

Average distance in interconnection networks via reduction theorems for vertex-weighted graphs

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Abstract

Average distance is an important parameter for measuring the communication cost of computer networks. A popular approach for its computation is to first partition the edge set of a network into convex components using the transitive closure of the Djoković-Winkler's relation and then to compute the average distance from the respective invariants of the components. In this paper we refine this idea further by shrinking the quotient graphs into smaller weighted graph called reduced graph, so that the average distance of the original graph is obtained from the reduced graphs. We demonstrate the significance of this technique by computing the average distance of butterfly and hypertree architectures. Along the way a computational error from [European J. Combin. 36 (2014) 71–76] is corrected.

Keywords: average distance; Wiener index; vertex-weighted graph; butterfly network; hypertree network.

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1 Introduction and preliminaries

The average distance is an important concept in mathematics, computer science, and cheminformatics, to mention just some central areas of interest. Moreover, for a very recent application of the average distance in phylogenetics see [34]. For further information on the average distance we refer to [3, 28] and references therein, see also [7] for the average distance in weighted graphs. In computer science, the average distance is used as a fundamental parameter to measure the communication cost of networks. The average distance can be (and in chemistry mostly is) equivalently studied as the sum of the distances between all pairs of vertices, this invariant being known as the Wiener index or the network distance.

Fast computation of the average distance or the Wiener index is hence a fundamental task. While it is straightforward that it can be computed in polynomial time, cf. [27], scientific community has been active in designing linear or even sub-linear algorithms for its computation on applicable families of graphs. For instance, in [5] a linear algorithm was presented for the Wiener index of benzenoid graphs while in [6] a sub-linear time algorithm for the same task was designed; it runs in time proportional to the number of the vertices on the circuit bounding a given benzenoid graph. Of course, a closed formula for the Wiener index of a given graph family also yields such a sub-linear time algorithm.

A well-established technique that often leads to fast algorithms is to partition the edge set of a given graph into (convex) components using the Djoković-Winklers relation Θ [8, 32] or its transitive closure Θ^* , and then to compute the Wiener index from the cardinalities and/or related properties of the (convex) components. This method was introduced in [17] and later used, rediscovered, and extended in numerous papers, see for instance [12, 13, 14, 15, 16, 19, 22, 25, 35], and the survey paper [23] for a complete picture. The key idea of this technique is to first shrink the original graph into smaller weighted graphs called quotient graphs and then to compute the Wiener index of the original graph by computing the Wiener index of the weighted quotient graphs. In this paper, we refine this idea further. Our technique shrinks the quotient graphs yet further into smaller weighted graphs called reduced graphs. During this shrinking process, a part of the Wiener index of the bigger graph is added as a corresponding weight to the smaller graph. At the end of the process, the Wiener index of the original graph is calculated by means of the Wiener index of the weighted reduced graphs.

For the new method, the Wiener index of weighted graphs is the key concept. It was introduced in [20] in order to express the Wiener index of the so-called phenylenes as the Wiener index and the weighted Wiener index of two smaller graphs, respectively. Setting the weights of vertices to be their degrees, the weighted Wiener index reduces to the Gutman index, cf. [26]. The weighted Wiener index turned out to be essential for several extensions of the cut-method [18, 21, 22] and was recently investigated on trees in [10]. For an application of the weighted Wiener index in biology see [1].

The paper is organized as follows. In the rest of this section concepts needed are formally introduced and an earlier result recalled. In the next section two reduction theorems are stated and proved. By their assistance an error from [21] is corrected.

In Section 3 we effortlessly reprove the recent result from [30] which have produced a sub-linear algorithm for the Wiener index of butterfly networks by applying some complex logics. In the subsequent section the method developed in the paper is applied to obtain closed formulas for the average distance of hypertrees which form another family of interconnection networks.

The graphs considered in this paper are simple. If x is a vertex of a graph G , then its open neighborhood $N_G(x)$ is the set of vertices adjacent to x . The *closed neighborhood* of x is $N_G[x] = N_G(x) \cup \{x\}$. Let $d_G(u, v)$ or $d(u, v)$ denote the length of the shortest path between the vertices u and v in G . The *Wiener index* of a graph G is defined as

$$W(G) = \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} d_G(u, v),$$

and the *average distance* of G is

$$\mu(G) = \frac{W(G)}{\binom{|V(G)|}{2}}.$$

Sometimes $\mu(G)$ is also defined as $W(G)/(|V(G)|(|V(G)|-1))$, so that it differs from our definition by factor 2, cf. [34]. A *weighted graph* (G, w) is a graph $G = (V(G), E(G))$ together with the weight function $w : V(G) \rightarrow \mathbb{R}^+$. The *Wiener index* $W(G, w)$ of (G, w) was introduced in [20] as follows:

$$W(G, w) = \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} w(u) w(v) d_G(u, v).$$

If G is a graph, then the Djoković-Winkler's relation Θ [8, 32] is a binary relation defined on $E(G)$ as follows: $e = xy$ is in relation Θ with $f = uv$ if $d(x, u) + d(y, v) \neq d(x, v) + d(y, u)$. Relation Θ is reflexive and symmetric, hence its transitive closure Θ^* is an equivalence relation. The partition of $E(G)$ induced by Θ^* -classes is called the Θ^* -*partition*. Let G be a connected graph and let E_1, \dots, E_k be the Θ^* -partition of $E(G)$. If $i \in \{1, \dots, k\}$, then the *quotient graph* G/E_i is the graph whose vertices are the connected components of $G - E_i$, two vertices C and C' being adjacent if there exist vertices $x \in C$ and $y \in C'$ such that $xy \in E_i$. If $\mathcal{X} = \{X_1, \dots, X_r\}$ and $\mathcal{Y} = \{Y_1, \dots, Y_s\}$ are partitions of a set X , then we say \mathcal{X} is *coarser than* \mathcal{Y} if each set X_i is the union of one or more sets from \mathcal{Y} . Using these concepts, the main result of [21] reads as follows.

Theorem 1.1 [21, Theorem 3.3] *Let (G, w) be a connected, weighted graph, and let $\mathcal{E} = \{E_1, \dots, E_k\}$ be a partition of $E(G)$ coarser than the Θ^* -partition. Then*

$$W(G, w) = \sum_{i=1}^k W(G/E_i, w_i),$$

where $w_i : V(G/E_i) \rightarrow \mathbb{R}^+$ is defined by $w_i(C) = \sum_{x \in C} w(x)$, for all connected components C of $G - E_i$.

Finally, if X is a set and k a non-negative integer, then $\binom{X}{k}$ denoted the set of the k -subsets of X .

2 Reduction theorems and an error correction

For our first main result the following relation is convenient. If G is a graph, then vertices x and y are in *relation* R if $N_G(x) = N_G(y)$. It is easy to see that R is an equivalence relation on $V(G)$ and that (since the graphs considered are without loops) an R -equivalence class induces an edgeless graph. The R -equivalence class containing x will be denoted with $[x]_R$.

Theorem 2.1 *Let (G, w) be a connected, weighted graph, $a \in V(G)$, and $A = [a]_R$. Let (G', w') be defined with $G' = G - (A - \{a\})$, $w'(a) = \sum_{x \in A} w(x)$, and $w'(x) = w(x)$ for any $x \notin A$. Then*

$$W(G, w) = W(G', w') + \sum_{\{x, y\} \in \binom{A}{2}} 2w(x)w(y).$$

Proof. If $|A| = 1$, then $(G', w') = (G, w)$, hence the result is trivial in this case. Assume now that $A = \{a_1, \dots, a_k\}$, where $a_1 = a$ and $k \geq 2$. Then we infer that

- $d_G(a_i, x) = d_G(a_j, x)$ holds for any $a_i, a_j \in A$ and any $x \notin A$;
- $d_G(a_i, a_j) = 2$ holds for any $a_i, a_j \in A$, $i \neq j$; and
- $d_{G'}(x, y) = d_G(x, y)$ holds for any vertices $x, y \in V(G')$.

Using these facts we can compute the Wiener index as follows:

$$\begin{aligned} W(G, w) &= \sum_{i=1}^k \sum_{x \notin A} w(a_i)w(x)d_G(a_i, x) + \sum_{\substack{\{x, y\} \\ x, y \in V(G) - A}} w(x)w(y)d_G(x, y) \\ &\quad + \sum_{\{a_i, a_j\} \in \binom{A}{2}} w(a_i)w(a_j)d_G(a_i, a_j) \\ &= \sum_{x \notin A} w(x) \sum_{i=1}^k w(a_i)d_G(a, x) + \sum_{\substack{\{x, y\} \\ x, y \in V(G) - A}} w(x)w(y)d_G(x, y) \\ &\quad + \sum_{\{a_i, a_j\} \in \binom{A}{2}} 2w(a_i)w(a_j) \\ &= \sum_{x \notin A} w'(x)w'(a)d_{G'}(a, x) + \sum_{\substack{\{x, y\} \\ x, y \in V(G) - A}} w'(x)w'(y)d_{G'}(x, y) \\ &\quad + \sum_{\{a_i, a_j\} \in \binom{A}{2}} 2w(a_i)w(a_j) \\ &= W(G', w') + \sum_{\{a_i, a_j\} \in \binom{A}{2}} 2w(a_i)w(a_j). \quad \square \end{aligned}$$

An important special case occurs when the weight function w is constant on an R -equivalence class, as is for instance the case when considering the standard Wiener index. From this reason and for later use we state:

Corollary 2.2 *Using the notation of Theorem 2.1 and if $w(x) = c$ for any $x \in A$, then*

$$W(G, w) = W(G', w') + 2c^2 \left[\frac{|A|(|A| - 1)}{2} \right].$$

We next define a relation similar to relation R . Vertices x and y of a graph G are in *relation* S if $N_G[x] = N_G[y]$. Relation S is again an equivalence relation on $V(G)$, its equivalence classes induce complete graphs. The S -equivalence class containing x is denoted with $[x]_S$.

Theorem 2.3 *Let (G, w) be a connected, weighted graph, $a \in V(G)$, and $A = [a]_R$. Let (G', w') be defined with $G' = G - (A - \{a\})$, $w'(a) = \sum_{x \in A} w(x)$, and $w'(x) = w(x)$ for any $x \notin A$. Then*

$$W(G, w) = W(G', w') + \sum_{\{x, y\} \in \binom{A}{2}} w(x)w(y).$$

Proof. Arguments are parallel to the proof of Theorem 2.1, the only difference is that now $d_G(a_i, a_j) = 1$ holds for any $a_i, a_j \in A$, $i \neq j$. This leads to the omission of factor 2. \square

Corollary 2.4 *Using the notation of Theorem 2.1 and if $w(x) = c$ for any $x \in A$, then*

$$W(G, w) = W(G', w') + c^2 \left[\frac{|A|(|A| - 1)}{2} \right].$$

Relations R and S are important elsewhere. The reader can check their intrinsic importance for graph products in [11, Chapters 7 and 8]. Vertices that are in one of these relations are often called *twins*. For instance, a graph admits an identifying code if and only if it is twin-free, that is, if and only if the relation S is trivial, cf. [4]. For the role of twins in theoretical computer science we refer to the book [31].

In the rest of the section we re-consider the family of graphs G_n , $n \geq 3$, from [21]. In Fig. 1 the graph G_5 is depicted, the definition of G_n should be then clear by saying that G_n has four inner pentagonal faces and $2(n - 2)$ inner hexagonal faces, so that the total number of inner faces of G_n is $2n$.

In [21] the Θ^* -classes of G_n were wrongly determined and consequently the expression for $W(G_n)$. Now, G_n has $(n - 1)$ Θ^* -classes. First, there are $n - 2$ such classes, which respectively consist of the six edges of horizontal edges of two hexagons lying one above the other. Let E_1 be the union of these Θ^* -classes. In Fig. 2 the graph $G_5 - E_1$, the corresponding weighted quotient graph $(G_5/E_1, w)$ and the reduced graph of $(G_5/E_1, w)$ are shown.

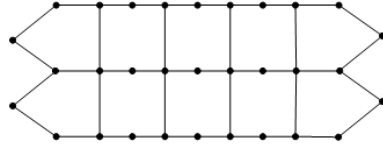


Figure 1: The graph G_5

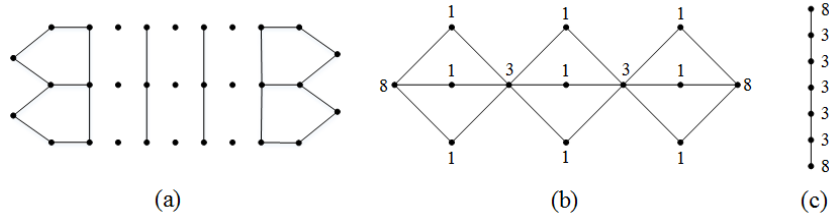


Figure 2: (a) Graph $G_5 - E_1$ (b) Weighted graph $(G_5/E_1, w)$ (c) Reduced graph of $(G_5/E_1, w)$ after applying Corollary 2.2

Note that in each subgraph $K_{2,3}$ of G_n/E_1 , the three vertices of weight 1 form an R -equivalence class. Hence applying Corollary 2.2 $(n - 2)$ times yields

$$W(G_n/E_1, w) = W(P_{2n-3}, w') + 6(n - 2),$$

where w' assigns 8 to the leaves of P_{2n-3} and 3 to all the inner vertices. It is straightforward to get $W(P_{2n-3}, w') = 12n^3 + 6n^2 - 82n + 44$, hence

$$W(G_n/E_1, w) = 12n^3 + 6n^2 - 76n + 32. \quad (1)$$

The remaining Θ^* -class contains the edges of the four pentagons and the vertical edges of the hexagons. It is denoted by E_2 , see Fig. 3 for $G_5 - E_2$ and the corresponding weighted quotient graph $(G_5/E_2, w)$.

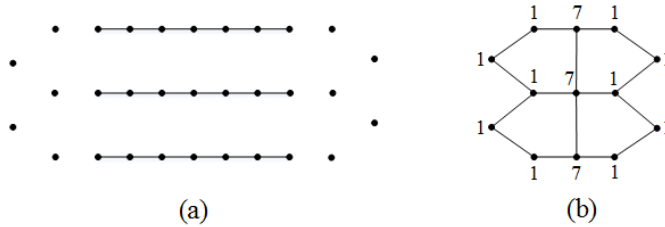


Figure 3: (a) Graph $G_5 - E_2$ (b) Weighted graph $(G_5/E_2, w)$

In general, $(G_n/E_2, w)$ is obtained from $(G_5/E_2, w)$ by replacing each of the three weights 7 with the weight $2n - 3$. By a direct computation we find out (as already computed in [21]) that

$$W(G_n/E_2, w) = 16n^2 + 76n - 28. \quad (2)$$

In view of (1), (2), and Theorem 1.1 we conclude that $W(G_n) = 12n^3 + 22n^2 + 4$. The false expression communicated in [21] for $W(G_n)$ was $12n^3 + 22n^2 - 2n + 8$.

3 Average distance of butterfly networks

The butterfly network is an important and well known topological structure used as an interconnection network. It is a bounded-degree derivative of the hypercube which aims at overcoming some drawbacks of hypercube and is in particular used to perform Fast Fourier Transform [24]. For a recent appealing application of butterfly networks see [2].

The r -dimensional butterfly $BF(r)$ has $n = 2^r(r + 1)$ nodes arranged in $r + 1$ levels of 2^r nodes each. Each node has a distinct label $\langle w, i \rangle$, where i is the *level* of the node ($1 \leq i \leq r + 1$) and w is a r -bit binary number that denotes the *column* of the node. All nodes of the form $\langle w, i \rangle$, $1 \leq i \leq r + 1$, are said to belong to column w . Similarly, the i^{th} level L_i consists of all of the nodes $\langle w, i \rangle$, where w ranges over all r -bit binary numbers. Two nodes $\langle w, i \rangle$ and $\langle w', i' \rangle$ are linked by an edge if $i' = i + 1$ and either w and w' are identical or w and w' differ only in the bit in position i' . The description of the butterfly networks just given is called the *normal representation*. Another appealing representation of these networks is the so-called diamond representation, cf. [24]. Fig. 4 illustrates the diamond and the normal representation of $BF(5)$. For basic properties see [33, Section 11.4]. In this section we add the following fundamental property.

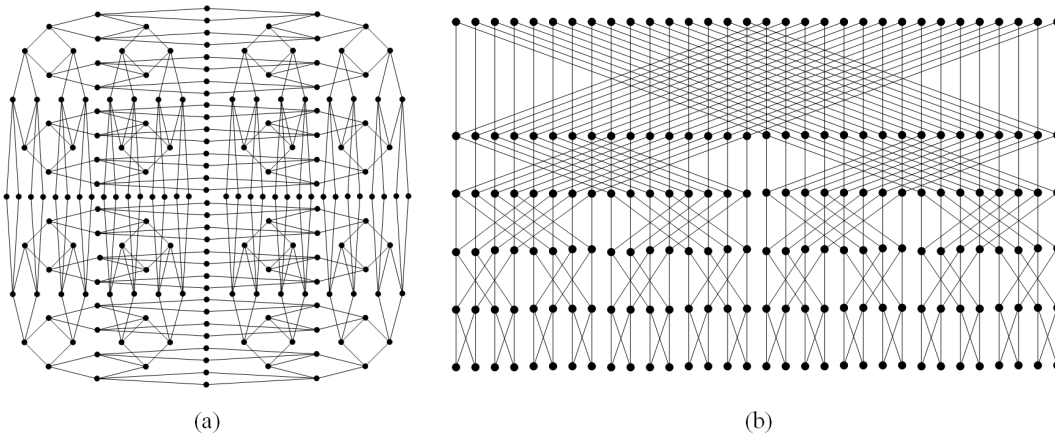


Figure 4: (a) Diamond representation of $BF(5)$ (b) Normal representation of $BF(5)$

Theorem 3.1 *If $r \geq 1$, then $W(BF(r)) = \frac{2^{2r}}{6} \left(5r^3 - 3r^2 + 28r - 36 + \frac{36}{2^r} \right)$.*

Since $|V(BF(r))| = 2^r(r + 1)$, Theorem 3.1 gives the following result which in turn implies that $\mu(BF(r))$ grows as $5r/3$.

Corollary 3.2 *If $r \geq 1$, then $\mu(BF(r)) = \frac{2^r (5r^3 - 3r^2 + 28r - 36) + 36}{2^r (3r^2 + 6r + 3) - 3r - 3}$.*

As already mentioned, Theorem 3.1 was very recently obtained in [30] using quite involved approach. In the rest of this section we are going to derive the result by the method of this paper. We first partition $E(BF(r))$ into r classes E_i , $1 \leq i \leq r$, where E_i contains the edges e such that one end-vertex of e lies in level i and the other end-vertex of e lies in level $i + 1$. (The levels refer to the normal representation of $BF(r)$.) This partition is coarser than the Θ^* -partition, hence we may apply Theorem 1.1. For the clarity of the exposition we will first compute $W(BF(5))$ and then proceed to the general case.

3.1 Computing $W(BF(5))$

In view of Theorem 1.1 we need to compute the weighted Wiener index of five weighted quotient graphs which can in turn be computed by applying Corollary 2.2 as follows.

BF(5)/E₁: The quotient graph **BF(5)/E₁** is isomorphic to $K_{2,32}$. Note that in the quotient graph $K_{2,32}$, the two vertices of weight 80 and the 32 vertices of weight 1 form two R -equivalence classes, see Fig. 5.

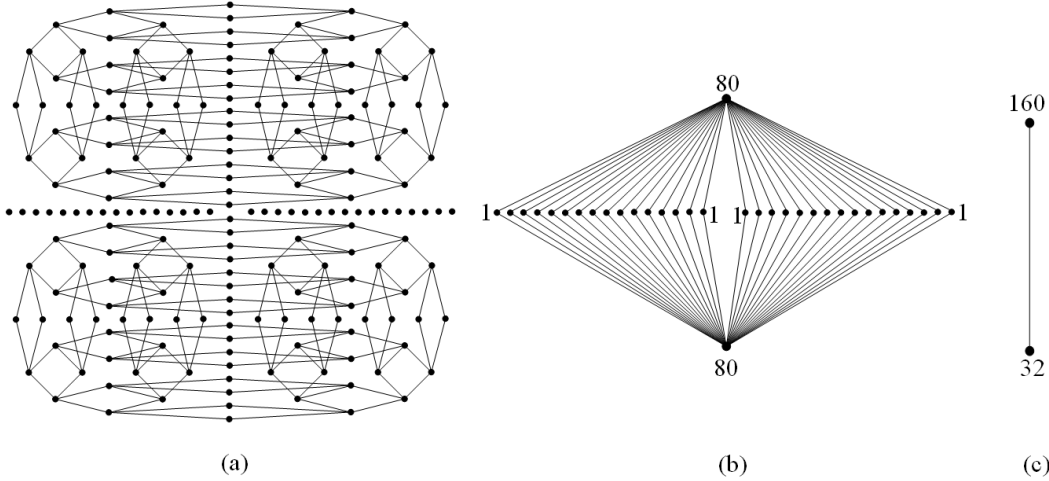


Figure 5: (a) Graph of $BF(5) - E_1$ (b) Weighted graph $(BF(5)/E_1, w)$ (c) Reduced graph of $(BF(5)/E_1, w)$ after applying Corollary 2.2

Therefore, applying Corollary 2.2 two times yields

$$W(BF(5)/E_1, w) = W(P_2, w') + 13792,$$

where w' assigns 160 and 32 to the vertices of P_2 . Since $W(P_2, w') = 5120$, we get

$$W(BF(5)/E_1, w) = 5120 + 13792 = 18912. \quad (3)$$

BF(5)/E₂: It is isomorphic to $K_{4,16}$, where the four vertices of weight 32 and the 16 vertices of weight 4 form two R -equivalence classes, see Fig. 6.

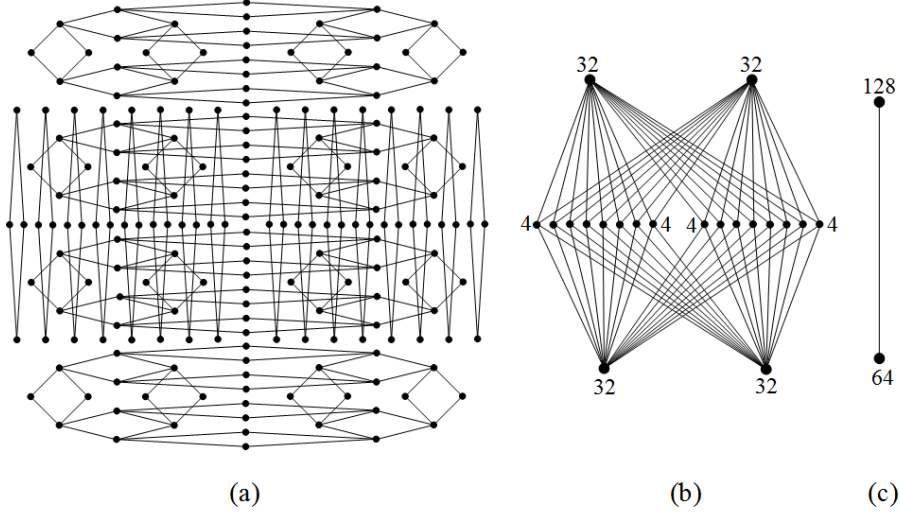


Figure 6: (a) Graph of $BF(5) - E_2$ (b) Weighted graph $(BF(5)/E_2, w)$ (c) Reduced graph of $(BF(5)/E_2, w)$ after applying Corollary 2.2

Hence applying Corollary 2.2 two times we get $W(BF(5)/E_2, w) = W(P_2, w') + 16128$, where w' assigns 128 and 64 to the vertices of P_2 . Since $W(P_2, w') = 8192$, we have

$$W(BF(5)/E_2, w) = 8192 + 16128 = 24320. \quad (4)$$

BF(5)/E₃: It is isomorphic to $K_{8,8}$. The eight vertices of weight 12 and the eight vertices of weight 12 form two R -equivalence classes, see Fig. 7.

Applying Corollary 2.2 two times we get $W(BF(5)/E_3, w) = W(P_2, w') + 16128$, where w' assigns 96 and 96 to the vertices of P_2 . Since $W(P_2, w') = 9216$ we conclude that

$$(BF(5)/E_3, w) = 9216 + 16128 = 25344. \quad (5)$$

BF(5)/E₄: It is isomorphic to $BF(5)/E_2$, see Fig. 8. It follows that

$$W(BF(5)/E_4, w) = W(BF(5)/E_2, w) = 24320. \quad (6)$$

BF(5)/E₅: It is isomorphic to the quotient graph of $BF(5) - E_1$, see Fig. 9. Hence,

$$W(BF(5)/E_5, w) = W(BF(5)/E_1, w) = 18912. \quad (7)$$

Inserting (3), (4), (5), (6), and (7) into Theorem 1.1 we conclude that $W(BF(5)) = 111808$.

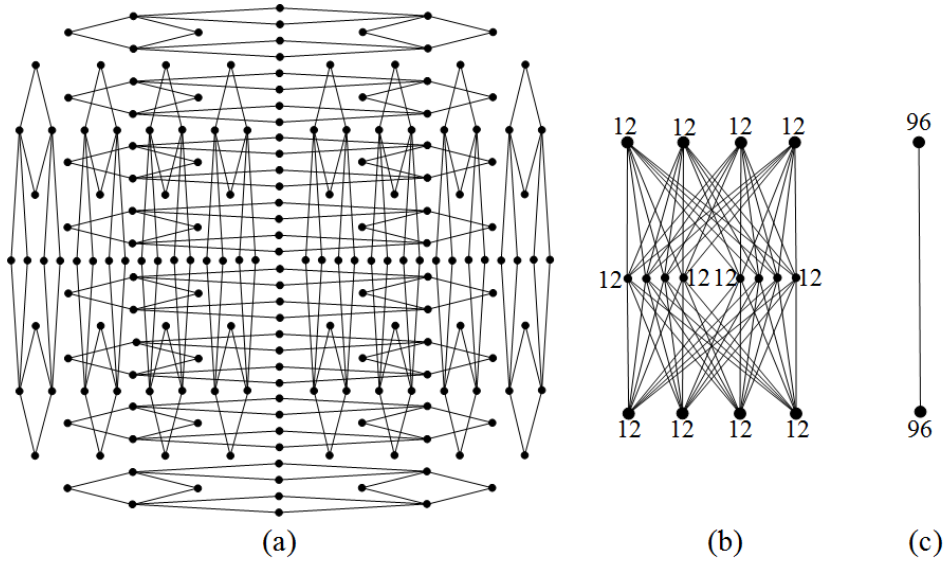


Figure 7: (a) Graph of $BF(5) - E_3$ (b) Weighted graph $(BF(5)/E_3, w)$ (c) Reduced graph of $(BF(5)/E_3, w)$ after applying Corollary 2.2

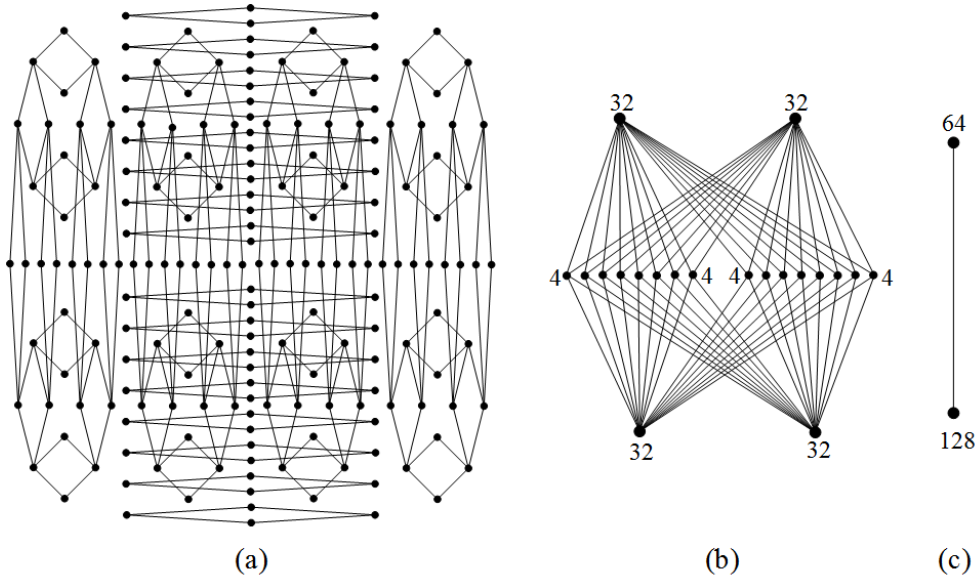


Figure 8: (a) Graph of $BF(5) - E_4$ (b) Weighted graph $(BF(5)/E_4, w)$ (c) Reduced graph of $(BF(5)/E_4, w)$ after applying Corollary 2.2

3.2 The general case

In the general case, the quotient graph $BF(r)/E_i$, $1 \leq i \leq r$, is isomorphic to $K_{2^i, 2^{r-i+1}}$. Note that in the quotient graph $K_{2^i, 2^{r-i+1}}$, $1 \leq i \leq r$, the 2^i vertices of weight $(r - i + 1)$

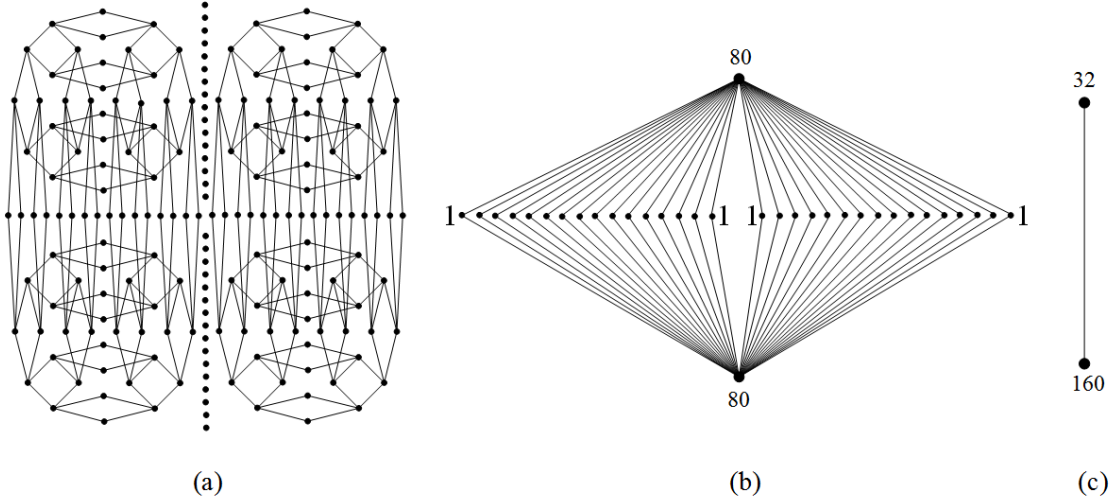


Figure 9: (a) Graph of $BF(5) - E_5$ (b) Weighted graph $(BF(5)/E_5, w)$ (c) Reduced graph of $(BF(5)/E_5, w)$ after applying Corollary 2.2

$1)2^{r-i}$ and the 2^{r-i+1} vertices of weight $i2^{i-1}$ form two R -equivalence classes. Then

$$\begin{aligned}
W(BF(r)/E_i, w) &= W(P_2, w') + i^2 2^{2i-2} 2^{r-i+1}(2^{r-i+1} - 1) \\
&\quad + 2^{2r-2i} 2^i(2^i - 1)(r - i + 1)^2 \\
&= i 2^{2r}(r - i + 1) + i^2 2^{2i-2} 2^{r-i+1}(2^{r-i+1} - 1) \\
&\quad + 2^{2r-2i} 2^i(2^i - 1)(r - i + 1)^2 \\
&= i 2^{2r}(r - i + 1) + i^2 2^{2i-2} 2^{r-i+1}(2^{r-i+1} - 1) \\
&\quad + 2^{2r-2i} 2^i(2^i - 1)(r - i + 1)^2,
\end{aligned}$$

where w' assigns $(r - i + 1)2^r$ and $i 2^r$ to the vertices of P_2 . Therefore,

$$\begin{aligned}
W(BF(r)) &= \sum_{i=1}^r (W(BF(r)/E_i, w) + i^2 2^{2i-2} 2^{r-i+1}(2^{r-i+1} - 1) \\
&\quad + 2^{2r-2i} 2^i(2^i - 1)(r - i + 1)^2) \\
&= \sum_{i=1}^r (i 2^{2r}(r - i + 1) + i^2 2^{2i-2} 2^{r-i+1}(2^{r-i+1} - 1) \\
&\quad + 2^{2r-2i} 2^i(2^i - 1)(r - i + 1)^2) \\
&= \frac{2^r}{6} (5r^3 2^r - 3r^2 2^r + 28r 2^r - 36 2^r + 36),
\end{aligned}$$

and Theorem 3.1 follows.

4 Average distance of hypertrees

The basic skeleton of a *hypertree* $HT(r)$ is a complete binary tree T_r , that is, T_r is a spanning subgraph of $HT(r)$. Its vertices are labeled as follows: The root node has label 1 and is said to be at level 1. The labels of the left (resp. right) children of a vertex are formed by appending 0 (resp. 1) to the label of the parent vertex, see Fig. 10(a).

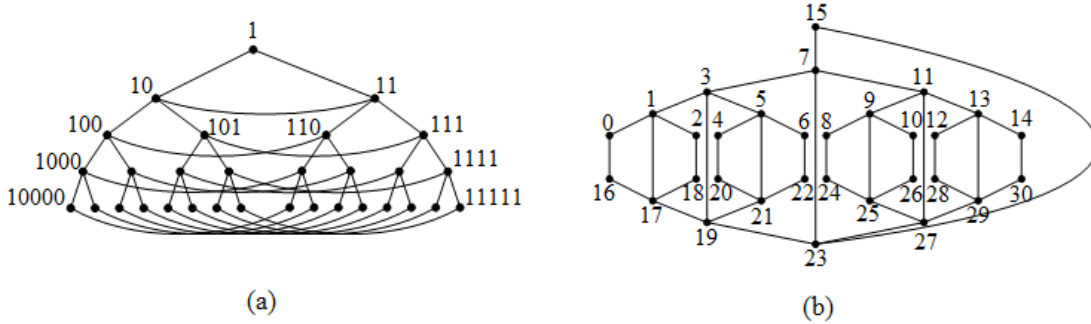


Figure 10: (a) $HT(5)$ with binary labels (b) $HT(5)$ with inorder labeling

In the corresponding decimal labelling of the hypertree, the children of the vertex x are labeled with $2x$ and $2x + 1$. Additional edges in a hypertree are horizontal, where two vertices in the same level i , $1 \leq i \leq r$, are joined by an edge if their label difference is 2^{i-2} . We denote the r -level hypertree with $HT(r)$, $r \geq 2$, [9].

We note that the hypertree $HT(r)$ has $2^r - 1$ vertices and $3(2^{r-1} - 1)$ edges. The diameter and connectivity of $HT(r)$ are $2r - 3$ and 2 respectively and it is a planar graph [29], see Fig. 10(b). The main result of this section asserts:

Theorem 4.1 *If $r \geq 1$, then*

$$W(HT(r)) = 2^{2r-2}(4r + 1) + 2^r(3r - 8 \sinh(r \log(2)) + 1) - 1.$$

Using the fact that $|V(HT(r))| = 2^r - 1$ we get:

Corollary 4.2 *If $r \geq 1$, then $\mu(HT(r)) = \frac{2^{2r}(4r - 15) + 2^{r+1}(3r + 1) + 12}{2^{2r+1} - 3 \cdot 2^{r+1} + 4}$.*

Hence $W(HT(r))$ grows as $2r$.

In the rest of the section we prove the theorem. After determining the Θ^* -classes of $HT(r)$ we will first compute $HT(5)$ and proceed to the general case at the end of the section.

4.1 Θ^* -classes of $HT(r)$

A removal of the horizontal edges of the hypertree $HT(r)$ leaves a complete binary tree T_r . Label the vertices of T_r using binary codes corresponding to the inorder labeling begin with 0, see Fig. 10(b). The graph $HT(r)$ contains $(2^{r-1} - 1)$ Θ^* -classes that can be described as follows.

- For $1 \leq i \leq r-2$, $1 \leq j \leq 2^{r-(i+1)}$ and j is odd, let S_j^i be the Θ^* -classes in $HT(r)$ containing the edges $\{(2^{i-1}(2j-1)-1, j2^i-1), (2^{r-1}+2^{i-1}(2j-1)-1, 2^{r-1}+j2^i-1)\}$.
- For $1 \leq i \leq r-2$, $1 \leq j \leq 2^{r-(i+1)}$ and j is even, let S_j^i be the Θ^* -classes in $HT(r)$ containing the edges $\{(2^{i-1}(2j-1)-1, 2^{i-1}(2j-2)-1), (2^{r-1}+2^{i-1}(2j-1)-1, 2^{r-1}+2^{i-1}(2j-2)-1)\}$.
- The remaining Θ^* -class, to be denoted E_{r-1} , containing all the remaining edges: $E_{r-1} = \{(k-1, k+2^{r-1}-1), (2^{r-2}-1, 2^{r-1}-1), (2^{r-1}-1, 2^{r-1}+2^{r-2}-1) : 1 \leq k \leq 2^{r-1}-1\}$. (In Fig. 10(b) these are all the vertical edges together with the edge (15, 23).)

In order to apply Theorem 1.1 we now construct a partition $\{E_1, \dots, E_{r-1}\}$ coarser than the Θ^* -partition as follows. We have already defined E_{r-1} above, while the other classes E_i are defined as:

$$E_i = \bigcup_{j=1}^{2^{r-(i+1)}} S_j^i, \quad 1 \leq i \leq r-2.$$

To reduce the quotient graphs $HT(r)/E_i$, we use the equivalence relation R for $1 \leq i \leq r-2$ (that is, we apply Corollary 2.2) and the equivalence relation S for $i = r-1$ (that is, we apply Corollary 2.4).

4.2 Computing $W(HT(5))$

To apply Theorem 1.1 we need to compute the weighted Wiener index of four weighted quotient graphs which can in turn be computed applying either Corollary 2.2 or Corollary 2.4 as follows.

HT(5)/E₁: It is isomorphic to $K_{1,8}$. Note that in the quotient graph $K_{1,8}$, the eight vertices of weight 2 form an R -equivalence class, see Fig. 11.

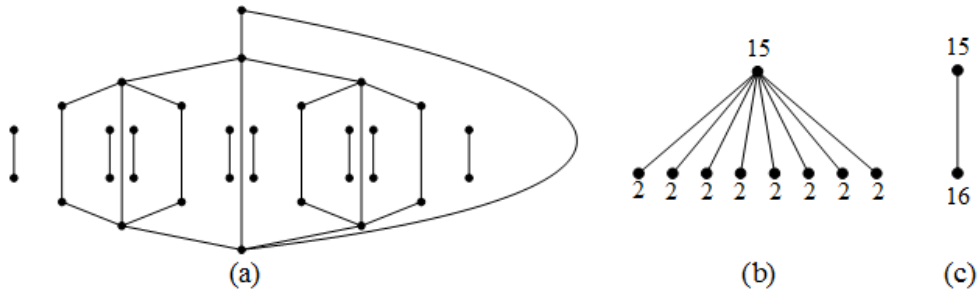


Figure 11: (a) Graph of $HT(5) - E_1$ (b) Weighted graph $(HT(5)/E_1, w)$ (c) Reduced graph of $(HT(5)/E_1, w)$ after applying Corollary 2.2

Corollary 2.2 yields

$$W(HT(5)/E_1, w) = W(P_2, w') + 224,$$

where w' assigns 15 and 16 to the vertices of P_2 . Since $W(P_2, w') = 240$ we get

$$W(HT(5)/E_1, w) = 240 + 224 = 464. \quad (8)$$

HT(5)/E₂: It is isomorphic to $K_{1,4}$. Note that in the quotient graph $K_{1,4}$, the four vertices of weight 6 form an R -equivalence class, see Fig. 12.

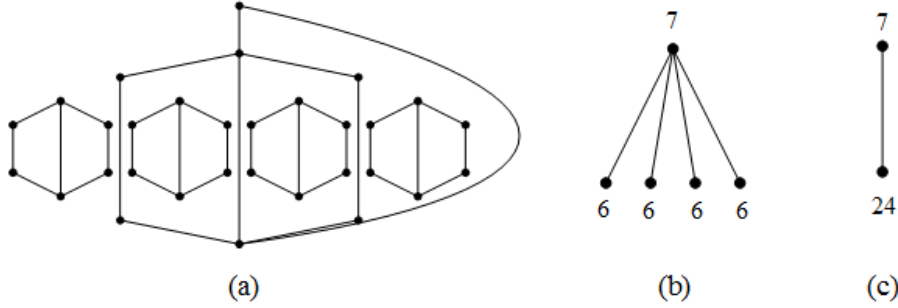


Figure 12: (a) Graph of $HT(5) - E_2$ (b) Weighted graph $(HT(5)/E_2, w)$ (c) Reduced graph of $(HT(5)/E_2, w)$ after applying Corollary 2.2

Applying Corollary 2.2 yields $W(HT(5)/E_2, w) = W(P_2, w') + 432$, where w' assigns 7 and 24 to the vertices of P_2 . Since $W(P_2, w') = 168$ we get

$$W(HT(5)/E_2, w) = 168 + 432 = 600. \quad (9)$$

HT(5)/E₃: It is isomorphic to $K_{1,2}$, where in the quotient graph $K_{1,2}$, the two vertices of weight 14 form an R -equivalence class, see Fig. 13.

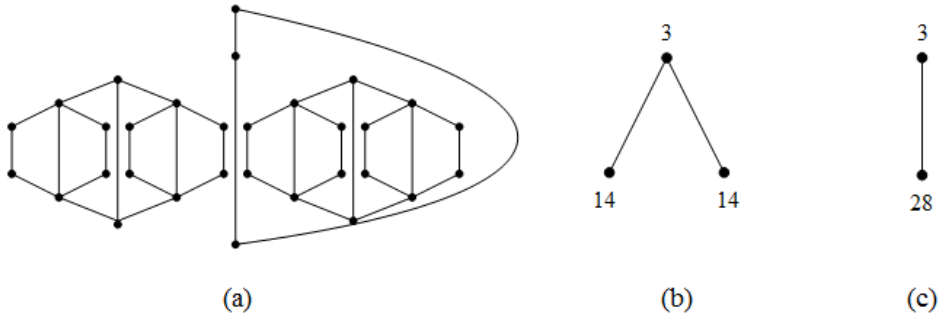


Figure 13: (a) Graph of $HT(5) - E_3$ (b) Weighted graph $(HT(5)/E_3, w)$ (c) Reduced graph of $(HT(5)/E_3, w)$ after applying Corollary 2.2

Applying Corollary 2.2 again we get $W(HT(5)/E_3, w) = W(P_2, w') + 392$, where w' assigns 3 and 28 to the vertices of P_2 . As $W(P_2, w') = 84$ we now have

$$W(HT(5)/E_3, w) = 84 + 392 = 476. \quad (10)$$

HT(5)/E₄: It is isomorphic to K_3 , where the two vertices of weight 15 form an S -equivalence class, see Fig. 14.

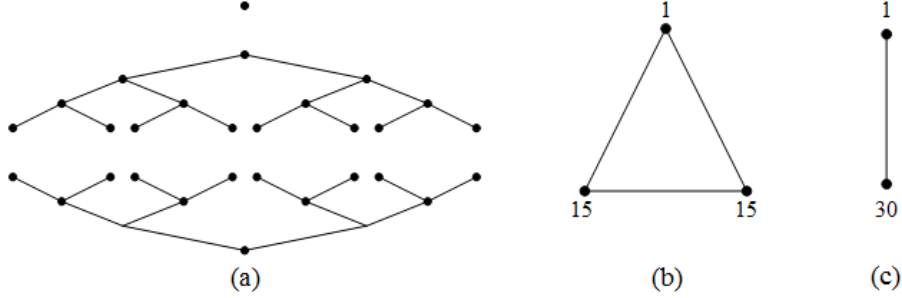


Figure 14: (a) Graph of $HT(5) - E_4$ (b) Weighted graph $(HT(5)/E_4, w)$ (c) Reduced graph of $(HT(5)/E_4, w)$ after applying Corollary 2.4

We now apply Corollary 2.4 to get $W(HT(5)/E_4, w) = W(P_2, w') + 225$, where w' assigns 1 and 30 to the vertices of P_2 . Since $W(P_2, w') = 30$, we get

$$W(HT(5)/E_4, w) = 30 + 225 = 255. \quad (11)$$

Inserting (8), (9), (10), and (11) into Theorem 1.1 we conclude that $W(HT(5)) = 1795$.

4.3 The general case

In general, $HT(r)/E_i$ is isomorphic to $K_{1,2^{r-i-1}}$, $1 \leq i \leq r - 2$. Note that in the quotient graph $K_{1,2^{r-i-1}}$, $1 \leq i \leq r - 2$, the 2^{r-i-1} vertices of weight $2^{i+1} - 2$ form an R -equivalence class. The last quotient graph $HT(r)/E_{r-1}$ is isomorphic to K_3 , where the two vertices of weight $2^{r-1} - 1$ form an S -equivalence class. Then

$$\begin{aligned} W(HT(r)/E_i, w) &= W(P_2, w') + (2^{i+1} - 2) (2^{i+1} - 2) 2^{r-i-1} (2^{r-i-1} - 1) \\ &= (2^{r-i} - 1)(2^r - 2^{r-i}) + (2^{i+1} - 2) (2^{i+1} - 2) 2^{r-i-1} (2^{r-i-1} - 1) \\ &= 2^r \left(1 - \frac{1}{2^i} \right) (2^r - 2^{i+1} + 1), \end{aligned}$$

where w' assigns $2^r - 2^{r-i}$ and $2^{r-i} - 1$ to the vertices of P_2 , $1 \leq i \leq r - 2$. Clearly, $W(K_3, w) = 2^{2r-2} - 1$, and hence by Theorem 1.1,

$$\begin{aligned} W(HT(r)) &= \sum_{i=1}^{r-2} (W(HT(r)/E_i, w) + 2^{2r-2} - 1) \\ &= \sum_{i=1}^{r-2} \left(2^r \left(1 - \frac{1}{2^i} \right) (2^r - 2^{i+1} + 1) \right) + 2^{2r-2} - 1 \\ &= 2^r (3r - 8 \sinh(r \log(2)) + r 2^r + 1) + 2^{2r-2} - 1. \end{aligned}$$

5 Conclusion

In this paper, we have further refined the cut method. We have demonstrate the power of the approach by effortlessly computing the Wiener index of butterfly networks and hypertree networks. On the other hand, the method is applicable only to those families of graphs that admit non-trivial partitions into Θ^* -classes and whose corresponding quotient graphs have a structure that enables an efficient computation of their weighted Wiener index. It would be very desirable to find a unified approach that is applicable to all classes of graphs, for instance, to all families of fullerenes and nanotubes. Further, computing other topological indices using the newly introduced technique is widely open.

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