Power domination in product graphs*

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Abstract

The power system monitoring problem asks for as few as possible measurement devices to be put in an electric power system. The problem has a graph theory model involving power dominating sets in graphs. The power domination number $\gamma_P(G)$ of G is the minimum cardinality of a power dominating set. Dorfling and Henning [2] determined the power domination number of the Cartesian product of paths. In this paper the power domination number is determined for all direct products of paths except for the odd component of the direct product of two odd paths. For instance, if n is even and C a connected component of $P_m \times P_n$, where m is odd or $m \geq n$, then $\gamma_P(C) = \lceil n/4 \rceil$. For the strong product we prove that $\gamma_P(P_n \boxtimes P_m) = \max\{\lceil n/3 \rceil, \lceil (n+m-2)/4 \rceil\}$, unless $3m - n - 6 \equiv 4 \pmod 8$. The power domination number is also determined for an arbitrary lexicographic product.

Key words: domination, power domination, electric power monitoring, graph products, paths

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1 Introduction

Electric power systems need to be continually monitored. One way to fulfill this task is to place phase measurement units at (carefully) selected locations in the system. The power system monitoring problem, as introduced in [1], asks for as few as possible measurement devices to be put in an electric power system.

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The power system monitoring problem has been formulated as a graph-theory domination problem by Haynes, Hedetniemi, Hedetniemi, and Henning in [5]. This problem is of somehow different flavor than standard domination type problems, since putting a phase measurement unit into a vertex of a graph can have global effects. For instance, if an electric power system can be modeled by a path, then a single measurement unit suffices to monitor the system no matter how long is the path.

Let G be a connected graph and S a subset of its vertices. Then we denote the *set monitored by* S with $M(S)$ and define it algorithmically as follows:

1. (*domination*)
   $M(S) \leftarrow S \cup N(S)$
2. (*propagation*)
   As long as there exists $v \in M(S)$ such that
   $N(v) \cap (V(G) - M(S)) = \{w\}$
   set $M(S) \leftarrow M(S) \cup \{w\}$.

The set $M(S)$ is thus obtained from S as follows. First put into $M(S)$ the vertices from the closed neighborhood of S. Then repeatedly add to $M(S)$ vertices w that have a neighbor v in $M(S)$ such that all the other neighbors of v are already in $M(S)$. After no such vertex w exists, the set monitored by S has been constructed.

The set S is called a *power dominating set* of G if $M(S) = V(G)$ and the *power domination number* $\gamma_P(G)$ is the minimum cardinality of a power dominating set.

In [5] these concepts have been introduced in a slightly more complicated way by treating both vertices and edges of a given graph. However, it is easily shown that both approaches are equivalent in the sense that the power dominating sets correspond. In fact, this is also (implicitly) observed in [2].

The power domination number has received considerable attention from the algorithmic point of view. As it turned out, the problem is quite difficult—it is NP-complete even when restricted to bipartite graphs and chordal graphs [5], to planar graphs and circle graphs [3] as well as to split graphs [3, 9]. On the other hand, the problem has efficient solutions on trees [5] and on interval graphs [9].

Dorfling and Henning [2] obtained closed formulas for the power domination numbers of grid graphs. This result is in a striking contrast to the fact a determination of such formulas for the usual domination number of grid graphs is a notorious open problem, see [6, Section 2.6.2]. Now, a natural description of a grid is as the Cartesian product of two paths. Besides the Cartesian one, there are three more standard graph products: the strong, the direct, and the lexicographic product [7]. Hence it is natural to ask whether the power domination number can also be determined for these products of paths. In this paper we indeed show to be the case.

We proceed as follows. In the rest of this section we give definitions used in this paper. In Section 2 we determine the power domination number for direct products of paths with the exception of the odd components of products of odd paths. For this case an upper bound is given for which we believe to be optimal. Then, in Section 3, we determine the power domination number for the strong product of paths, except that in one of the eight cases the power domination number is only bounded. But also in this case the bounds are almost tight. While it takes quite an effort to obtain the results for the direct and the strong product of paths, it is rather straightforward to deal with the lexicographic product. In fact, in the last section we show that the power domination number can be determined exactly for the lexicographic product of arbitrary graphs in terms of the (total) domination number of the factors.

For a vertex v of a graph G, let $N(v)$ denote the open neighborhood of v, and for a subset S of $V(G)$ let $N(S) = \cup_{v \in S} N(v) - S$. In the same way, the close neighborhood $N[S]$ of a subset S is the set $N[S] = N(S) \cup S$.

All of the four standard graph products constructed from graphs G and H have vertex set $V(G) \times V(H)$. Let (g,h) and (g',h') be two vertices in $V(G) \times V(H)$. They are adjacent in the *Cartesian product* $G \,\Box\, H$ if they are equal in one coordinate and adjacent in the other. They are adjacent in the *direct product* $G \times H$ if they are adjacent in both coordinates. The edge set of the *strong product* $G \boxtimes H$ is $E(G \,\Box\, H) \cup E(G \times H)$. Finally, (g,h) and (g',h') are adjacent in the *lexicographic product* $G \circ H$ if $gg' \in E(G)$, or if $g = g'$ and $hh' \in E(H)$. All these products are associative, all but the lexicographic product are also commutative; see [7] for more information on all the products.

Let $G * H$ be any of the four standard graph products. Then the subgraph of $G * H$ induced by $\{g\} \times V(H)$ is called a *H-fiber* and denoted ${}^gH$. Similarly one defines the *G-fiber* G^h for a vertex h of H. If $*$ is the Cartesian, the strong, or the lexicographic product, then fibers are isomorphic to the corresponding factors. For the direct product the fibers are discrete, that is, they induce edgeless graphs.

Finally, for the path P_m we will always assume that $V(P_m) = \{0, \ldots, m-1\}$, where vertices are adjacent in the natural way. Moreover, the vertices of $P_m * P_n$ will be denoted by (i,j) where $0 \le i < m$ and $0 \le j < n$.

2 The direct product

The direct product $P_m \times P_n$ is not connected. If m or n is even $P_m \times P_n$ consists of two isomorphic connected components, otherwise it has two non-isomorphic connected components, see [10], cf. also [4, 8]. We call the component of $P_m \times P_n$ containing $(0,0)$ the *even component* and the other component the *odd component*.

In this section we determine the power domination number of the direct product of two paths, except for the odd component of the product of two odd paths. In the first subsection we treat the case when at least one of the paths is even. In this case $\gamma_P(C) = \lceil n/4 \rceil$, where C is a connected component of $P_m \times P_n$ with n even and m odd or $m \ge n$. In the subsequent subsection we follow with products of odd paths proving that for the even component C, $\gamma_P(C) = \lceil (m+n)/6 \rceil$. We close by construction a power dominating set of the odd component of such products. We conjecture that the construction is optimal.

Let $p_1 : P_m \times P_n \to P_m$ and $p_2 : P_m \times P_n \to P_n$ be the natural projections onto the first and the second factor of $P_m \times P_n$, respectively. Then:

Lemma 2.1 *Let S be a power dominating set of a connected component C of $P_m \times P_n$. If m is odd, then for every subpath $P \subseteq P_n$ of length 3, $p_2(S) \cap P \neq \emptyset$.*

Proof. Suppose on the contrary that there exists a subpath $P = x_1x_2x_3x_4$ of length 3, such that $p_2(S) \cap P = \emptyset$.

Assume C is even. Then if x_1 is even, $P_m^{x_3} \cap C$ cannot be power dominated since any neighbor of a vertex from $P_m^{x_3} \cap C$ has two neighbors in $P_m^{x_3}$. Similarly we infer that if x_1 is odd then $P_m^{x_2} \cap C$ cannot be power dominated, a contradiction. Arguments for C being odd are analogous. □

2.1 At least one factor is even

To solve the case when at least one factor is an even path, we begin with two lemmas.

Lemma 2.2 *Let m and n be even and let S be a power dominating set of a connected component C of $P_m \times P_n$. If there is a subpath $P \subseteq P_m$ of length 3, such that $P \cap p_1(S) = \emptyset$ then for every subpath $Q \subseteq P_n$ of length 3, $Q \cap p_2(S) \neq \emptyset$.*

Proof. Suppose on the contrary that $P \subseteq P_m$ and $Q \subseteq P_n$ are paths of length 3 with $p_1(S) \cap P = p_2(S) \cap Q = \emptyset$. Assume that C is even and that all vertices from

$$F = \{(i, j) \,|\, i \notin P, j \notin Q\} \cap C$$

are monitored. We claim that all monitored vertices are contained in the neighborhood of F. Suppose that (u, v) is a vertex from F, such that all but one neighbors of (u, v) are contained in F (and thus monitored). Then clearly (u, v) lies on the boundary of $P_m \times P_n$. Hence all the newly obtained monitored vertices from $C \setminus F$ do not lie on the boundary of $P_m \times P_n$, and therefore have at least two neighbors not monitored. Thus all monitored vertices are contained in the neighborhood of F. Since $N[F] \neq C$ we find that S is not a power dominating set of C. □

Lemma 2.3 *Let m be even and let C be a connected component of $P_m \times P_n$. If all the vertices from $(P_m^u \cup P_m^v) \cap C$ are monitored, where u and v are adjacent vertices of P_n, then C is power dominated.*

Proof. Since m is even, both $P_m^u \cap C$ and $P_m^v \cap C$ contain a vertex that has exactly one neighbor not in $(P_m^u \cup P_m^v) \cap C$. It follows easily that the two neighboring fibers (or the neighboring fiber if there is only one such) are monitored. Induction completes the argument. □

Theorem 2.4 *Let n be even and C be a connected component of $P_m \times P_n$. If m is odd or $m \geq n$, then $\gamma_P(C) = \lceil n/4 \rceil$.*

Proof. We first consider the case when m is odd. Then, since n is even, either all vertices from $P_m^1 \cap C$ have two neighbors in $P_m^0 \cap C$ or all vertices from $P_m^{n-2} \cap C$ have two neighbors in $P_m^{n-1} \cap C$. It follows that any power dominating set $S \subseteq C$ has nonempty intersection with one of the fibers P_m^0, P_m^1, P_m^{n-2} and P_m^{n-1}.

By Lemma 2.1 a power dominating set S of C has the property that for any subpath $P \subseteq P_n$ of length 3, $P \cap p_2(S) \neq \emptyset$. Without loss of generality assume that $P_m^1 \cap S$ is nonempty. Since the fiber P_m^0 as well as the fibers P_m^{n-2} and P_m^{n-1} (but not P_m^{n-3}) might have empty intersection with S and at least every fourth fiber has nonempty intersection with S we find that

$$|S| \geq |\{1, 1+4, \ldots, 1+4\lceil (n-4)/4 \rceil\}| = \lceil n/4 \rceil \ .$$

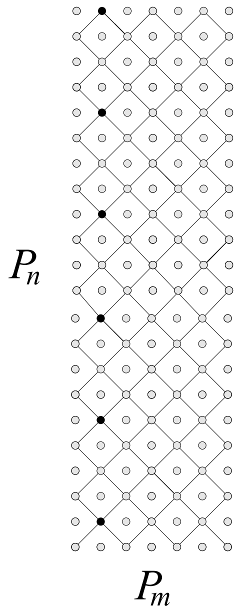
Hence $\gamma_P(G) \geq \lceil n/4 \rceil$. To prove the reverse inequality we need to construct a power dominating set S of C with $|S| = \lceil n/4 \rceil$. We may without loss of generality assume that C is even. Then let:

$$S = \{(1, 1+4\alpha) \,|\, \alpha = 0, \ldots, \lceil n/4 \rceil - 1\} \,. \tag{1}$$

Observe that all the vertices from ${}^0P_n$ are monitored since for every vertex from ${}^0P_n$ there is a neighboring vertex from S which dominates it; see the left-hand side of Fig. 1. Next observe that also all the vertices from ${}^1P_n$ are monitored, since for every vertex from ${}^1P_n$ there is a vertex from ${}^0P_n$ with all but one vertices in S. Since the consecutive fibers ${}^0P_n$ and ${}^1P_n$ are monitored and n is even, Lemma 2.3 implies that S is a power dominating set of C. This completes the proof for m being odd.

Let now m be even, $m \geq n$. Let S be a power dominating set of $P_m \times P_n$. By Lemma 2.2, there is either no subpath of P_m of length 3 disjoint with $p_1(S)$ or there is no subpath of P_n of length 3 disjoint with $p_2(S)$. Hence $|S| \geq \lfloor n/4 \rfloor$. This is the desired lower bound if $n = 4k$.

If $n = 4k+2$ and $|S| = k$ then either there is a subpath $P \subseteq P_n$ of length 3 disjoint with $p_2(S)$, or the first three or the last three P_m-fibers (that is, P_m^0, P_m^1 and P_m^2, or P_m^{n-3}, P_m^{n-2} and P_m^{n-1}) have empty intersection with $p_2(S)$. In the first case there is no subpath of P_m of length 3 disjoint with $p_1(S)$, therefore in this case $m = n$. But this means that S is disjoint with fibers ${}^0P_n, {}^1P_n$ and ${}^2P_n$ (or ${}^{m-3}P_n, {}^{m-2}P_n$ and ${}^{m-1}P_n$). Therefore ${}^0P_n$ is not monitored (see the right-hand side of Fig. 1), a contradiction. The second case is when P_m^0, P_m^1 and P_m^2 have empty intersection with $p_2(S)$. If also ${}^0P_n, {}^1P_n$ and ${}^2P_n$ have empty intersection with $p_1(S)$, then ${}^0P_n$ and P_m^0 are not monitored. Otherwise there is a subpath of P_m of length 3 disjoint with $p_1(S)$. This case leads to a contradiction similarly as shown on Fig.1. Thus if $n = 4k+2$ and $|S| = k$, then S cannot be a power dominating set. Hence in either of the cases $|S| \geq \lceil n/4 \rceil$. Finally, an optimal power dominating set S of cardinality $n/4$ for the case when m is even is again defined by (1). $\square$



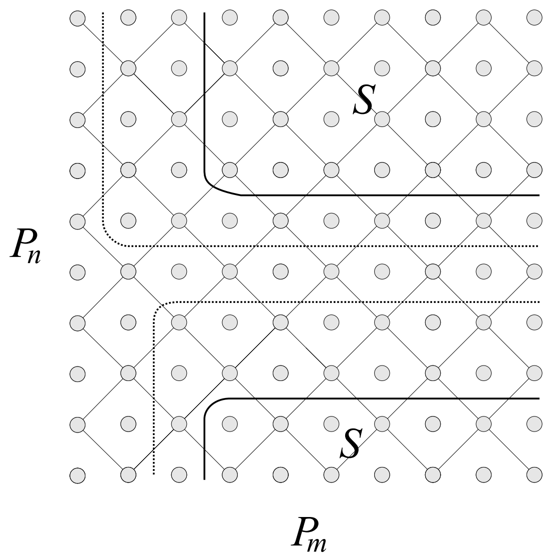


Figure 1: An optimal power dominating set of $P_m \times P_n$ is shown on the left. The maximal set of monitored vertices is marked with broken line on the right.

Note that Theorem 2.4 holds also for $n = 2$. Indeed, a connected component of $P_m \times P_2$ is isomorphic to P_m whence its power domination number is 1.

2.2 Both factors are odd

We continue with the odd times odd case. For this case we introduce the following concept on a grid $P_m \,\square\, P_n$.

Consider an initial set S of vertices. Let D be initialized by $D = S$. As long as there is a vertex v in $V \setminus D$ such that v has at least two neighbors in D, add v to D. If at the end of the process $D = V(P_m \,\square\, P_n)$, then we call the set S a *life winning set of* $P_m \,\square\, P_n$, LWS, for short. Let us call this concept the *life-like game*.

Theorem 2.5 *For any positive integers m, n, there exists a LWS set of cardinality $\lceil \frac{m+n}{2} \rceil$ and it is minimum.*

Proof. Suppose m and n are odd. Then let S consists of vertices $(0,0), (2,0), \ldots, (m-1,0), (m-1,2), (m-1,4), \ldots, (m-1,n-1)$. In the other cases S is constructed similarly. It is straightforward to verify that S is a LWS set of the desired cardinality.

Let us prove that S is minimum. For $X \subseteq V(P_m \,\square\, P_n)$ let $\Pi(X)$ be the *perimeter of* X, that is, $\Pi(X) = \sum_{v \in X}(4 - |N(v) \cap X|)$. If D' is a subset of $V(P_m \,\square\, P_n)$ obtained by propagation from D, then $\Pi(D') \leq \Pi(D)$. Indeed, any vertex added to D decreases

$\Pi(D)$ by at least 2 and adds at most 2 to $\Pi(D)$. At the end of the propagation, $\Pi(D) = 2(m+n)$. At the beginning, $\Pi(S) \leq 4|S|$. Hence, $|S| \geq (m+n)/2$. □

The above theorem might be of independent interest, but for our purposes it is important that during the propagation process the perimeter of the monitored set does not increase.

In the following we will consider the direct product $P_m \times P_n$ and at the same time the Cartesian product $P_m \,\square\, P_n$, where the vertex sets of both products are the same.

Lemma 2.6 *Let m, n be odd integers and P a power dominating set of the even component of $P_m \times P_n$. Let S be the set monitored by P after the domination step of the algorithm, then S is a LWS of $P_m \,\square\, P_n$.*

Proof. We first remark that all the vertices of the even component either have both their coordinates odd or both even. We will call them odd and even vertices, respectively. Since both m and n are odd, the set of even vertices is a LWS of $P_m \,\square\, P_n$.

Let M and D, initialized to S, be the sets that are propagating for power domination and life-like game respectively. We will prove by induction that any even vertex in M is in D. From the choice $M = D = S$, this condition is initially true.

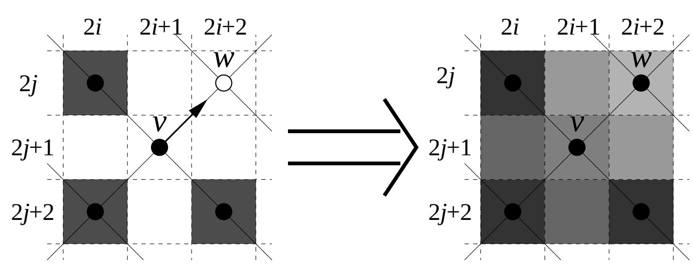


Figure 2: Propagation from an odd vertex to an even

Suppose an even vertex w is added to M. Then, it is monitored by propagation from an odd vertex v. Any odd vertex has 4 neighbors, all even. So all the 3 other neighbors of v are monitored and belong to D. In Fig. 2, we show how w is thus also put in D by the rule of the life-like game. □

Theorem 2.7 *Let m and n be odd and C the even component of $P_m \times P_n$. Then*

$$\gamma_P(C) = \max\left\{\left\lceil \frac{n}{4} \right\rceil, \left\lceil \frac{m+n}{6} \right\rceil\right\}.$$

Proof. Since any power dominating set of C contains a vertex from P_m^0 or P_m^1, and a vertex from P_m^{n-2} or P_m^{n-1}, Lemma 2.1 implies that $\gamma_P(C) \geq n/4$.

Let P be a power dominating set of $P_m \times P_n$. Let S be the set monitored after the domination step. Let us consider the life-like game starting on S. Every vertex from P contributes at most 5 vertices to S which propagate to a 3×3 box in the grid. Denoting the obtained set S' we infer that every vertex in P contributes at most 12 to $\Pi(S')$. The set S' is a LWS of the grid, so $12\,|P| \geq 2(m+n)$. (Here we have used the fact that during the propagation process the perimeter of the monitored set does not increase.) Therefore $|P| \geq \lceil (n+m)/6 \rceil$.

To establish the upper bound we construct a corresponding power dominating set P of C as follows. Suppose first that $m \leq n \leq 2m$ and set

$$\begin{aligned} S_1 &= \{(1+2i, 1+4i) \mid 0 \leq i \leq (2n-m-3)/6\} \\ S_2 &= \{((2n-m)/3+4j, (4n-2m-3)/3+2j) \mid 1 \leq j \leq (2m-n-3)/6\}\,. \end{aligned}$$

Suppose that $n+m \equiv 0 \pmod 6$ (the perfect case). Then set $P = S_1 \cup S_2$. Otherwise, construct P from a maximal perfect case contained in $P_m \times P_n$ by adding the vertex $(m-2, n-2)$ to it. It is straightforward to verify that in all cases P is a power dominating set. Moreover, $((2n-m-3)/6+1)+(2m-n-3)/6 = (m+n)/6$, hence $|P| = \lceil (m+n)/6 \rceil$.

Let $n > 2m$. Then set

$$\begin{aligned} S_1 &= \{(1+2i, 1+4i) \mid 0 \leq i \leq (m-3)/2\} \\ S_2 &= \{(1, 1+4j) \mid (m-3)/2 < j \leq (n-3)/4\}\,. \end{aligned}$$

Now the perfect case is when $n \equiv 3 \pmod 4$, in which case we set $P = S_1 \cup S_2$. Otherwise, proceed as above by adding $(m-2, n-2)$ to the power dominating set of a maximal perfect case contained in $P_m \times P_n$. Since $(m-3)/2+1+((n-3)/4-(m-3)/2) = (n+1)/4$, in this case $|P| = \lceil n/4 \rceil$. $\square$

The construction for the first case from the previous proof is illustrated in Fig. 3 with black dots. Moreover, another optimal power dominating set is presented with black squares in order to illustrate that such a set can be unusual.

Theorem 2.8 *Let m and n be odd and C the odd component of $P_m \times P_n$. Then*

$$\gamma_P(C) \leq \max\left\{ \left\lceil \frac{n-2}{4} \right\rceil, \left\lceil \frac{m+n-2}{6} \right\rceil \right\}.$$

Proof. We give a construction of a power dominating set of the given cardinality. Suppose $m \leq n \leq 2m+2$ and $m+n-2 \equiv 0 \pmod 6$ (the perfect case), then the construction is as follows. Let

$$\begin{aligned} S_1 &= \{(3+2i, 4i) \mid 1 \leq i \leq (2n-m-7)/6\} \text{ and} \\ S_2 &= \{(2n-m-1)/3+4j, (4n-2m-11)/3+2j) \mid 1 \leq j \leq (2m-n-7)/6\}\,. \end{aligned}$$

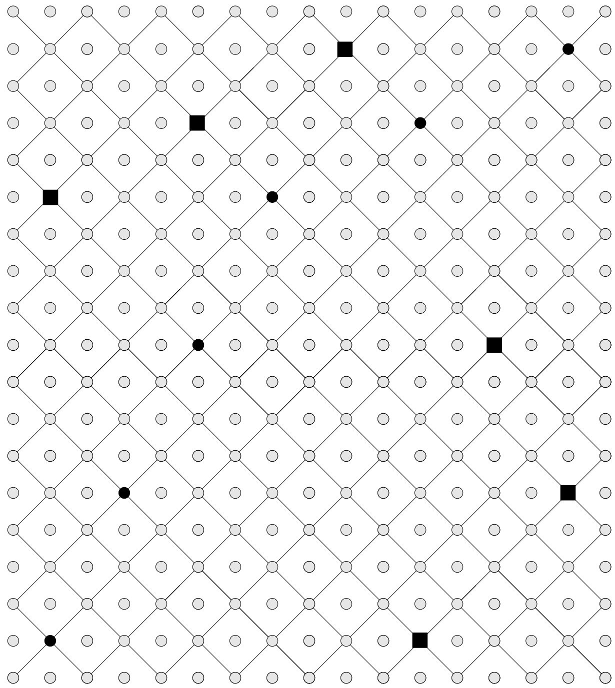


Figure 3: Two optimal power dominating sets of the even component of $P_{17} \times P_{19}$

Then $P = S_1 \cup S_2 \cup \{(2,1),(m-2,n-3)\}$ is a power dominating set of C. Since $(2n-m-7)/6 + (2m-n-7)/6 + 2 = (m+n-2)/6$, this construction has the claimed cardinality.

In the non perfect case, construct S_1 and S_2 for a minimal perfect case containing $P_m \times P_n$. Add to these sets the vertices $(2,1)$ and $(m-2,n-3)$ (thus, only this last vertex differs). This set clearly has the cardinality $\lceil (m+n-2)/6 \rceil$.

If $n > 2m+2$ and $n \equiv 1 \pmod 4$ then let

$$\begin{aligned} S_1 &= \{(1+2i, 2+4i) \mid 0 \le i \le (m-3)/2\} \text{ and} \\ S_2 &= \{(1, 2+4j) \mid (m-3)/2 \le j \le (n-5)/4\}. \end{aligned}$$

In the non perfect case ($n \equiv 3 \pmod 4$), construct S_1 and S_2 for $P_m \times P_{n+2}$ and it will form a PDS of $P_m \times P_n$. Since $(m-3)/2 + 1 + ((n-5)/4 - (m-3)/2) = (n-1)/4$, in this case $|P| = \lceil (n-2)/4 \rceil$.

$\square$

We think that the upper bound of Theorem 2.8 is optimal. However, proving that this is the case seems harder than it was for the even component.

3 The strong product

In this section we prove the following theorem.

Theorem 3.1 *Let $n \geq m \geq 1$. Then*

$$\gamma_P(P_n \boxtimes P_m) = \max\left\{\left\lceil\frac{n}{3}\right\rceil, \left\lceil\frac{n+m-2}{4}\right\rceil\right\}$$

unless $3m - n - 6 \equiv 4 \pmod 8$ *in which case*

$$\max\left\{\left\lceil\frac{n}{3}\right\rceil, \left\lceil\frac{n+m-2}{4}\right\rceil\right\} \leq \gamma_P(P_m \boxtimes P_n) \leq \max\left\{\left\lceil\frac{n}{3}\right\rceil, \left\lceil\frac{n+m-2}{4}\right\rceil + 1\right\}.$$

In the first subsection we prove the lower bounds, while in the subsequent two subsections corresponding power dominating sets are constructed. It is easily verified that the result holds if $n < 4$ and $m < 4$. Hence in the rest of this section we assume that $n \geq m \geq 4$.

3.1 Proof of the lower bound

Lemma 3.2 $\gamma_P(P_m \boxtimes P_n) \geq \left\lceil\frac{n}{3}\right\rceil$.

Proof. Let S be a power dominating set of $P_m \boxtimes P_n$. Suppose there is a P_m-fiber that does not contain any vertex of $N[S]$. Then, the vertices of this P_m-fiber have to be monitored during the propagation step. Consider the first vertex v of this P_m-fiber that is monitored, and let w be the vertex from whom it is monitored. w has at least one other neighbor in the same P_m-fiber than v. This neighbor is not yet monitored, so the propagation cannot happen. Thus S must dominate at least one vertex of each P_m-fiber which implies that $\gamma_P(P_m \boxtimes P_n) \geq \lceil n/3\rceil$. □

Lemma 3.3 $\gamma_P(P_m \boxtimes P_n) \geq \dfrac{n+m-2}{4}$.

Proof. Let S be a minimum power dominating set of $P_m \boxtimes P_n$. We can suppose that S contains no vertices on the boundary (that is, the vertices of $P_m \boxtimes P_n$ of degree 3 or 5), since any closed neighborhood of a vertex on the boundary is contained in the closed neighborhood of a vertex not on the boundary.

Let M be any set of monitored vertices of $P_m \boxtimes P_n$ during a propagation step of the algorithm. Let $B(M)$ be the set of vertices of M that have less than 8 neighbors in M. We claim that during the propagation step, $|B(M)|$ cannot increase.

Suppose that during the propagation step we monitor a new vertex w, where w is the only neighbor not in M of some vertex v from M. Then w may now be added to $B(M)$. We distinguish two cases.

Case 1. the degree of v is 8.
Then v now has all its 8 neighbors in M, so v is removed from $B(M)$. Some other vertices may be removed from $B(M)$, if w was their eighth neighbor to enter $B(M)$, but none other may be added. So $|B(M)|$ can not be increasing.

Case 2. the degree of v is less than 8.
In this case we check that w could have been monitored with a propagation from a vertex not on the boundary. The cases to be considered are shown in Fig. 4.

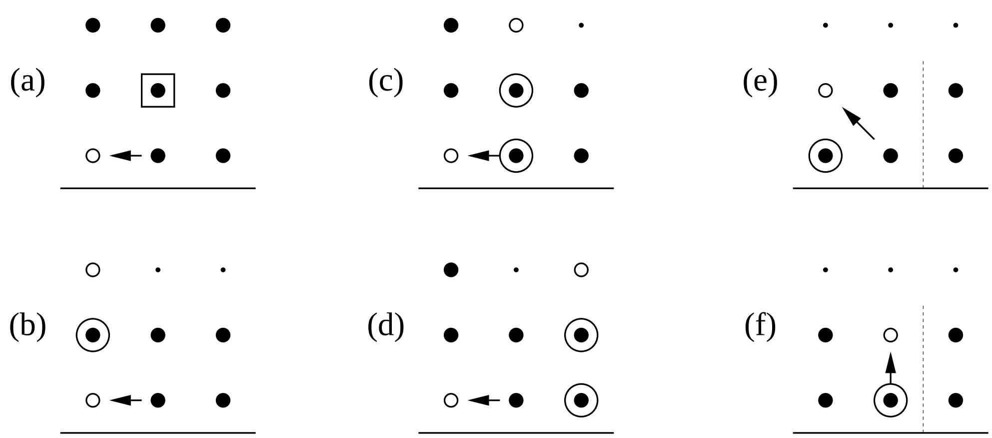


Figure 4: Propagation from the border

In case (a), we can consider that the propagation comes from the vertex in a box, which has 8 neighbors. Cases (b), (e), and (f) cannot occur, because the vertices in circles must have been monitored with some propagation, which is impossible. In cases (c) and (d), the two vertices in circles must have been monitored by propagation, but none could have been before the other was. The case of the corner is induced by cases e and f. This proves the claim.

Hence $|B(M)|$ cannot increase during a propagation step of the algorithm. After the domination step of the algorithm, $|B(M)|$ is not greater than $8\,|S|$. Moreover, at the end of the algorithm, when all the vertices are monitored, $|B(M)|$ is equal to $2n+2m-4$. So we must have $|S| \geq (n+m-2)/4$. $\square$

3.2 A construction for $n \geq 3m-6$

In this subsection we construct power dominating sets for the case when one path is much longer than the other, more precisely, when $n \geq 3m-6$. Then

$$\left\lceil \frac{n}{3} \right\rceil \geq \left\lceil \frac{3n}{12} + \frac{3m-6}{12} \right\rceil = \left\lceil \frac{n+m-2}{4} \right\rceil,$$

hence the power domination number depends only of the size of the longer path. Thus we need to construct a power dominating set S of cardinality $\lceil n/3 \rceil$. We do this as follows. Set $S = S_1 \cup S_2$, where

$$S_1 = \{(1+i, 1+3i) \mid 0 \leq i \leq m-3\}$$

and

$$S_2 = \begin{cases} \{(1, 3m-9+3j) \mid 1 \leq j \leq \lceil (n-3m+6)/3 \rceil\}; & n \equiv 1 \pmod 3, \\ \{(1, 3m-8+3j) \mid 1 \leq j \leq \lceil (n-3m+6)/3 \rceil\}; & \text{otherwise.} \end{cases}$$

The construction is illustrated in Fig. 5.

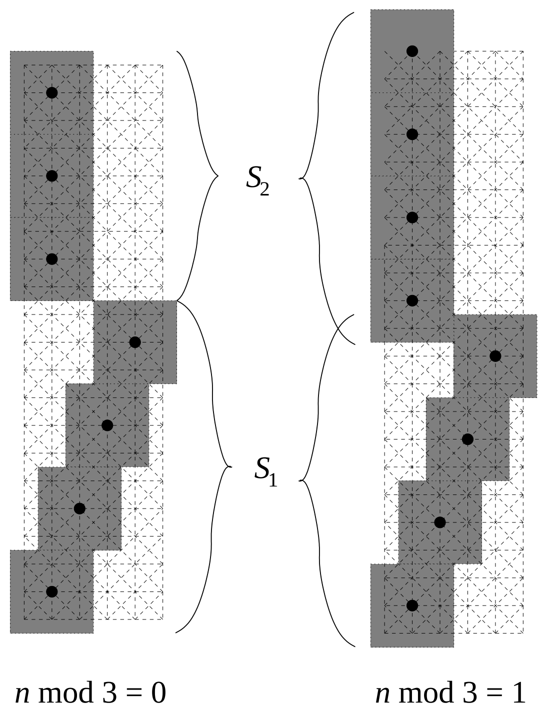


Figure 5: Optimal power dominating sets when one path is relatively long

It is easy to check that the set S will power dominate the whole strong product. Moreover, $|S| = |S_1| + |S_2| = m - 2 + \lceil (n-3m+6)/3 \rceil = \lceil n/3 \rceil$.

3.3 A construction for $n \leq 3m - 6$

In this case m and n are comparable. Since $\lceil n/3 \rceil \leq \lceil (n+m-2)/4 \rceil$ we need to construct power dominating sets S of cardinality $\lceil (n+m-2)/4 \rceil$. We start with the case when 8 divides $3m - n - 6$, and modify it in all the other cases.

Case 1. $3m - n - 6 \equiv 0 \pmod 8$.
Set $S = S_1 \cup S_2$, where

$$\begin{aligned} S_1 &= \{(1+3i, 1+i) \mid 0 \le i \le (3m-n-6)/8\} \\ S_2 &= \{(1+3(3m-n-6)/8+j, 1+(3m-n-6)/8+3j) \mid \\ &\quad 1 \le j \le (n+m-6)/4-(3m-n-6)/8\}\,. \end{aligned}$$

The construction is illustrated in Fig. 6.

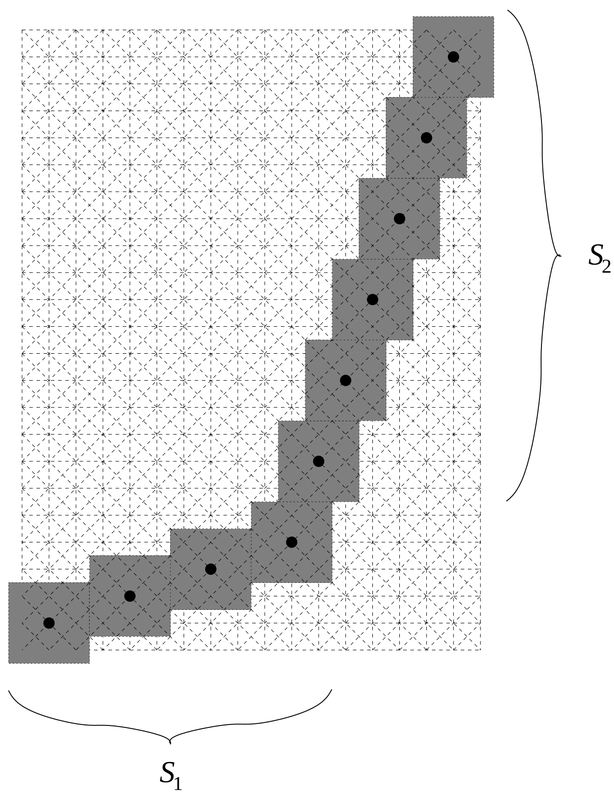


Figure 6: Power dominating sets of products of path of similar length

For coherence of the above construction we should remark that:

- $\frac{3m-n-6}{8}$ and $\frac{n+m-6}{4} = m - 3 - \frac{3m-n-6}{4}$ are integers.
- $\frac{n+m-6}{4} - \frac{3m-n-6}{8} = \frac{3n-m-6}{8}$ is positive since $n \ge m$ and $n \ge 4$.
- The coordinates of the last vertex which is furthest down and right are $(m-2, n-2)$, so all the vertices are in the graph.

- The set S does not contain any vertex on the border of the grid. (This remark will be very useful in the following.)

The set S contains $(n+m-2)/4$ vertices and is therefore optimal.

Case 2. $3m - n - 6 \equiv 1 \pmod 8$.
In this case $3m - (n+1) - 6 \equiv 0 \pmod 8$. Then we construct the power dominating set S of $P_m \boxtimes P_{n+1}$ from Case 1. Removing the fiber P_m^0 from $P_m \boxtimes P_{n+1}$ we get a graph $G = P_m \boxtimes P_n$. Since S does not contain any border vertex, the set S restricting to G does not change. Consequently, $|S| = (n+1+m-2)/4 = \lceil (n+m-2)/4 \rceil$.

Case 3. $3m - n - 6 \equiv 2 \pmod 8$.
Then $3m - (n+2) - 6 \equiv 0 \pmod 8$, hence construct the power dominating set S of $P_m \boxtimes P_{n+2}$ as in Case 1 and restrict it to the graph $P_m \boxtimes P_{n+2}$ with the fibers P_m^0 and P_m^{n+1} removed. Now $|S| = (n+2+m-2)/4 = \lceil (n+m-2)/4 \rceil$.

Case 4. $3m - n - 6 \equiv 3 \pmod 8$.
Since $3(m+2) - (n+1) - 6 \equiv 0 \pmod 8$ we take the power dominating set S of $P_{m+2} \boxtimes P_{n+1}$ and remove the fibers ${}^0P_n$, ${}^{m+1}P_n$, and P_m^0. Then $|S| = (n+1+m+2-2)/4 = \lceil (n+m-2)/4 \rceil$.

Case 5. $3m - n - 6 \equiv 4 \pmod 8$.
As $3(m+2) - (n+2) - 6 \equiv 0 \pmod 8$ we proceed as before, removing the fibers ${}^0P_n$, ${}^{m+1}P_n$, P_m^0, and P_m^{n+1}. Then $|S| = (n+2+m+2-2)/4 = \lceil (n+m-2)/4 \rceil + 1$. Note that in this special case we miss the lower bound by 1.

Case 6. $3m - n - 6 \equiv 5 \pmod 8$.
Then $3(m+1) - n - 6 \equiv 0 \pmod 8$ so we construct the power dominating set S of $P_{m+1} \boxtimes P_n$ and remove the fiber ${}^0P_n$. Then $|S| = (n+m+1-2)/4 = \lceil (n+m-2)/4 \rceil$.

Case 7. $3m - n - 6 \equiv 6 \pmod 8$.
Then $3(m+1) - (n+1) - 6 \equiv 0 \pmod 8$ and we start from the power dominating set S of $P_{m+1} \boxtimes P_{n+1}$, and remove the fibers ${}^0P_n$ and P_m^0. Now $|S| = (n+1+m+1-2)/4 = \lceil (n+m-2)/4 \rceil$.

Case 8. $3m - n - 6 \equiv 7 \pmod 8$.
Then $3(m+1) - (n+2) - 6 \equiv 0 \pmod 8$. Take the power dominating set S in $P_{m+1} \boxtimes P_{n+2}$ and remove the fibers ${}^0P_n$, P_m^0, and P_m^{n+1}. We have $|S| = (n+2+m+1-2)/4 = \lceil (n+m-2)/4 \rceil$.

4 The lexicographic product

The remaining standard graph product to consider is the lexicographic one. For it the power domination problem is much easier. In fact we will determine the power domination number for any lexicographic product in terms of the domination number and the total domination number of its factors.

Recall that for a graph $G = (V, E)$, a set $S \subseteq V$ is a *dominating set* if each vertex in $V \setminus S$ is adjacent to at least one vertex of S. If in addition each vertex of S has a neighbor in S, then S is called a *total dominating set*. The *domination* (resp. *total domination*) *number* $\gamma(G)$ (resp. $\gamma_t(G)$) of G is the minimum cardinality of a dominating (resp. total dominating) set. Then we have:

Theorem 4.1 *For any nontrivial graphs G and H,*

$$\gamma_P(G \circ H) = \begin{cases} \gamma(G); & \gamma_P(H) = 1\,, \\ \gamma_t(G); & \gamma_P(H) > 1\,. \end{cases}$$

Proof. Suppose $\gamma_P(H) = 1$. Let $\{v\}$ be a power dominating set of H and D a dominating set of G. Then $\{v\} \times D$ is a power dominating set of $G \circ H$. Indeed, for any vertex u of G, if $u \notin D$, any vertex of ${}^uH$ is in the neighborhood of (u, v), so it is monitored. If $u \in D$, any neighbor of a vertex of ${}^uH$ not in ${}^uH$ is monitored, and $\{(u, v)\} \cup N(u, v)$ is monitored. So since $\{v\}$ is a power dominating set of H, the fiber is monitored. Therefore $\gamma_P(G \circ H) \leq \gamma(G)$.

Assume that there is a power dominating set S of $G \circ H$ that contains less than $\gamma(G)$ vertices. Then there is an H-fiber ${}^uH$ that contains no vertex of $S \cup N(S)$. So the vertices of ${}^uH$ are monitored by the propagation. The first vertex of ${}^uH$ that is monitored must be the only neighbor not monitored by some vertex not in ${}^uH$. But this vertex is also adjacent to all the other vertices of ${}^uH$, which are not monitored yet. Since H is not trivial this implies that there can be no propagation on ${}^uH$. Hence $\gamma_P(G \circ H) \geq \gamma(G)$.

Suppose now $\gamma_P(H) > 1$. Let D be a total dominating set of G. Then for any vertex v of H, $D \times \{v\}$ is a dominating set of $G \circ H$ and so a power dominating set of H of cardinality $|D| = \gamma_t(G)$. Thus $\gamma_P(G \circ H) \leq \gamma_t(G)$.

Let P be a minimum power dominating set of $G \circ H$. Suppose that there exists an H-fiber ${}^uH$ such that for any neighbor v of u in G, ${}^vH \cap P$ is empty. Thus, the vertices monitored by P in ${}^uH$ are exactly the vertices monitored by $P \cap {}^uH$. Since there is no power dominating set of H of cardinality 1, $P \cap {}^uH$ contains at least two vertices. Let P' be obtained from P by removing all vertices from $P \cap^u H$ but one and adding an arbitrary vertex of ${}^vH$, where v is a neighbor of u in G. Then P' is a power dominating set of $G \circ H$ with $|P'| \leq |P|$. By repeating this process we construct a minimum power dominating set P^* of $G \circ H$ such that any vertex $u \in V(G)$ has a neighbor v in G whose H-fiber ${}^vH$ contains a vertex of P^*. So the set $\{v \in V(G) \mid {}^vH \cap P^* \neq \emptyset\}$ is a total dominating set of G and we conclude that $\gamma_P(G \circ H) = |P^*| \geq \gamma_t(G)$. □

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