(s+3)$ is nonzero for $s \geq 3$, the number of pairs a, b is finite. We may conclude that for each fixed $s \geq 3$, the number of 2-stable trees $T_{s,t,d}$ is finite.

Finally, by a direct computation one can verify that the trees

$$T_{2d-6,2d^3-17d^2+45d-36,d}, \; d \geq 6,$$

are 2-stable. □

Using (3) and computer, all 2-stable trees from Fig. 3 with $3 \leq s \leq 50$ are presented in Table 2. For each of these trees T, the table gives its parameters s and t, the order n, the diameter d, and the peripheral Wiener index PW(T).

Table 2: All possible solutions of (3) for $3 \leq s \leq 50$.

s	d	t	n	PW(T)	s	d	t	n	PW(T)
4	10	42	140	6870	26	16	4524	13636	63269440
6	6	54	176	10590	28	17	5642	16995	98166670
8	7	132	415	59380	30	18	6930	20864	147797850
10	8	260	804	223000	32	19	8400	25279	216763984
11	5	165	518	90475	34	20	10064	30276	310667860
12	9	450	1379	655014	36	21	11934	35891	436249890
14	10	714	2176	1627570	38	22	14022	42160	601532590
16	11	1064	3231	3580840	40	23	16340	49119	815973700
18	12	1512	4580	7181100	42	24	18900	56804	1090627944
20	13	2070	6259	13387450	44	25	21714	65251	1438317430
21	6	855	2609	2299080	46	26	24794	74496	1873810690
22	14	2750	8304	23527174	48	27	28152	84575	2414010360
24	15	3564	10751	39379740	50	28	31800	95524	3078149500

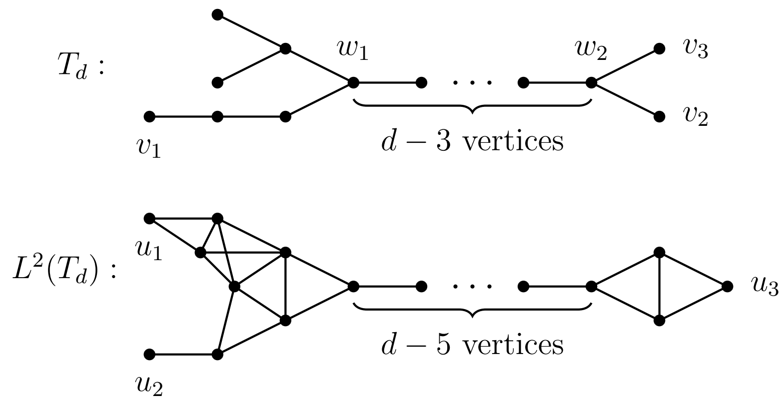


Figure 4: 2-stable tree T_d, $d \geq 6$.

Let T_d, $d \geq 6$, be the trees as shown in Fig. 4. Noticing that T_6, T_7, T_8, and T_9 are respectively isomorphic to the trees 11_1:14, 12_1:16, 13_3:18, and 14_1:20 from Table 1, this leads to the following infinite family of 2-stable trees.

Proposition 3.2. *If $d \geq 6$, then T_d is 2-stable.*

Proof. Using the vertex labeling from Fig. 4 and the assumption $d \geq 6$, we infer that $P(T) = \{v_1, v_2, v_3\}$ and $P(L^2(T)) = \{u_1, u_2, u_3\}$. Hence,

$$\mathrm{PW}(L^2(T_d)) = 2(d-1) + 4 = 2d + 2 = \mathrm{PW}(T_d)\,,$$

which we wanted to verify. $\square$

In the rest of the section we aim to find families of k-stable trees where $k \geq 3$. For this sake consider first the trees T'_d, $d \geq 5$, as shown in Fig. 5.

Proposition 3.3. *If $d \geq 5$, then T'_d is 3-stable.*

Proof. Let H_v and H_w be subgraphs of T'_d spanned by the pendant edges incident to vertices v and w, respectively, see Fig. 5. Then $L(H_v) \cong K_3 \cong L(H_w)$ and $L^k(H_v) \cong K_3 \cong L^k(H_w)$ for $k \geq 1$. Moreover, the vertices in $L^k(H_v)$ and $L^k(H_w)$ are peripheral in $L^k(T'_d)$. Thus $|P(T)| = |P(L^3(T'_d))| = 8$, cf. Fig. 5 where the structure of $L^3(T'_d)$ is schematically presented. The direct calculation yields $\mathrm{PW}(T'_d) = 12d{+}52 = \mathrm{PW}(T)$, as required. $\square$

Finally, k-stable trees are constructed for every $k \geq 4$. For $s \geq 2$ and $d \geq 2s$, let $T_{s,d}$ denote the tree shown in Fig. 6.

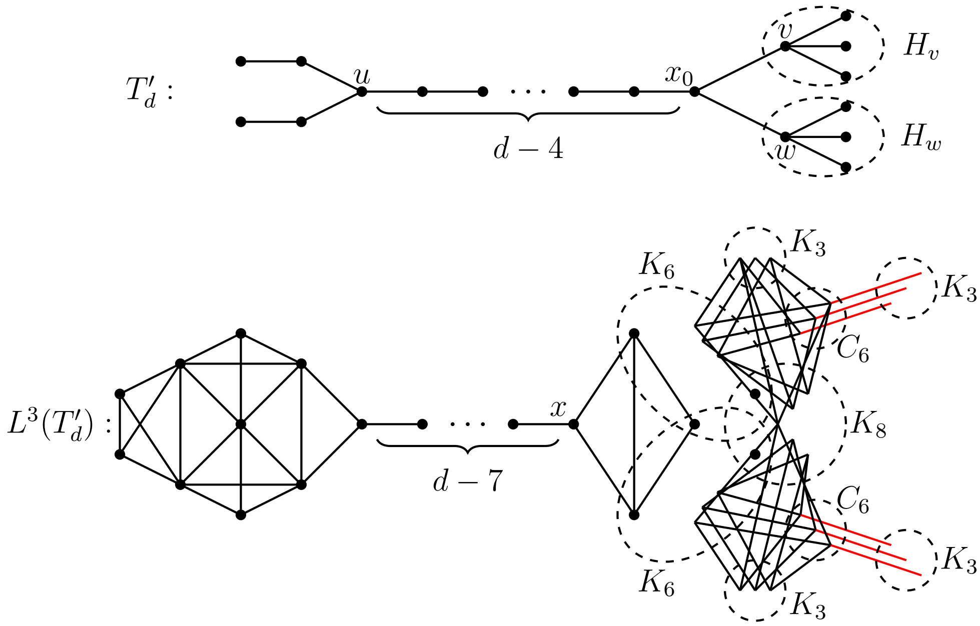


Figure 5: 3-stable trees T'_d.

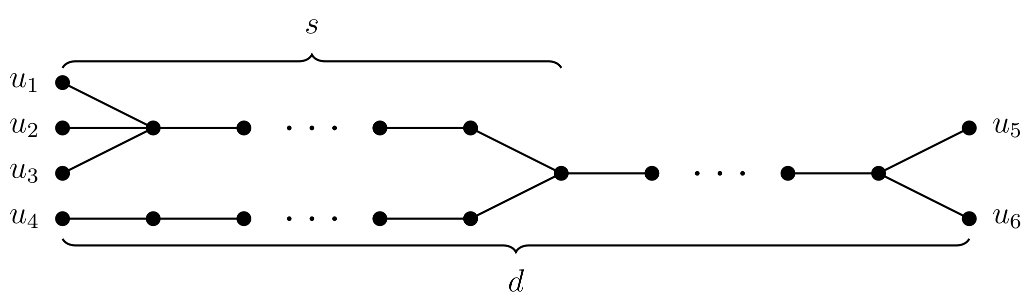


Figure 6: k-stable tree $T_{s,d}$,for $d \geq 2s$ and $k \geq 4$.

Theorem 3.4. *If $s \geq k+1 \geq 5$, then $T_{s,6s-9k+29}$ is k-stable.*

Proof. Let $k \geq 4$, $s \geq k+1$, and $d \geq 2s$ be integers. Consider the tree $T_{s,d}$ as presented in Fig. 6. For simplicity, denote $T_{s,d}$ by T. Since $d \geq 2s$, we have $P(T) = \{u_i : \ i \in [6]\}$ and consequently

$$\mathrm{PW}(L(T)) = 8d + 6s + 8\,.$$

Consider next $L^3(T)$, see Fig. 7. We infer that $\mathrm{diam}(L^3(T)) = d = \mathrm{diam}(T)$.

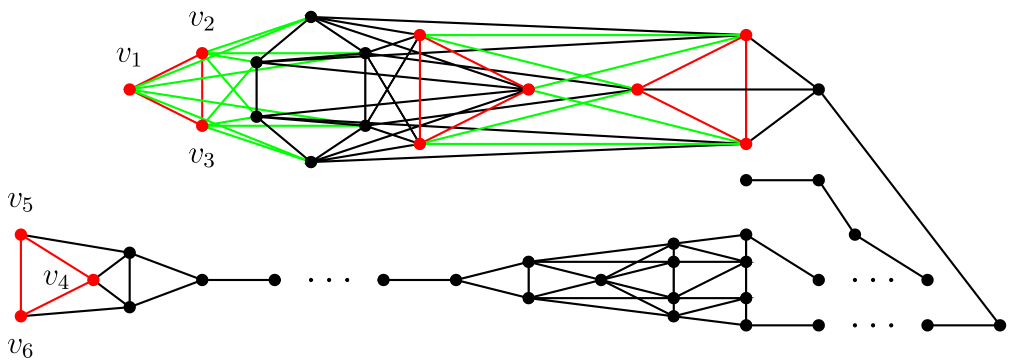


Figure 7: $L^3(T_{s,d})$, where $T_{s,d}$ is the tree from Fig. 6.

Let $V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, v_5, v_6\}$, where v_i's are the vertices in $L^3(T)$ which constitute the two end triangles as shown in Fig. 7. For $k \geq 4$, $L^{k-3}\left(L^3(T)[V_1]\right) \cong L^{k-3}\left(L^3(T)[V_2]\right) \cong K_3$, $V(B^k_{k-3}(v^k_i)) \subseteq V_1$, and $V(B^k_{k-3}(v^k_i)) \subseteq V_2$, where v^k_i and v^k_j are vertices in the triangles $L^{k-3}\left(L^3(T)[V_1]\right)$ and $L^{k-3}\left(L^3(T)[V_2]\right)$, respectively, $1 \leq i \leq 3$ and $4 \leq j \leq 6$. Then, by Lemma 2.1,

$$d_{L^k(T)}(v^k_i, v^k_j) = (k-3) + d_{L^3(T)}(v_i, v_j) = d + k - 3\,,$$

where, $1 \leq i \leq 3$ and $4 \leq j \leq 6$.

Since $\mathrm{diam}(L^k(T)) \leq k - 3 + \mathrm{diam}(L^3(T))$,

$$V\left(L^{k-3}\left(L^3(T)[V_1 \cup V_2]\right)\right) \subseteq P(L^k(T))\,.$$

Let $E^* = E(L^3(T)) \setminus [E\left(L^3(T)[V_1 \cup V_2]\right)]$. Since $s \geq k+1$, for any two vertices u, w in $L^4(T)$ corresponding to edges in E^*, we have $d_{L^4(T)}(u, w) \leq d - 2$. Consequently,

$$d_{L^k(T)}(u', w') \leq k - 3 + d - 2$$

for all vertices $u', w' \in V(L^k(T)) \setminus \left[V\left(L^{k-3}\left(L^3(T)[V_1 \cup V_2]\right)\right)\right]$ and $k \geq 4$. This in turn implies that

$$P(L^k(T)) \subseteq V\left(L^{k-3}\left(L^3(T)[V_1 \cup V_2]\right)\right).$$

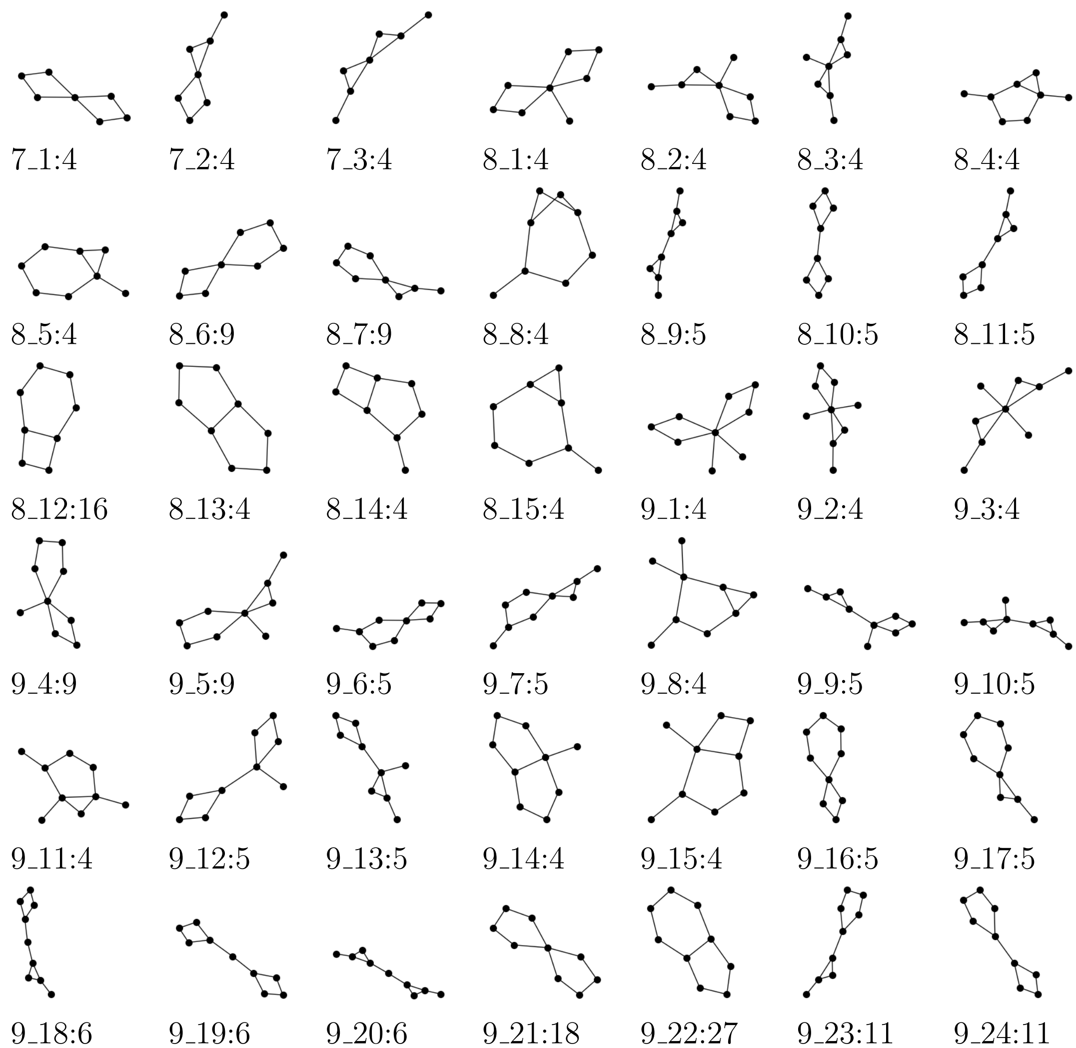
Hence, $P(L^k(T)) = V\left(L^{k-3}\left(L^3(T)[V_1 \cup V_2]\right)\right)$, and thus $PW(L^k(T)) = 9d + 9k - 21$. Hence, $\mathrm{PW}(L^k(T)) = \mathrm{PW}(T)$ if and only if $d = 6s - 9k + 29$, $k \geq 4$, $s \geq k+1$. $\square$

To conclude this section, it should be noted that an exhaustive search establishes that no non-trivial tree T of order $n \leq 18$ exist which satisfies $W(L^4(T)) = W(T)$.

4 Families of k-stable bicyclic graphs

This section focuses on k-stable bicyclic graphs. The first result presents an infinite family of 2-stable bicyclic graphs. The second establishes that, for every $k \geq 3$ and sufficiently large n relative to k, there exists a k-stable bicyclic graph of order n. The section concludes with a complete list of 2-stable unicyclic graphs of order at most 10. But first, an exhaustive computer search determines all 2-stable bicyclic graphs of order at most 9, which are presented in Table 3.

Table 3: List of 2-stable bicyclic graphs of order up to 9.



Theorem 4.1. *If $n \geq 7$, then there exists a 2-stable bicyclic graph G of order n.*

Proof. Let $G(p, q)$, $p, q \geq 3$, be the graph obtained from cycles C_p and C_q by identifying a vertex of C_p with a vertex of C_q, see Fig. 8.

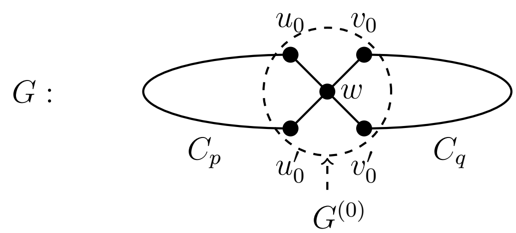


Figure 8: Bicyclic graph $G(p, q)$

Let $n \geq 7$, and let $p \geq 4$ and $q \geq 4$ be fixed integers such that $p + q = n - 1$. Seting $G = G(p, q)$ for the rest of the proof, we claim that G is 2-stable. Let w be the common vertex of C_p and C_q in G, and let u_0, u_0', v_0, and v_0' be vertices in G, as shown in Fig. 8. Let further $P^{(0)}$ be the u_0, u_0'-path not containing w and $Q^{(0)}$ the v_0, v_0'-path not containing w. The lengths of $P^{(0)}$ and $Q^{(0)}$ are $p - 2$ and $q - 2$, respectively. The periphery of G is the union of the centers of the paths $P^{(0)}$ and $Q^{(0)}$, and $\mathrm{diam}(G) = \left\lfloor \frac{p-2}{2} \right\rfloor + 1 + 1 + \left\lfloor \frac{q-2}{2} \right\rfloor = \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor$.

Let $G^{(0)}$ be the subgraph of G induced by the vertices w, u_0, u_0', v_0, and v_0'. Then, $G^{(1)} = L(G^{(0)}) = K_4$. Let u_1, u_1', v_1, and v_1' be the vertices of $G^{(1)}$ which correspond to the edges of $G^{(0)}$ incident to u_0, u_0', v_0, and v_0', respectively. Let $P^{(1)}$ be the u_1, u_1'-path containing $L(P^{(0)})$, and let $Q^{(1)}$ be the v_1, v_1'-path containing $L(Q^{(0)})$, see Fig. 9. The lengths of $P^{(1)}$ and $Q^{(1)}$ are $p - 3 + 2 = p - 1$ and $q - 3 + 2 = q - 1$, respectively.

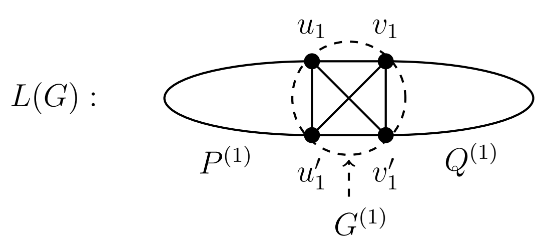


Figure 9: Some notation for $L(G)$

Let further u_2 and u_2' be the vertices of $L^2(G)$ corresponding to the edges on $P^{(1)}$ incident to u_1 and u_1', and let v_2 and v_2' be the vertices of $L^2(G)$ corresponding to the edges on $Q^{(1)}$ incident to v_1 and v_1', respectively. Let $P^{(2)} = L(P^{(1)})$ and

$Q^{(2)} = L(Q^{(1)})$ be the paths as shown in Fig. 10. The paths $P^{(2)}$ and $Q^{(2)}$ have respective lengths $p-2$ and $q-2$, and $d_{L^2(G)}(P^{(2)}, Q^{(2)}) = 2$.

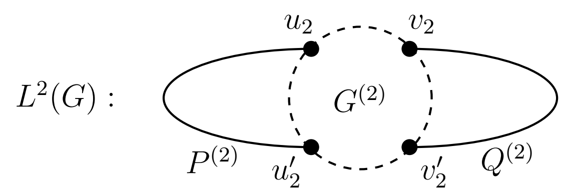


Figure 10: Some notation for $L^2(G)$

The periphery of $L^2(G)$ is the union of the centers of $P^{(2)}$ and $Q^{(2)}$, and

$$\operatorname{diam}(L^2(G)) = \left\lfloor \frac{p-2}{2} \right\rfloor + 2 + \left\lfloor \frac{q-2}{2} \right\rfloor = \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor = \operatorname{diam}(G)\,.$$

In conclusion, the numbers of peripheral vertices of G and $L^2(G)$ are equal, and because the length of $P^{(0)}$ and $P^{(2)}$ is $p-2$, and the length of $Q^{(0)}$ and $Q^{(2)}$ is $q-2$, we can conlcude that $\mathrm{PW}(L^2(G)) = \mathrm{PW}(G)$. $\square$

Theorem 4.2. *If $k \geq 3$ and $n \geq 2k+4$, then there exists a k-stable bicyclic graph of order n.*

Proof. We distinguish two cases depending on the parity of n.

Case 1: n is odd.
If $p \geq k$ is an even integer, and $q \geq k$ an odd integer, then let $G'(p,q)$ be the graph formed by identifying an edge of the cycle C_{p+3} with an edge of the cycle C_{q+3}, see Fig. 11. Clearly, $G'(p,q)$ is a bicyclic graph of odd order $n = p+q+4 \geq 2k+4$.

Fix k, an odd n, an even p, and an odd q, and set $G = G'(p,q)$ for the rest of Case 1. Let us further use the notation of vertices as shown in Fig. 11, and let P and Q be the u_0, u_0'-path and the v_0, v_0'-path, respectively, that do not contain the vertices w and w'. Then P and Q have respective lengths p and q. Hence,

$$\operatorname{diam}(G) = \left\lfloor \frac{p}{2} \right\rfloor + 2 + \left\lfloor \frac{q}{2} \right\rfloor = \frac{p}{2} + \frac{q-1}{2} + 2.$$

Since the edge ww' joins the two cycles, the three middle vertices of P together with the two middle vertices of Q consist the periphery of G.

By Corollary 2.2, the paths $P^{(k)}$ from u_k to u_k' and $Q^{(k)}$ from v_k to v_k' in $L^k(G)$ corresponding to the paths P and Q in G have lengths $p-k$ and $q-k$, respectively.

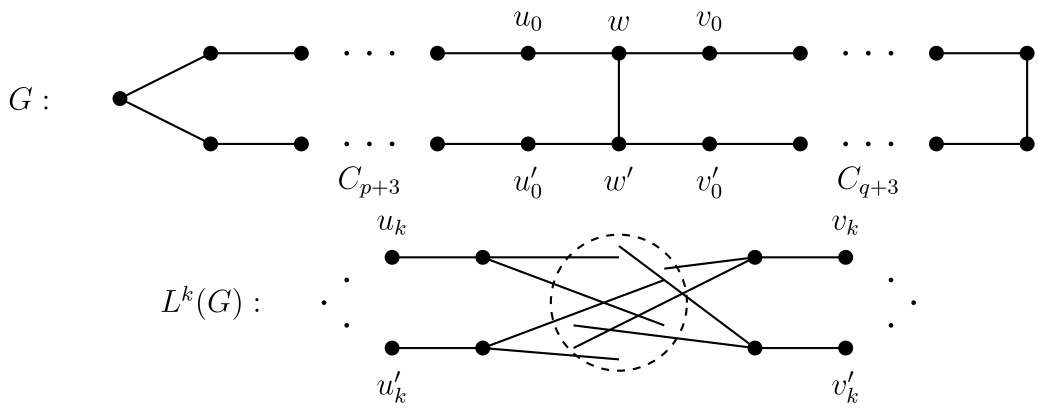


Figure 11: Bicyclic graph $G = G'(p, q)$ and $L^k(G)$

In view of Lemma 2.1 it follows that

$$\begin{aligned}\operatorname{diam}(L^k(G)) &= \left\lfloor \frac{p-k}{2} \right\rfloor + \operatorname{diam}(L^k(G))(u_k, v_k) + \left\lfloor \frac{q-k}{2} \right\rfloor \\ &= \left\lfloor \frac{p-k}{2} \right\rfloor + k + 2 + \left\lfloor \frac{q-k}{2} \right\rfloor .\end{aligned}$$

Consequently, if k is odd, then

$$\begin{aligned}\operatorname{diam}(L^k(G)) =& \frac{p-k-1}{2} + k + 2 + \frac{q-k}{2} \\ =& \frac{p}{2} + 2 + \frac{q-1}{2} = \operatorname{diam}(G) ,\end{aligned}$$

and if k is even, then

$$\begin{aligned}\operatorname{diam}(L^k(G)) =& \frac{p-k}{2} + k + 2 + \frac{q-k-1}{2} \\ =& \frac{p}{2} + 2 + \frac{q-1}{2} = \operatorname{diam}(G) .\end{aligned}$$

By an argument analogous to the one for the paths P and Q in G, the corresponding paths $P^{(k)}$ and $Q^{(k)}$ in $L^k(G)$ have the property that the three middle vertices of the odd path and the two middle vertices of the even path are precisely the peripheral vertices of $L^k(G)$. Hence, G is k-stable.

Case 2: n is even.
In this case let $p \geq k$ be an even integer, let $q \geq k$ an odd integer, and let $G''(p, q)$

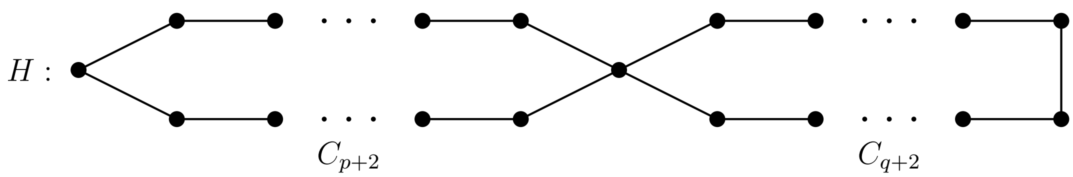


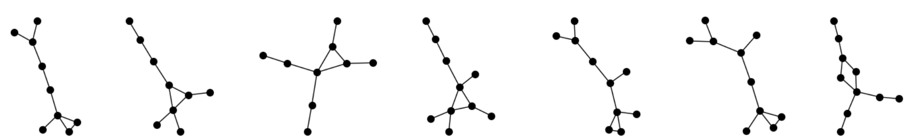
Figure 12: Bicyclic graph $H = G''(p, q)$.

be the graph obtained by identifying a vertex of C_{p+2} with a vertex of C_{q+2}, see Fig. 12.

Fix now k, an even n, an even p, and an odd q, and set $H = G''(p, q)$ for the rest of Case 2. Then H is a bicyclic graph of order $p + q + 3 \geq 2k + 3$. We now proceed analogously as in Case 1. Indeed, the corresponding paths in H and in $L^k(H)$ satisfy the same distance properties as those established for G in Case 1. Consequently, the central vertex of the odd path together with the two central vertices of the even path are precisely the peripheral vertices of H. Likewise, the corresponding central vertices of the associated paths in $L^k(H)$ constitute the periphery of $L^k(H)$. Therefore, $P(H) \cong P(L^k(H))$. $\square$

To wrap up this section, let's take a quick look also at unicyclic graphs. Using a computer, we have detected all 2-stable unicyclic graphs of order up to 10. These graphs (where cycles are excluded) are presented in Table 4.

Table 4: List of all 2-stable unicyclic graphs, but cycles, of order up to 10.



9_1:37 9_2:23 9_3:23 10_1:23 10_2:37 10_3:37 10_4:16

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Declaration of interests

The authors declare that they have no conflict of interest.

Data availability

Our manuscript has no associated data.

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