Generalized power domination: propagation radius and Sierpiński graphs

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Abstract

The recently introduced concept of k-power domination generalizes domination and power domination, the latter concept being used for monitoring an electric power system. The k-power domination problem is to determine a minimum size vertex subset S of a graph G such that after setting $X = N[S]$, and iteratively adding to X vertices x that have a neighbour v in X such that at most k neighbours of v are not yet in X, we get $X = V(G)$. In this paper the k-power domination number of Sierpiński graphs is determined. The propagation radius is introduced as a measure of the efficiency of power dominating sets. The propagation radius of Sierpiński graphs is obtained in most of the cases.

Keywords: power domination; electrical network monitoring; domination; Sierpiński graph; propagation radius

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1 Introduction

The motivation for the power domination in graphs is the problem of monitoring an electric power system by placing as few measurement devices in the system as possible [3]. Actually, several hundreds of measurement units for monitoring an electric power system have been installed world wide. For a detailed description of the application in power networks and for related references see the very instructive introduction of [1].

The problem of monitoring an electric power system was formulated in graph theory terms as follows [13]. Let G be a graph and $S \subseteq V(G)$. Then the set $M(S)$, called the set of vertices monitored by S, is defined as follows. Initially, the set $M(S)$ consists of all vertices dominated by S. After that, we repeatedly add to $M(S)$ vertices x that have a neighbour v in $M(S)$ such that all the other neighbours of v are already in $M(S)$. We continue this process until no such vertex x exists, $M(S)$ being the obtained set. The set S is a power dominating set of G, PD-set for short, if $M(S) = V(G)$. The power domination number $\gamma_{\rm P}(G)$ of G is the minimum cardinality of a PD-set in G. Actually, the formulation of the power domination problem as just given is not the original definition but an equivalent simplification of it that was independently proposed in [8, 9]. Recently, the power domination was extended to the so-called generalized power domination [5]. The generalization is that a non-negative integer k is given and then a vertex x is added to the set $M(S)$ of already observed vertices provided that x has a neighbour v in $M(S)$ such that at most k neighbours of v are not yet in $M(S)$.

After the power domination was proposed as a graph theory problem, it has received a lot of attention, especially from the algorithmic point of view. For complexity results and related topics (like parametrized power domination complexity and approximation algorithms) see [1, 2, 4, 12, 13, 21]. In particular, the problem of deciding whether there exists a power dominating set of a given size is NP-complete for planar bipartite graphs [4]. On the other hand, linear-time algorithms for finding a minimum power dominating set were given for trees [13], for block graphs [27], and for interval graphs [22]. The exact value of the power domination number was determined for some products of graphs in [8, 9], bounds for the power domination numbers of connected graphs and of claw-free cubic graphs are given in [31]. As already mentioned, the k-power domination was first studied in [5]. It was further investigated in [7].

Sierpiński graphs form a two parametric family of graphs introduced in [18], motivated by the Tower of Hanoi problem and studies of certain universal topological spaces. See recent books [14, 24] for many connections between Sierpiński graphs and these two topics and [15] for the problem of when Sierpiński graphs embed as spanning subgraphs into the corresponding Tower of Hanoi graphs. In addition, Sierpiński graphs were studied from numerous other points of view, recent investigations include [11, 16, 17, 20, 23, 28, 29, 30]. We also point out that earlier than the Sierpiński graphs, the so-called WK-recursive networks were introduced in [6], see also [10]. WK-recursive networks are very similar to Sierpiński graphs—they can be obtained from Sierpiński graphs by adding a link (an open edge) to each of its extreme vertices. In addition, Sierpiński graph were independently studied in [26].

We proceed as follows. In the next section concepts needed in this paper are introduced and some known results recalled. Then, in Section 3, we determine the k-power domination number of Sierpiński graphs. This seems to be the first class of graphs of fractal nature, for which the power domination number is determined exactly. Actually, for only few non-trivial families of graphs the exact power domination number is known. In the subsequent section we introduce the concept of a propagation radius as the minimum number of propagation steps over all kPD-sets. We were in particular motivated by the investigations of Aazami in [1], where the problem of determining a minimum size power dominating set is studied under the additional condition that the number of propagation steps in bounded by a fixed constant. In the final section the propagation radius of Sierpiński graphs S_p^n is determined for most of the parameters p and n.

2 Preliminaries

All graphs $G = (V(G), E(G))$ considered are finite and simple, that is, $|V(G)| \in \mathbb{N}$ and there are neither multiple edges nor loops. The *open neighbourhood of* a vertex v of G, denoted by $N_G(v)$, is the set of vertices adjacent to v. The *closed neighbourhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. The *open* (resp. *closed*) *neighbourhood* $N_G(S)$ (resp. $N_G[S]$) of a set $S \subseteq V(G)$ is the union of the open (resp. closed) neighbourhoods of its elements. When G is clear from context, we may use N instead of N_G. The *degree* of a vertex v, denoted by $d(v)$, is the order of its open neighbourhood, the maximum degree of G is denoted by $\Delta(G)$.

A *dominating set* of a graph G is a set of vertices S such that $N[S] = V(G)$. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set of G.

The *distance* $d_G(u, v)$ between vertices u and v of a connected graph G is the number of edges on a shortest u, v-path. The *eccentricity* of a vertex u is $\max_{x \in V(G)} d_G(u, x)$. The *radius* of G, denoted by $\mathrm{rad}(G)$, is the minimum eccentricity of the vertices of G.

Let $k \geq 0$. If G is a graph and $S \subseteq V(G)$, then the sets $\left(\mathcal{P}^i_{G,k}(S)\right)_{i \geq 0}$ of vertices *monitored* by S at step i are defined as follows:

$$\mathcal{P}^0_{G,k}(S) = N[S], \text{ and}$$
$$\mathcal{P}^{i+1}_{G,k}(S) = \bigcup\{N[v] \colon v \in \mathcal{P}^i_{G,k}(S) \text{ such that } \left|N[v] \setminus \mathcal{P}^i_{G,k}(S)\right| \leq k\}.$$

Clearly, $\mathcal{P}^i_{G,k}(S) \subseteq \mathcal{P}^{i+1}_{G,k}(S) \subseteq V(G)$ holds for any $i \geq 0$. It is also clear (cf. [5]) that if $\mathcal{P}^{i_0}_{G,k}(S) = \mathcal{P}^{i_0+1}_{G,k}(S)$ for some i_0, then $\mathcal{P}^j_{G,k}(S) = \mathcal{P}^{i_0}_{G,k}(S)$ for every $j \geq i_0$; accordingly define $\mathcal{P}^\infty_{G,k}(S) = \mathcal{P}^{i_0}_{G,k}(S)$. When the graph G is clear from the context, we will simplify the notation to $\mathcal{P}^i_k(S)$ and $\mathcal{P}^\infty_k(S)$.

Here are the key definitions from [5]. Let k, G, and S be as above. If $\mathcal{P}^{\infty}_{G,k}(S) = V(G)$, then S is called a *k-power dominating set* of G, abbreviated *kPD-set*. The minimum cardinality of a kPD-set in G is the *k-power domination number* $\gamma_{\mathrm{P},k}(G)$ of G. A *$\gamma_{\mathrm{P},k}(G)$-set* is a kPD-set in G of cardinality $\gamma_{\mathrm{P},k}(G)$.

We now turn to Sierpiński graphs. For $n \in \mathbb{N}$, let $[n]_0 = \{0, \dots, n-1\}$ and $[n] = \{1, \dots, n\}$. If $p, n \in \mathbb{N}$, then the *Sierpiński graph* S_p^n is defined as follows. The vertex set of S_p^n is the set $[p]_0^n$, whose elements we denote by $s_n \dots s_1$. Two vertices s and t are adjacent if and only if there exists a $\delta \in [n]$ such that

(i) $s_d = t_d$, for $d \in [n] \setminus [\delta]$;

(ii) $s_\delta \neq t_\delta$;

(iii) $s_d = t_\delta$ and $t_d = s_\delta$ for $d \in [\delta - 1]$.

Note that $S_1^n \cong K_1$ $(n \geq 1)$, $S_2^n \cong P_{2^n}$ $(n \geq 1)$, and $S_p^1 \cong K_p$ $(p \geq 1)$. See Fig. 1 for S_5^3.

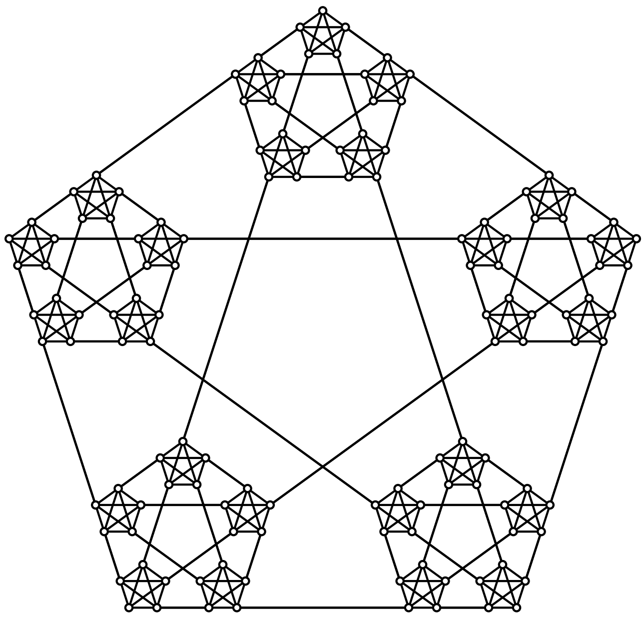

Figure 1: Sierpiński graph S_5^3

Vertices of the form $k \dots k = k^n$ are called *extreme vertices* of S_p^n. Clearly, S_p^n contains p extreme vertices and they are of degree $p-1$; all the other vertices are of degree p. If $r \in [n-1]$, then let $s_n \dots s_{r+1} S_p^r$ denote the subgraph of S_p^n induced by vertices with prefix $s_n \dots s_{r+1}$. Note that $s_n \dots s_{r+1} S_p^r$ is isomorphic to S_p^r, in particular, $s_n \dots s_2 S_p^1$ is isomorphic to K_p.

Finally, we recall two related results on Sierpiński graphs.

Theorem 2.1 ([19, Theorem 3.8]) *For any $n \geq 1$ and any $p \geq 1$,*

$$\gamma_{\mathrm{P},0}(S_p^n) = \gamma(S_p^n) = \begin{cases} p\frac{p^{n-1}+1}{p+1}; & n \text{ even}, \\ \frac{p^n+1}{p+1}; & n \text{ odd}. \end{cases}$$

Theorem 2.2 ([25, Theorem 3.1]) *For any $n \geq 1$ and any $p \geq 1$,*

$$\mathrm{rad}(S_p^n) = \begin{cases} 2^n - 1; & n < p, \\ 2^{n-p+1}(2^{p-1} - 1); & n \geq p. \end{cases}$$

In the rest of the paper we will consider the k-power domination for $k \geq 1$. Hence, until stated otherwise, we will throughout assume that $k, p, n \in \mathbb{N}$.

3 The k-power domination number of Sierpiński graphs

In this section we prove:

Theorem 3.1 *We have*

$$\gamma_{\mathrm{P},k}(S_p^n) = \begin{cases} 1; & p = 1 \text{ or } p = 2 \text{ or } n = 1 \text{ or } p \leq k+1, \\ p - k; & n = 2 \text{ and } p \geq k+2, \\ (p-k-1)p^{n-2}; & n \geq 3 \text{ and } p \geq k+2. \end{cases}$$

Proof. Recall that $S_1^n \cong K_1$, that $S_2^n \cong P_{2^n}$, and that $S_p^1 \cong K_p$. Hence $\gamma_{\mathrm{P},k}(G) = 1$ for any of these Sierpiński graphs G. Note also that since $\Delta(S_p^n) = p$, it follows that $\gamma_{\mathrm{P},k}(S_p^n) = 1$ when $k \geq p - 1 = \Delta(S_p^n) - 1$ (see [5, Lemma 7]).

Now let $n \geq 3$ and $p \geq k+2$. Let S be a kPD-set of S_p^n and let $w \in [p]_0^{n-2}$. We claim that

$$|S \cap V(wS_p^2)| \geq p - k - 1. \tag{1}$$

Assume first that $|S \cap V(wS_p^2)| = p - k - 2$ and that S has exactly one vertex in p-cliques wiS_p^1 for $i \in \{i_1, \ldots, i_{p-k-2}\}$. Then $S \cap V(wi'S_p^1) = \emptyset$ holds for $k+2$ coordinates i'. Let wjS_p^1 be an arbitrary such subgraph. Let $X = \{wji_1, \ldots, wji_{p-k-2}\} \cup \{wjj\}$, and observe that $\mathcal{P}_k^1(S) \cap wjS_p^1 \subseteq X$. Note that this conclusion holds for any $j \in J = [p]_0 \setminus \{i_1, \ldots, i_{p-k-2}\}$, and thus that the set of vertices $\{wjj' : j \in J, j' \in J, j' \neq j\}$ has an empty intersection with $\mathcal{P}_k^1(S)$. Since every vertex in $V(S_p^n)$ has either 0 or $k+1$ neighbours in this set, no vertex from this set may get monitored later on, a contradiction. Assume next that $|S \cap V(wS_p^2)| < p - k - 2$ or that S intersects some wiS_p^2 in more than one vertex. Then we can analogously conclude that not all vertices of wjS_p^1 will be monitored. Hence (1) is proved.

Since $V(S_p^n)$ partitions into p^{n-2} sets $V(wS_p^2)$, where $w \in [p]_0^{n-2}$, we obtain:

$$|S| = \sum_{w\in[p]_0^{n-2}} |S \cap V(wS_p^2)| \geq \sum_{w\in[p]_0^{n-2}} (p-k-1) = (p-k-1)p^{n-2}\,.$$

We next show that $\gamma_{\mathrm{P},k}(S_p^n) \leq (p-k-1)p^{n-2}$ whenever $n \geq 3$ and $p \geq k+2$. To this end recall from [18] that S_p^n is hamiltonian for any $p \geq 3$. Using this result together with the fact that contracting each of the subgraphs of S_p^n of the form wS_p^2 into a single vertex yields a graph isomorphic to S_p^{n-2}, we can arrange the subgraphs wS_p^2 into a circuit such that there is exactly one edge between the consecutive subgraphs. We now construct a set S which contains $p-k-1$ vertices of each subgraph wS_p^2.

Let wS_p^2, $w'S_p^2$, and $w''S_p^2$ be arbitrary but fixed consecutive subgraphs in the selected Hamiltonian order. Let uu' be the edge between wS_p^2 and $w'S_p^2$, where $u \in wS_p^2$, and let $x'x''$ be the edge between $w'S_p^2$ and $w''S_p^2$, where $x' \in w'S_p^2$. Let $S \cap w'S_p^2$ consist of the vertex x' and of $p-k-2$ additional vertices, no two lying in the same subgraph $w'iS_p^1$, and no one lying in the p-clique Q that contains u'. Do this in parallel for any subgraph yS_p^2. As wS_p^2, $w'S_p^2$, and $w''S_p^2$ are arbitrary, the set S is precisely defined in this way. In particular, the vertex u is put into S when considering wS_p^2. Observe now that $p-k$ vertices of Q lie in $\mathcal{P}_k^1(S)$: one of these vertices is u', the other $p-k-1$ are those vertices of Q that have a neighbour in the p-cliques that contain the $p-k-1$ vertices of S. But then the remaining k vertices of Q lie in $\mathcal{P}_k^2(S)$ and it is then straightforward that all the vertices of $w'S_p^2$ lie in $\mathcal{P}_k^\infty(S)$. We conclude that S is a kPD-set. Since $|S| = (p-k-1)p^{n-2}$, the proof is complete for the case $n \geq 3$ and $p \geq k+2$.

The remaining case to consider is when $n = 2$. The arguments in this case are similar to those that we used above when $p \geq k+2$. The only difference is that now we have only one subgraph of the form wS_p^2 (that is, the one where w is the empty word). Hence no vertex of S_p^2 is monitored from outside through one of its extremal vertices and thus we need at least $p-k$ vertices in a kPD-set instead of $p-k-1$. It is then easy to verify that kPD-sets of order $p-k$ indeed exist. $\square$

4 Propagation radius

In practice, besides the minimum size of a kPD-set, the information in how many propagation steps the graph is monitored from a given kPD-set could also be important. For instance, in the path graph, its central vertex seems to be “the best” candidate for the power dominating set, as it propagates to the whole path in the shortest time. We hence introduce the *k-propagation radius* of a graph G defined as

$$\mathrm{rad}_{\mathrm{P},k}(G) = 1 + \min\{i :\ \mathcal{P}_{G,k}^i(S) = V(G),\ S\ k\text{PD-set of } G,\ |S| = \gamma_{\mathrm{P},k}(G)\}\,.$$

In [13], trees T for which $\gamma_{\rm P,1}(T) = \gamma(T)$ holds were characterized. This result naturally leads to the more general question for which k and which G, $\gamma_{\rm P,k}(G) = \gamma(G)$ holds. From this point of view we observe the following:

Proposition 4.1 *Let G be a graph. Then $\gamma_{\rm P,k}(G) = \gamma(G)$ if and only if $\mathrm{rad}_{\rm P,k}(G) = 1$.*

Proof. Suppose $\mathrm{rad}_{\rm P,k}(G) = 1$. Then a $\gamma_{\rm P,k}(G)$-set is also a dominating set, hence $\gamma(G) \le \gamma_{\rm P,k}(G)$. Since in general $\gamma_{\rm P,k}(G) \le \gamma(G)$ holds, we conclude that $\gamma_{\rm P,k}(G) = \gamma(G)$.

Suppose $\mathrm{rad}_{\rm P,k}(G) \ge 2$. Since $\mathcal{P}^0_{G,k}(S) = V(G)$ holds for any γ-set of G, we must have $\gamma_{\rm P,k}(G) < \gamma(G)$. □

Note that $\gamma_{\rm P,k}(G)$ can be any positive integer less than $\gamma(G)$ as soon as $\mathrm{rad}_{\rm P,k}(G) > 1$. Indeed, if $n \ge 2$, then let T_n be the tree obtained from the star $K_{1,n}$ by subdividing each of its edges. Then $\gamma_{\rm P,k}(T_n) = 1$ and $\gamma(T_n) = n$.

Recall from [5, Lemma 7] that if $\Delta(G) \le k+1$, then $\gamma_{\rm P,k}(G) = 1$ and that any vertex of G forms a power dominating set. From the proof of this result, it readily follows:

Lemma 4.2 *If $\Delta(G) \le k+1$, then $\mathrm{rad}_{\rm P,k}(G) = \mathrm{rad}(G)$.*

On the other hand, to see that $\gamma_{\rm P,k}(G) = 1$ in general does not imply that $\mathrm{rad}_{\rm P,k}(G) = \mathrm{rad}(G)$ consider the following example. For $n, k \ge 1$ define the *peacock* P_n^k as follows. Start with a path on n vertices $v_1, \ldots, v_n$ and a vertex x not on the path. Add edges xv_i, $i \in [n]$, and for all $i \in [n-1]$ subdivide the edge v_iv_{i+1} by a vertex w_i. Add $k-1$ pendant vertices at v_i for $i = 2, 3, \ldots, n-1$, and add k pendant vertices at the vertex v_n. Finally, add to the graph a disjoint path P_{k+1} on (consecutive) vertices $u_1, \ldots, u_{k+1}$ and connect u_i with x for every $i \in [k+1]$. The peacock P^3_{10} is shown in Fig. 2.

Proposition 4.3 *For any n and k, $\gamma_{\rm P,k}(P_n^k) = 1$ and $\mathrm{rad}_{\rm P,k}(P_n^k) = n+1$.*

Proof. For $i = 1, \ldots, n-1$, let $v_{i,j}$, $2 \le j \le k-1$, be the $k-1$ leaves attached to v_i and let $v_{n,j}$, $j \in [k]$, be the k leaves attached to v_n. Clearly, $\mathcal{P}^0_k(\{x\}) = \{x\} \cup \{v_1, \ldots, v_n\} \cup \{u_1, \ldots, u_{k+1}\}$. Note further that $\mathcal{P}^1_k(\{x\}) = \mathcal{P}^0_k(\{x\}) \cup \{w_1\}$ and that $\mathcal{P}^i_k(\{x\}) = \mathcal{P}^{i-1}_k(\{x\}) \cup \{w_i, v_{i,1}, \ldots, v_{i,k-1}\}$ holds for $i = 2, \ldots, n-1$. Finally, $\mathcal{P}^n_k(\{x\}) = V(P_n^k)$. If follows that $\{x\}$ is a kPD-set $\gamma_{\rm P,k}(P_n^k) = 1$.

To conclude that $\mathrm{rad}_{\rm P,k}(P_n^k) = n+1$ it suffices to observe that no vertex different from x is a PD-set. This is true for any vertex u_i because then no propagation would be possible from x having $n+k > k$ neighbours not yet monitored. Similarly, no

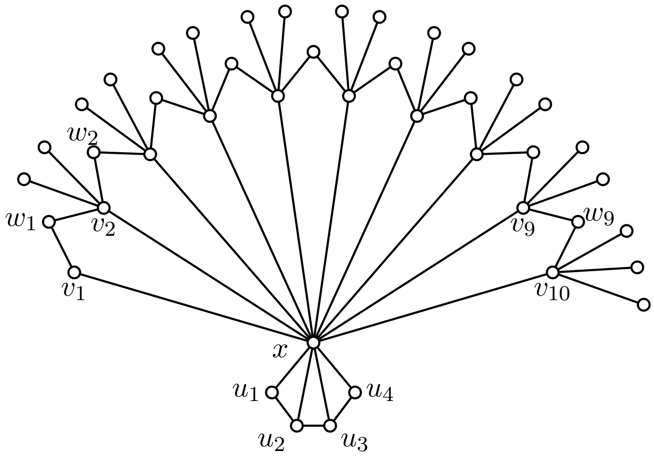

Figure 2: The peacock P_{10}^3

vertex v_i, w_i, or $v_{i,j}$ can form a kPD-set, because (at least) vertices $u_1, \ldots, u_{k+1}$ would not get monitored. □

Note that the arguments of the proof of Proposition 4.3 apply to any graph that is constructed as P_n^k, except that the path on vertices $u_1, \ldots, u_{k+1}$ is replaced by an arbitrary graph of order at least $k+1$. (Using this fact, a graph theorist with artistic gift has many options to draw a fine picture of the peacock's body.)

Corollary 4.4 *For any positive integers k and t, there exists a graph G with $\gamma_{\mathrm{P},k}(G) = 1$ such that* $\mathrm{rad}_{\mathrm{P},k}(G) - \mathrm{rad}(G) = t$.

Proof. By Proposition 4.3, $\mathrm{rad}_{\mathrm{P},k}(P_{t+1}^k) - \mathrm{rad}(P_{t+1}^k) = (t+2) - 2 = t$. □

All the graphs from the proof of Corollary 4.4 have radius 2. To construct graphs with arbitrary radius that lead to the same conclusion, let $P_{n,r}^k$ be the graph obtained from the disjoint union of the peacock $P_{n,r}^k$ and a path P_{2r}, $r \geq 2$, by identifying a leaf of P_{2r} with u_1. Then $\mathrm{rad}(P_{n,r}^k) = r+1$ and $\mathrm{rad}_{\mathrm{P},k}(P_{n,r}^k) = \max\{n+1, 2r\}$. Therefore, for any $n \leq 2r-1$,

$$\mathrm{rad}_{\mathrm{P},k}(P_{n,r}^k) - \mathrm{rad}(P_{n,r}^k) = 2r - (r+1) = r-1\,.$$

5 Propagation radius of Sierpiński graphs

In this section, we compute the propagation radius of the Sierpiǹski graphs S_p^n. We first consider the easier case when $n \leq 2$. First note that if $p = 1$ or $n = 1$, then the graph is a complete graph and the propagation radius is 1.

Theorem 5.1 *If $p \geq 2$, then*

$$\mathrm{rad}_{\mathrm{P},k}(S_p^2) = \begin{cases} 2; & k = 1 \text{ or } p = 2\,, \\ 3; & k \geq 2 \text{ and } p \geq 3\,. \end{cases}$$

Proof. If $k \geq p-1$, then $k+1 \geq \Delta(S_p^2)$ so $\mathrm{rad}_{\mathrm{P},k}(S_p^2) = \mathrm{rad}(S_p^2)$ which is 2 if $p = 2$ and 3 if $p \geq 3$.

Now for $k \in [p-2]$ consider the set $D = \{i0 : \ k \leq i \leq p-1\}$. This set is of order $p - k = \gamma_{\mathrm{P},k}(S_p^2)$. If $k = 1$, this set dominates all of S_p^2 except vertex 00, so $\mathcal{P}_1^1(D) = V(S_p^2)$ and $\mathrm{rad}_{\mathrm{P},1}(S_p^2) \leq 2$. Note that the propagation radius is not 1 because $\gamma(S_p^2) = p > \gamma_{\mathrm{P},1}(S_p^2)$. Suppose now that $k \geq 2$. Observe that $\mathcal{P}_k^0(D) = N[D] = V(S_p^2) \setminus \big(\{0i : \ i \in [k]_0\} \cup (V(iS_p^1))_{i \in [k]_0}\big)$, then that $\mathcal{P}_k^1(D) = V(S_p^2) \setminus \{ij : i \in [k-1], j \in [k]_0\}$ and finally that $\mathcal{P}_k^2(D) = V(S_p^2)$. Thus $\mathrm{rad}_{\mathrm{P},k}(S_p^2) \leq 3$. The proof of $\mathrm{rad}_{\mathrm{P},k}(S_p^2) \geq 3$ is deferred to the following more general lemma which will turn out to be useful later as well. □

Lemma 5.2 *Let $n \geq 2$ and $k \in [p-2]$. If $k \geq 2$ or $n \geq 3$, $\mathrm{rad}_{\mathrm{P},k}(S_p^n) \geq 3$.*

Proof. Suppose that $k \geq 2$ or $n \geq 3$, and let D be a minimum kPD-set of S_p^n. In both cases, $\gamma_{\mathrm{P},k}(S_p^n) \leq (p-2)p^{n-2}$ so there exists some wS_p^2 (w possibly null) containing at most $p-2$ vertices of D. Thus there exist in wS_p^2 two copies of S_p^1 not containing any vertex of D, say $w0S_p^1$ and $w1S_p^1$. We prove that $w01$ or $w10$ is not in $\mathcal{P}_k^1(D)$. Clearly, $w01$ and $w10$ are not in $\mathcal{P}_k^0(D)$. Moreover, $|(w0S_p^1 \cup w1S_p^1) \cap \mathcal{P}_k^0(D)|$ contains no more than $p-k$ vertices if $n = 2$ or than $p-k-1+2$ vertices if $n \geq 3$. So in both cases, $|(w0S_p^1 \cup w1S_p^1) \setminus \mathcal{P}_k^0(D)| \geq 2p - (p-k+1) = p+k-1$ and since $k \leq p-2$, this is at least $2k+1$. Therefore, in $w0S_p^n$ or in $w1S_p^n$, there are more than k unmonitored vertices to which any neighbour of $w01$ or $w10$ respectively is adjacent, preventing any propagation to this vertex on that step. Thus $w01$ or $w10$ is not in $\mathcal{P}_k^1(D)$ and $\mathrm{rad}_{\mathrm{P},k}(S_p^n) \geq 3$. □

We now state the theorem for $n \geq 3$, that we then prove with a sequence of lemmas. (The radius of S_p^n that appears in the statement can be found in [25] and is recalled in Theorem 2.2.)

Theorem 5.3 *If $n \geq 3$, then*

$$\mathrm{rad}_{\mathrm{P},k}(S_p^n) = \begin{cases} 3; & p \geq 2k+3\,, \\ 4 \text{ or } 5; & 2k+2 \geq p \geq k+1+\sqrt{k+1}\,, \\ 5; & k+1+\sqrt{k+1} > p \geq k+2\,, \\ \mathrm{rad}(S_p^n); & p \leq k+1\,. \end{cases}$$

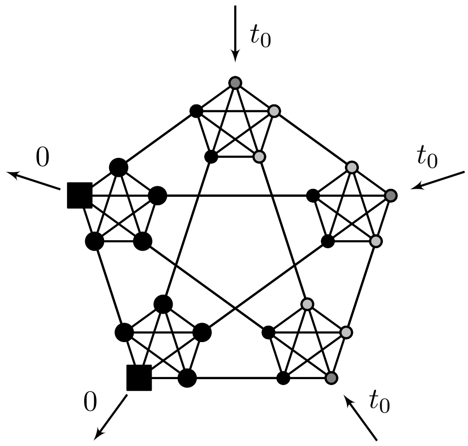

Figure 3: Propagation for getting radius 3 in S_5^2, $k = 2$.

First note that since the maximum degree of S_p^n is p, $\mathrm{rad}_{\mathrm{P},k}(S_p^n) = \mathrm{rad}(S_p^n)$ whenever $k \geq p-1$ as we already mentioned in Lemma 4.2. The following lemmas, together with Lemma 5.2, give the other parts of the proof.

Lemma 5.4 *Let $n \geq 3$ and $k \in [p-2]$. Then $\mathrm{rad}_{\mathrm{P},k}(S_p^n) = 3$ if and only if $p \geq 2k+3$.*

Proof. Recall that $\mathrm{rad}_{\mathrm{P},k}(S_p^n) \geq 3$ by Lemma 5.2. We first prove that if $\mathrm{rad}_{\mathrm{P},k}(S_p^n) \leq 3$, then $p \geq 2k+3$. Let D be a minimum kPD-set of S_p^n that has radius at most 3. By Theorem 3.1, $|D| = (p-k-1)p^{n-2}$ and thus there exist a copy wS_p^2 with at most $p-k-1$ vertices. We consider the possible configurations of sequences of propagations within wS_p^2 (see Fig. 3). Renaming the vertices if necessary, we can assume that the vertices in $D' = D \cap wS_p^2$ are located in the copies wiS_p^1 with $k < i \leq p-1$. Therefore, all vertices of wiS_p^1 for $k < i \leq p-1$ are possibly in $\mathcal{P}_k^0(D')$. Moreover, in each copy $wjS_p^1, j \in [k+1]_0$, no more than $p-k-1$ vertices are in $\mathcal{P}_k^t(D')$ for $t \geq 1$ and the set D' does not power dominate wS_p^2 by itself: it needs that some vertices be monitored from a neighbouring S_p^2.

Observe that if the configuration wS_p^2 gets at most k vertices monitored by propagation from neighbouring copies of S_p^2, and the earliest such propagation is at time t_0, then its vertices are monitored on step t_0+3 at the earliest. Indeed, in such a case, at least one copy among $wjS_p^1, j \in [k+1]_0$ does not have a vertex monitored by a neighbouring copy of S_p^2, say one is wkS_p^1. Say also that the vertex $w00$ is one of the vertices monitored by an adjacent $w'S_p^2$ on step t_0. Then, all vertices of $w0S_p^1$ should get monitored just after, that is $w0S_p^1 \subset \mathcal{P}_k^{t_0+1}$. They then can propagate on step t_0+2 to the vertices $wj0$ for $j \in [k]$, and especially wkk gets monitored no earlier than step t_0+3.

Yet in that case, even if $t_0 = 0$, the graph gets monitored only on step 3 so would have propagation radius 4. Thus, we know that each copy of S_p^2 must have at least $k+1$ vertices monitored by a neighbouring copy of S_p^2. To summarize, each wS_p^n monitors at most $p-k-1$ vertices in neighbouring copies and need to have at least $k+1$ vertices monitored from outside. Moreover, the p copies $i^{n-2}S_p^n$ have only $p-1$ neighbouring copies. Thus, we must have

$$(k+1)p^{n-2} \leq (p-k-1)p^{n-2} - p\,.$$

We infer that $k+1 < p-k-1$ and thus $2k+2 < p$.

Assume now that $p \geq 2k+3$, we give an explicit construction of a kPD-set that has radius 3. For S_p^3, take the set $D = \{ijj :\ i \in [p]_0, i \leq j \leq i+p-k-1\}$ where values are considered modulo p. Then each copy of iS_p^2 gets all its vertices ijj for $i+p-k-1 < j \leq i+p-1$ monitored in $\mathcal{P}_k^0(D)$ by neighbouring copies jS_p^2. As described earlier, the whole S_p^3 thus gets monitored at step 2 (see Fig. 3). For larger n, one can just reproduce this set on each copy wS_p^3. $\square$

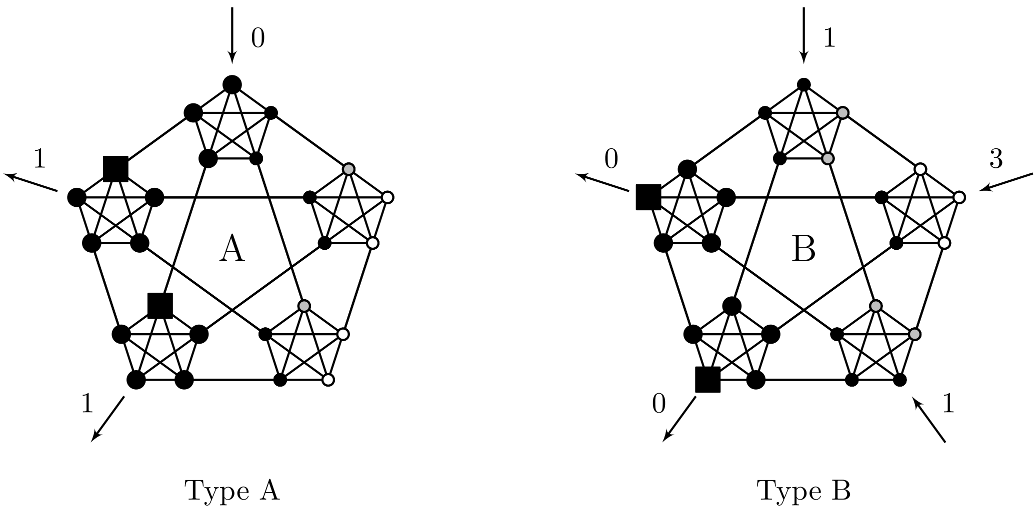

Figure 4: Various types of propagation in S_5^2 when $k=2$.

Lemma 5.5 *Let $n \geq 3$ and $k \in [p-2]$. If $p < k+1+\sqrt{k+1}$, then* $\mathrm{rad}_{\mathrm{P},k}(S_p^n) \geq 5$.

Proof. We prove this lemma by contradiction; assume $\mathrm{rad}_{\mathrm{P},k}(S_p^n) \leq 4$. Let D be a minimum kPD-set of S_p^n that has radius 4. Again consider the possible configurations of propagation within a copy wS_p^2 containing $p-k-1$ vertices of D; they are slightly different (see Fig. 4). Renaming the vertices if necessary, we assume that the vertices in $D' = D \cap wS_p^2$ are located in the copies wiS_p^1 with $k < i \leq p-1$. Again, all vertices of wiS_p^1 for $k < i \leq p-1$ are in $\mathcal{P}_k^0(D')$. Moreover, in each copy $wjS_p^1, j \in [k+1]_0$, exactly $p-k-1$ vertices are in $\mathcal{P}_k^t(D')$ for $t \geq 1$ and, the set D' does not power dominate wS_p^2 by itself, it needs that some vertices be monitored from a neighbouring S_p^2.

As observed in the proof of the previous lemma, if such a configuration gets at most k vertices monitored by propagation from neighbouring copies of S_p^2, and the earliest is at time t_0, then its vertices all get monitored only at step t_0+3. Therefore, this situation may happen in our graph only if $t_0 = 0$. Also, it is required that $w0S_p^1 \subset \mathcal{P}_k^{t_0+1}(D)$, thus that at least $p-k-1$ vertices of $w0S_p^1$ be in $\mathcal{P}_k^0(D \cap wS_p^2)$. The vertices in $D \cap wS_p^2$ are thus necessarily the vertices $wi0$ for $k < i \leq p-1$ (see Fig. 4(A) for an example on S_5^2). As a consequence, it means also that this copy wS_p^2 may propagate only to $p-k-1$ neighbouring copies $w'S_p^2$ on step 1 and to other copies on step 4, when the graph should already be completely monitored. We call similar copies of S_p^2 copies of type (A) and remember that they need one propagation on step 0 and give at most $p-k-1$ on step 1.

Consider now the case when a copy of S_p^2 has at least $k+1$ vertices monitored by neighbouring copies of S_p^2 (such a configuration can be monitored by step 3 as shown, e.g., in Fig. 4(B).) We call such a configuration type (B). We remember that these configurations need $k+1$ propagations from neighbouring $w'S_p^n$ and give at most $p-k-1$ propagations, possibly on step 0. Denote by a the number of wS_p^2 of type (A) and by b the number of type (B) in our graph. Since every copy of type (A) needs to have a vertex monitored on step 0 by some copy of type (B), we have

$$a \leq (p-k-1)b\,. \tag{2}$$

Moreover, each copy of type (B) needs at least $k+1$ propagations from either type (A) or type (B) that are not already given to a type (A) copy. Since the total number of propagations required before step 4 must be less than the total number of propagations given, we get

$$a + (k+1)b \leq (p-k-1)a + (p-k-1)b \tag{3}$$

and combining (2) with (3) we get $0 \leq ((p-k-1)^2 - (k+1))b$. So since b is positive, we must have $(p-k-1)^2 \geq k+1$ or S_p^n has radius at least 5. $\square$

We conclude simply by proving that no S_p^n with $n \geq 3$ and $k \in [p-2]$ has k-power domination radius greater than 5.

Lemma 5.6 *If $n \geq 3$ and $k \in [p-2]$, then* $\mathrm{rad}_{\mathrm{P},k}(S_p^n) \leq 5$.

Proof. As observed in the proof of the previous lemma, if a copy wS_p^2 of type A has one vertex monitored from an adjacent copy on step t_0, it gets fully monitored on step t_0+3. So to monitor S_p^n, on each wS_p^2 we select initial vertices in the power-dominating set so that it is of type A. We need that each copy gets one vertex monitored on step 1 by a neighbouring copy and they all can monitor two such vertices. Arranging the copies so that they monitor vertices of adjacent copies along a Hamiltonian cycle in S_p^{n-2}, we easily find a minimum power dominating set with radius 5. $\square$

6 Concluding remarks

In this paper we have determined the k-power domination number of Sierpiński graphs and introduced the propagation radius of a graph. We have determined this radius for all Sierpiński graphs S_p^n except when $2k+2 \geq p \geq k+1+\sqrt{k+1}$ and $n \geq 3$. In these cases the propagation radius is either 4 or 5. Other conditions than those from the proof of Lemma 5.5 can impose that the radius is at least 5. With better counting, one could get further conditions on k and p for the radius to be 4, but we did not find any concise formula. In particular, we do not give here any reason for the 3-power domination radius of S_6^n or for the 6-power domination radius of S_{10}^n to be greater than 4, though it can be proved that it is at least 5.

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