On the remoteness function in median graphs*

Kannan Balakrishnan† Boštjan Brešar‡ Manoj Changat§
Wilfried Imrich¶ Sandi Klavžar‖ Matjaž Kovše**
Ajitha R. Subhamathi††

Abstract

A profile on a graph G is any nonempty multiset whose elements are vertices from G. The corresponding remoteness function associates to each vertex $x \in V(G)$ the sum of distances from x to the vertices in the profile. Starting from some nice and useful properties of the remoteness function in hypercubes, the remoteness function is studied in arbitrary median graphs with respect to their isometric embeddings in hypercubes. In particular, a relation between the vertices in a median graph G whose remoteness function is maximum (antimedian set of G) with the antimedian set of the host hypercube is found. While for odd profiles the antimedian set is an independent set that lies in the strict boundary of a median graph, there exist median graphs in which special even profiles yield a constant remoteness function. We characterize such median graphs in two ways: as the graphs whose periphery transversal number is 2, and as the graphs with the geodetic number equal to 2. Finally, we present an algorithm that, given a graph G on n vertices and m edges, decides in $O(m \log n)$ time whether G is a median graph with geodetic number 2.

Keywords: hypercube; median graph; median set; remoteness function; geodetic number; periphery transversal

*Work supported by the Ministry of Science of Slovenia and by the Ministry of Science and Technology of India under the bilateral India-Slovenia grants BI-IN/06-07-002 and DST/INT/SLOV-P-03/05, respectively.

†Department of Computer Applications, Cochin University of Science and Technology, Cochin-22, India, bkannan@cusat.ac.in

‡Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia, bostjan.bresar@uni-mb.si

§Department of Futures Studies, University of Kerala, Trivandrum-695034, India, mchangat@gmail.com

¶Chair of Applied Mathematics, Montanuniversität Leoben, Austria, imrich@unileoben.ac.at

‖Faculty of Mathematics and Physics, University of Ljubljana, Slovenia and Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia, sandi.klavzar@fmf.uni-lj.si

**Faculty of Natural Sciences and Mathematics, University of Maribor, matjaz.kovse@gmail.com

††Department of Futures Studies, University of Kerala, Trivandrum-695034, India, ajithars@gmail.com

AMS Subj. Class. (2000): Primary: 05C85; Secondary: 05C12, 90B80;

1 Introduction

A *profile* $\pi = (x_1, \ldots, x_k)$ on a graph G is a finite sequence of vertices of G, and $k = |\pi|$ is called the *size* of the profile π. Note that in a profile a vertex may be repeated. Given a profile π on G and a vertex u of G, the *remoteness* $D(u, \pi)$ (see [16]) is

$$D(u, \pi) = \sum_{x \in \pi} d(u, x)\,,$$

where d stands for the usual (shortest paths) distance in G.

In the location theory one quests for the location of (un)desirable facilities, so the following definitions are significant. The vertex u is called a *median (antimedian) vertex for* π if $D(u, \pi)$ is minimum (maximum), and the *median (antimedian) set* $M(\pi, G)$ $(AM(\pi, G))$ of π in G is the set of all median (antimedian) vertices for π. The problem of locating median sets for profiles on graphs was considered by many authors; see, for example, [1, 3, 4, 17, 18]. On the other hand, not much work has been done so far on the antimedian problem for profiles on graphs, and though the two problems look similar, there are important differences. For instance, while it is clear that any vertex can be in the median set of a graph for some profile, this is not always true for the antimedian set.

In this paper we give a closer look at the remoteness function in median graphs with the aim to shed more light on the antimedian problem in this class. Median graphs form a closely investigated and well understood class of graphs, and are probably the most important class of graphs in metric graph theory (we refer to a comprehensive survey on median graphs [15]). Hence it is not surprising that they were investigated also in location theory [3, 17, 22]. For instance, it is known that in median graphs median sets are always intervals between two vertices [3], and in particular, for odd profiles they consist of exactly one vertex [17]. On the other hand, there are no results on the antimedian set in median graphs, and only a few observations about their remoteness function as such. In this paper we are trying to clear up this grey area in the location theory, and also believe that the problem is of similar applicability as its median set counterpart. First let us introduce some important notions used throughout this paper.

We consider only finite, undirected, simple and connected graphs. A shortest path between vertices u and v in a graph G will be called a u, v-*geodesic*, and the number of edges on such a path is the *distance* $d(u, v)$ between u and v (denoted also $d_G(u, v)$ if the graph G is not understood from the context). The set of vertices on all u, v-geodesics is called the *interval* between u and v, denoted $I(u, v)$. For a graph G and subsets of vertices $X, Y \subseteq V(G)$ we will write $d(X, Y) = \min\{d(x, y) \mid x \in X, y \in Y\}$. In particular, for a vertex u of G and a set of vertices X we have

$d(u, X) = \min\{d(u, x) \mid x \in X\}$. A set S of vertices in a graph G is called the *geodetic set* of G if for every vertex $x \in V(G)$ there exist $u, v \in S$ such that $x \in I(u, v)$. The *geodetic number* $g(G)$ of a graph G is the least size of a set of vertices S such that any vertex from G lies on a u, v-geodesic, where $u, v \in S$. We refer to [5, 8] for surveys on geodetic sets in graphs.

A (connected) graph G is a *median graph* if for any three vertices x, y, z there exists a unique vertex that lies in $I(x, y) \cap I(x, z) \cap I(y, z)$. Two of the most important classes of median graphs are trees and hypercubes. For a graph G and an edge xy of G we denote $W_{xy} = \{w \in V(G) \mid d(x, w) < d(y, w)\}$. Note that if G is a bipartite graph then $V(G) = W_{ab} \cup W_{ba}$ holds for any edge ab. Next, for an edge xy of G let U_{xy} denote the set of vertices u that are in W_{xy} and have a neighbor in W_{yx}. Sets in a graph that are U_{xy} for some edge xy will be called *U-sets*. Similarly we define *W-sets*. If for some edge xy, $W_{xy} = U_{xy}$, we call the set U_{xy} *peripheral set* or *periphery*. A subset S of vertices in a graph G is *convex* in G if $I(u, v) \subseteq S$ for any $u, v \in S$. It is clear that peripheries in median graphs are convex.

In Section 2 we start with the core example of median graphs—hypercubes. We show that the antimedian set of profiles on a hypercube is precisely the set of antipodal vertices of the median set. In addition, we extend this result to Hamming graphs. In Section 3 we deal with the remoteness function in arbitrary median graphs. A connection between antimedian sets on a median graph G and antimedian sets on the hypercube, into which G is embedded isometrically, is established. In Section 4, we obtain some additional properties of antimedian sets in median graphs, in particular for the case of odd profiles. It turns out that only in the case when the profile is even, it is possible that the (anti)median set is the whole vertex set of a median graph. Graph in which this can happen (for some very special even profiles) are precisely the median graphs with geodetic number 2. They were studied previously in [6], where several characterizations of these graphs were obtained. In Section 5 we prove two more characterizations, one of which is used in the algorithm for the recognition of median graphs with geodetic number 2.

Section 6 is concerned with the algorithm. Median graphs are a subclass of the class of isometric subgraphs of hypercubes. The complexity of recognizing whether a given graph G with n vertices and m edges is such a graph is $O(mn)$ in general. For median graphs this essentially reduces to $O(m\sqrt{n})$; see [11]. There is little hope to reduce it further in general, since it is closely related to that of recognizing triangle-free graphs (see [14]). However, in special cases the complexity is much lower. For example, it is $O(m)$ for planar median graphs. Here we show that median graphs with geodetic number 2 can be recognized in $O(m \log n)$ time.

2 Remoteness in hypercubes

In this section we study the remoteness function in hypercubes which form the fundamental example of median graphs. The *hypercube* or *n-cube* Q_n, $n \geq 1$, is the graph with vertex set $\{0,1\}^n$, two vertices being adjacent if the corresponding tuples differ in precisely one position. A vertex u of Q_n will be written in its coordinate's form as $u = u^{(1)} \dots u^{(n)}$. A natural generalization of hypercubes are *Hamming graphs*, whose vertices are m-tuples $u = u^{(1)} \dots u^{(m)}$, such that $0 \leq u^{(i)} \leq m_i - 1$, where $m_i \geq 2$ for each i, and adjacency is defined in the same way (that is, two vertices are adjacent precisely when they differ in exactly one coordinate). Note that the distance between vertices in Hamming graphs coincides with the Hamming distance (that is, the number of coordinates in which the m-tuples differ).

For a vertex x of Q_n let $\overline{x}$ be its antipodal vertex, that is, the vertex that is obtained from x by reversing the roles of zeros and ones. Let $X \subseteq V(Q_n)$. Then

$$\overline{X} = \{\overline{x} \mid x \in X\}$$

is called the *antipodal set of* X. Since $\overline{x} \neq \overline{y}$ for $x \neq y$ it follows that $\overline{\overline{X}} = X$.

Let $\pi = (x_1, \dots, x_k)$ be a profile on Q_d. For $i = 1, \dots, k$ let $n_0^{(i)}$ and $n_1^{(i)}$ be the number of vertices from π with the ith coordinate equal 0 and 1, respectively. More formally,

$$n_0^{(i)}(\pi) = |\{x \in \pi \mid x^{(i)} = 0\}|$$

and

$$n_1^{(i)}(\pi) = |\{x \in \pi \mid x^{(i)} = 1\}|\,.$$

Define Majority(π) as the set of vertices $u = u^{(1)} \dots u^{(d)}$ of Q_d, where

$$u^{(i)} \begin{cases} = 0; & n_0^{(i)}(\pi) > n_1^{(i)}(\pi), \\ = 1; & n_0^{(i)}(\pi) < n_1^{(i)}(\pi), \\ \in \{0,1\}; & n_0^{(i)}(\pi) = n_1^{(i)}(\pi)\,. \end{cases}$$

We say that vertices $u \in$ Majority(π) are obtained by the *majority rule.* Minority(π) and the *minority rule* are defined analogously. It is easy to verify (using that the distance between vertices in hypercubes coincides with their Hamming distance) that $M(\pi, Q_n) =$ Majority(π), and similarly $AM(\pi, Q_n) =$ Minority(π). We now infer:

Lemma 2.1 *Let π be a profile on Q_n. Then $M(\pi, Q_n)$ induces a subcube of Q_n. Moreover, $AM(\pi, Q_n) = \overline{M(\pi, Q_n)}$.*

Let Q and Q' be two subcubes of Q_n. Then we say that Q and Q' are *parallel* if they are of the same dimension, say r, and if vertices v_i of Q and v_i' of Q' can be ordered such that $d(v_i, v_i') = s$ for some integer s and for any $i = 1, 2, 3 \dots, 2^r$, where the mapping $v_i \mapsto v_i'$ is an isomorphism $Q \to Q'$.

Proposition 2.2 *Let π be a profile on Q_n and let Q be a subcube parallel to the subcube induced by $M(\pi, Q_n)$. Then the function $D(\cdot, \pi)$ is constant on Q.*

Proof. If $|M(\pi, Q_n)| = 1$ there is nothing to be proved. Assume in the rest that $|M(\pi, Q_n)| > 1$, hence $|\pi|$ must be even. By Lemma 2.1, $M(\pi, Q_n)$ induces a subcube Q' and let $x'y'$ be an edge of Q'. Partition the profile π into subprofiles π_1 and π_2, where vertices of π_1 lie in $W_{x'y'}$ and vertices of π_2 in $W_{y'x'}$. Since $x', y' \in M(\pi, Q_n)$, we have $D(x', \pi) = D(y', \pi)$. Therefore, the following reasoning

$$\begin{aligned} D(x', \pi) &= D(x', \pi_1) + D(x', \pi_2) \\ &= D(y', \pi_1) - |\pi_1| + D(y', \pi_2) + |\pi_2| \\ &= D(y', \pi) - |\pi_1| + |\pi_2| \end{aligned}$$

implies that $|\pi_1| = |\pi_2|$.

Let $d(Q, Q') = s$ and let xy be the edge of Q with $d(x, x') = d(y, y') = s$. Then, it can be easily verified that $d(x, y') = d(x', y) = s + 1$, and consequently $W_{xy} = W_{x'y'}$ and $W_{yx} = W_{y'x'}$. From the definition of W_{xy} and because $|\pi_1| = |\pi_2|$ it follows that $D(x, \pi) = D(y, \pi)$. By the connectivity of Q we conclude that D must be a constant function on Q. □

We can generalize the concept of antipodes from hypercubes to Hamming graphs, noting that an antipode of a vertex x is any vertex that is farthest from x. In the case of hypercubes this vertex is unique, but not in general Hamming graphs. Hence for a vertex x of a Hamming graph H its *antipodal vertex* is any vertex y such that $y^{(i)} \neq x^{(i)}$ for all $i = 1, \ldots, m$. For $X \subseteq V(H)$, let the *antipodal set* $\overline{X}$ *of* X be the set of all antipodal vertices over all vertices of X.

Theorem 2.3 *A Hamming graph H is a hypercube if and only if for any profile π*

$$AM(\pi, H) = \overline{M(\pi, H)}.$$

Proof. Suppose H is a hypercube. Then $AM(\pi, H) = \overline{(M(\pi, H))}$ for any profile π by Lemma 2.1.

For the converse suppose that a Hamming graph H is not a hypercube and let j be the index (coordinate) with $m_j \geq 3$. Consider the following profile $\pi = (x, y)$ of size 2 such that $x^{(i)} = y^{(i)} = 0$ for all $i \neq j$ and let $x^{(j)} = 0$, $y^{(j)} = 1$. Then $M(\pi, H) = \{x, y\}$, and $\overline{M(\pi, H)}$ consists of vertices z with $z^{(i)} > 0$ for $i \neq j$. On the other hand $AM(\pi, H)$ consists of vertices z with $z^{(i)} > 0$ for $i \neq j$ and $z^{(j)} > 1$. Hence $AM(\pi, H) \subset \overline{M(\pi, H)}$ and the inclusion is strict, by which the theorem is proved. □

3 Remoteness in median graphs embedded into hypercubes

In this section we obtain some properties of the remoteness function in arbitrary median graphs, by using their isometric embedding into hypercubes. Since the properties of median sets have already been studied in several papers, we restrict mainly to the properties of antimedian sets in median graphs.

A subgraph H of a (connected) graph G is an *isometric subgraph* if $d_H(u,v) = d_G(u,v)$ holds for any vertices $u, v \in H$. Let G be an isometric subgraph of some hypercube. An important structural result due to Mulder [19] asserts that every median graph G can be isometrically embedded in a hypercube such that the median of every profile π of cardinality three in G on the hypercube coincides with the median of π in G.

A vertex v of G is called a *local minimum* of a function $D(x,\pi)$ if $D(v,\pi) \leq D(u,\pi)$ for any neighbor u of v. It was proved by Bandelt and Chepoi [4] that in a graph G the set $M(\pi, G)$ is connected for any profile π on G if and only if for any π the function $D(x,\pi)$ has the property that every local minimum is a global minimum. Since median graphs have the property that $M(\pi, G)$ is connected for every π, we derive that in median graphs every local minimum is a global minimum.

For antimedian vertices, that is, vertices achieving global maximum of $D(x,\pi)$, the analogous result is not true in median graphs. Consider for example the 3×4 grid, and one of the two vertices of degree 4 as the only vertex of the profile π (all four vertices of degree 2 achieve a local maximum, but only two of them are also global). Thus there are local maxima which are not global maxima and, moreover, antimedians need not be connected.

Restricting to hypercubes the fact that local minima are global minima can be strengthened as follows. First recall that by Lemma 2.1, the median of π is a subcube in Q_n, and the antimedian is its antipodal (hence parallel) subcube. By Proposition 2.2, $D(x,\pi)$ is constant on every subcube parallel to them. Hence on any two shortest paths from $M(\pi, Q_n)$ to $AM(\pi, Q_n)$, the two corresponding sequences of values of the remoteness function are the same. (Note also that any two distinct intervals from vertices in $M(\pi, Q_n)$ to their (unique) closest vertices in $AM(\pi, Q_n)$ are disjoint, and every vertex of G lies on some shortest path from $M(\pi, Q_n)$ to $AM(\pi, Q_n)$.)

Lemma 3.1 *Let π be a profile on Q_n and let xx' be an edge of Q_n such that $d(x', AM(\pi, Q_n)) < d(x, AM(\pi, Q_n))$. Then $D(x,\pi) < D(x',\pi)$.*

Proof. Let $k = |\pi|$ and let $m_j = \min\{n_0^{(j)}(\pi), n_1^{(j)}(\pi)\}$ and $M_j = \max\{n_0^{(j)}(\pi), n_1^{(j)}(\pi)\}$. Since $AM(\pi, Q_n)$ can be obtained by the minority rule, for all $a \in AM(\pi, Q_n)$, we have

$$D(a,\pi) = \sum_{j=1}^{n} M_j.$$

Let $d(x', AM(\pi, Q_n)) = d(x', a_{x'}) = l$, where $a_{x'}$ is the unique closest vertex to x' from $AM(\pi, Q_n)$. Then

$$\begin{aligned} D(x',\pi) &= D(a_{x'},\pi) - \sum_{p=1}^{l} M_{i_p} + \sum_{p=1}^{l} m_{i_p}, \\ &= D(a_{x'},\pi) - \sum_{p=1}^{l} (M_{i_p} - m_{i_p}) \end{aligned}$$

where x' and $a_{x'}$ differ at coordinates i_p, $p = 1, \ldots, l$. Since x, x' are adjacent and $d(x, a_{x'}) = d(x', a_{x'}) + 1$ there exists a coordinate p_{l+1}, distinct from all coordinates i_p, $1 \leq p \leq l$, such that

$$D(x,\pi) = D(a_{x'},\pi) - \sum_{p=1}^{l+1} (M_{i_p} - m_{i_p})$$

and $D(x, \pi) < D(x', \pi)$. □

Theorem 3.2 *Let G be a median graph embedded isometrically into Q_n, and let π be a profile on G. Let $a \in AM(\pi, G)$ and let a' be the closest vertex to a in $AM(\pi, Q_n)$. Then*

$$I(a, a') \cap V(G) = \{a\}\,.$$

Proof. Let b be the closest vertex to a' in $M(\pi, Q_n)$. From Lemma 2.1 we find that b is unique (as subcubes of a cube are gated; see [15], if necessary). In addition, Lemma 3.1 implies that $D(x, \pi)$ is strictly increasing on any shortest path from b to a'. Since $I(a, a') \subseteq I(b, a')$, it follows that $D(x, \pi)$ is strictly increasing on any shortest path from a to a'. Thus $c \in I(a, a') \cap V(G)$, $c \neq a$, would imply that $D(c, \pi) > D(a, \pi)$, a contradiction with $a \in AM(\pi, G)$. Hence $I(a, a') \cap V(G) = \{a\}$. □

In Fig. 1 we give an illustration of the above theorem. Vertices of a median graph G are darkened, and G is isometrically embedded into the 3-cube. Let the profile π consist of all five vertices of G. Then $AM(\pi, Q_3)$ consists of the vertex w, where $D(w, \pi) = 10$. Vertices u and v are the only vertices from G that enjoy the condition from the theorem, that is $I(a, w) \cap V(G) = \{a\}$. Hence u and v are the only candidates to be antimedian vertices with respect to G, and both achieve the

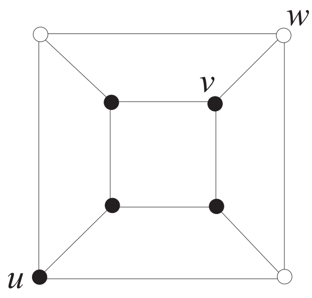

Figure 1: Example on antimedians

local maximum of $D(\cdot, \pi)$ with respect to G. Since $D(u, \pi) = 8$ and $D(v, \pi) = 7$, we infer that $AM(\pi, G) = \{u\}$. Note that even though v is closer to $AM(\pi, Q_n)$ (that is, to w) than u, it is not an antimedian vertex.

We proved in [2] that $M(\pi, Q_n) \cap V(G) \neq \emptyset$ holds for any profile π which is used in an efficient algorithm for computing median sets in median graphs. In the events when $AM(\pi, Q_n) \cap V(G) \neq \emptyset$ we have $AM(\pi, G) = AM(\pi, Q_n) \cap V(G)$, and then the antimedian set is also connected and it induces isometric subgraph of G. Unfortunately $AM(\pi, Q_n) \cap V(G) \neq \emptyset$ is not true in general, as can be seen in the example from Fig. 1. Nevertheless, Theorem 3.2 could occasionally be helpful in finding the antimedian set for profiles on median graphs, since it can considerably reduce the number of candidates for the antimedian set to the vertices that achieve the condition from the theorem.

4 Antimedian sets of some particular profiles in median graphs

In this section we study the remoteness function in median graphs for two types of profiles: profiles whose size is odd, and profiles that consist of all vertices of a graph with no repetitions. These cases indicate that the antimedian set is in many cases restricted to a rather small subset of the vertex set – the strict boundary of a graph. A vertex v of a graph G is a *strict boundary vertex (with respect to v')* of G if there exists a vertex v' such that for any neighbor u of v, $d(v', v) > d(v', u)$. (In other words, the neighborhood of v is contained in $I(v, v')$.) The *strict boundary* $\overline{\partial}G$ of a graph G is the set of strict boundary vertices in G.

Every vertex can clearly be in some median set of a graph (e.g., by taking this vertex as the unique vertex in the profile). The antimedian case is different, as one can readily verify on trees which are not paths (i.e. in any such tree, only leaves can be in the antimedian set for any profile). We will consider in Section 5 the case of median graphs with geodetic number 2 which are somewhat special, in the same sense as paths are special trees. Note that by taking as the profile both leaves of a path, the resulting remoteness function is constant, hence all vertices of the path are (anti)median.

We suspect that the following question has affirmative answer.

Question 4.1 *Let G be a median graph and $g(G) > 2$. Is it true that there exists a vertex in G that is not in $AM(\pi, G)$ for all profiles π on G?*

We present two partial results that confirm this.

For an edge uv in a median graph G and a profile π, we let $\pi_{uv} = W_{uv} \cap \pi$. As usually, $|\pi_{uv}|$ denotes the size of the profile π in W_{uv}. Note that $|\pi_{uv}| > |\pi_{vu}|$ implies that the median set of π on G lies in W_{uv} which in turn implies that if u and v are both in a median set then $|\pi_{uv}| = |\pi_{vu}|$. These observations are a basis for several strategies to find median sets in median-like graphs, see [1, 18].

Lemma 4.2 *Let π be an odd profile in a median graph G, then every vertex in $AM(\pi, G)$ is a strict boundary vertex.*

Proof. Let $v \in AM(\pi, G)$ and $|\pi|$ be odd. Since for every neighbor u_i of v, $|\pi_{u_i v}| > |\pi_{v u_i}|$ we infer that

$$|\pi_{u_i v}| > \frac{|\pi|}{2}.$$

Hence $\pi_{u_i v}$ and $\pi_{u_j v}$ intersect for any neighbors u_i,u_j of v, where $i \neq j$. Since $\pi_{u_i v} \subseteq W_{u_i v}$, the sets $W_{u_i v}$ also pairwise intersect for all neighbors u_i of v. Since W-sets are convex, by the *Helly property* for convex sets in median graphs (that is, any family of pairwise intersecting convex sets has a common intersection), there exists a vertex

$$v' \in \bigcap_{u_i \in N(v)} W_{u_i v}.$$

Hence u_i is strictly closer to v' than v for any i, and so v is a strict boundary vertex (with respect to v'). $\square$

From the proof of the lemma above we also see that no neighbor of $v \in AM(\pi, G)$ achieves $D(v, \pi)$, hence we derive the following result.

Proposition 4.3 *Let π be an odd profile in a median graph G. Then $AM(\pi, G)$ is an independent set in G and $AM(\pi, G) \subseteq \overline{\partial} G$.*

Note that in the case of even profiles the antimedian vertices need not be in a strict boundary, even if $g(G) > 2$. For instance, let G be obtained from the 3×3 grid (that is the Cartesian product $P_3 \Box P_3$) so that to the central vertex another vertex a is attached, and let the profile π consist of two vertices u, v of degree two such that

$d(u, v) = 2$. Then $AM(\pi, G) = \{x, y, z, a\}$, where x and y are another two vertices of degree two (different from u and v), and z is their common neighbor. Note that z is not a strict boundary vertex in G, even though all the antimedian vertices are peripheral. For an additional example consider the graph on Fig. 2. For the profile (a, b) the antimedian vertices are darkened. In particular, x is an antimedian vertex that is neither in the boundary nor in the periphery of the graph.

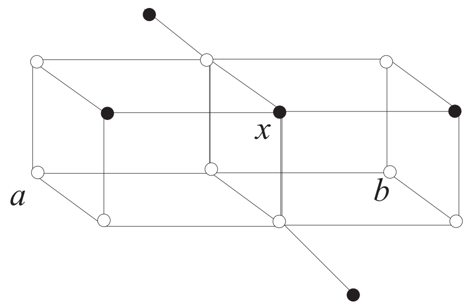

Figure 2: Vertex x is an antimedian vertex for the profile (a, b).

Now, we consider the remoteness function when the profile is the whole vertex set, each vertex appearing exactly once. This problem is known in the literature as the obnoxious center problem, and has been quite well studied, cf. [7, 21, 23, 25]. We prove a result similar to Proposition 4.3.

Proposition 4.4 *Let G be a median graph, and let π be the profile, consisting of vertices of $V(G)$ (with no repetitions). If $v \in AM(\pi, G)$ then v is a strict boundary vertex.*

Proof. Let $v \in AM(\pi, G)$. We infer that for every neighbor u_i of v, $|W_{u_i v}| \geq |W_{v u_i}|$, hence

$$|W_{u_i v}| \geq \frac{|V(G)|}{2} \; .$$

Let $u_1, \ldots, u_t$ be the neighbors of v. If $t = 1$, that is, v has only one neighbor, then v is clearly a strict boundary vertex with respect to any other vertex. Suppose that u_i, u_j are neighbors of v and $i \neq j$. Then by the above

$$|W_{u_i v}| + |W_{u_j v}| \geq |V(G)|.$$

Since $v \notin W_{u_i v}$, for any i, we find that $W_{u_i v}$ and $W_{u_j v}$ intersect. Since W-sets are convex, we infer by the Helly property for convex sets that there exists a vertex

$$v' \in \bigcap_{i=1}^{t} W_{u_i v} \; .$$

Hence v is a strict boundary vertex with respect to v' which completes the proof of the proposition. $\square$

5 Median graphs with geodetic number two

As mentioned in the previous section, median graphs with geodetic number two are somehow a special case which is excluded in Question 4.1. Before we present characterizations of these graphs, one of which also considers the remoteness function of some even profiles, we need to introduce a few more natural concepts on median graphs. The first one concerns peripheries in median graphs.

Let G be a median graph. We say that a set S is a *periphery transversal* if every peripheral subgraph of G contains a vertex of S. It was proved in [6] that every geodetic set is a periphery transversal. Let $\mathrm{pt}(G)$ denote the size of a minimum periphery transversal in a median graph G. Then, clearly, $\mathrm{pt}(G) \leq g(G)$ for any median graph G. On the other hand, it may happen that any minimum geodetic set of a median graph G must contain some vertices that are not in a peripheral subgraph. For instance, in the graph G obtained from the 3-cube by attaching a leaf to 3 independent vertices we have $\mathrm{pt}(G) = 3 < 4 = g(G)$.

The next concept is a generalization of the partition of the edge-set into parallel classes in hypercubes to more general graphs. Edges $e = xy$ and $f = uv$ of a graph G are in the Djoković-Winkler relation Θ [10, 24] if $d_G(x,u) + d_G(y,v) \neq d_G(x,v) + d_G(y,u)$. Relation Θ is reflexive and symmetric. If G is bipartite, then Θ can be defined as follows: $e = xy$ and $f = uv$ are in relation Θ if $d(x,u) = d(y,v)$ and $d(x,v) = d(y,u)$. It is well-known that the relation Θ is transitive in isometric subgraphs of hypercubes [24], and so it is an equivalence relation on the edge set of every median graph. Note that peripheral sets are precisely the U-sets that induce a connected component of $G - F$ for some Θ-class F.

The following result from [6] will be used in the main theorem of this section.

Theorem 5.1 *Let G be a median graph. Then $g(G) = 2$ if and only if there exist vertices $a, b \in V(G)$ and an a,b-geodesic that contains edges from all Θ-classes of G.*

We also need the following easy facts, see [13].

Lemma 5.2 *Let G be a median graph, C a cycle, P a geodesic, and F a Θ-class of G. Then*

(i) $F \cap C \neq \emptyset \Rightarrow |F \cap C| \geq 2$;

(ii) $F \cap P \neq \emptyset \Rightarrow |F \cap P| = 1$.

Combining Lemma 5.2 with Theorem 5.1 we infer that if a and b are as in the theorem, then on any geodesic from a to b all Θ-classes appear. Conversely, $g(G) > 2$ implies that for any two vertices a and b in G there exists a Θ-class whose edges are outside $I(a, b)$.

Theorem 5.3 *For a median graph G the following statements are equivalent.*

(i) $g(G) = 2$*,*

(ii) $\mathrm{pt}(G) = 2$*,*

(iii) $D(x, \pi)$ *is constant on G for some profile π.*

Proof. (i)$\Rightarrow$(ii): Let G be a median graph with $g(G) = 2$. As every W-set in a median graph contains a periphery, we infer that $\mathrm{pt}(G) \geq 2$. We have already observed that in general $\mathrm{pt}(G) \leq g(G)$, hence $\mathrm{pt}(G) = 2$.

(ii)$\Rightarrow$(i): Let G be a median graph with $\mathrm{pt}(G) = 2$, and assume to the contrary that $g(G) > 2$. Then for any two vertices $a, b \in V(G)$, $I(a, b) \neq V(G)$, and by Theorem 5.1 we infer that there exists a Θ-class F that lies outside $I(a, b)$. Then there also exists a W-set W_{xy} that has an empty intersection with $I(a, b)$. In addition, W_{xy} contains a periphery that does not contain a and b. Thus $\{a, b\}$ is not a periphery transversal, and since a and b were chosen arbitrarily we infer that $\mathrm{pt}(G) > 2$, a contradiction.

(i)$\Rightarrow$(iii): Let a and b be vertices in G such that $I(a, b) = V(G)$. Set $\pi = (a, b)$. Since for any $x \in V(G)$ we have $d(a, x) + d(x, b) = d(a, b) = \mathrm{diam}(G)$ we get $D(x, \pi) = \mathrm{diam}(G)$.

(iii)$\Rightarrow$(i): For this direction we recall a result by Bandelt and Barthélemy [3, Proposition 6] which says that for any profile π on a median graph G, the median set $M(\pi, G)$ coincides with the interval $I(\alpha(\pi), \beta(\pi))$ (where $\alpha(\pi)$ and $\beta(\pi)$ are two vertices in G obtained by a formula in the associated median semilattice). Hence, if $D(x, \pi)$ is constant on G for a profile π, then $V(G) = M(\pi, G) = I(\alpha(\pi), \beta(\pi))$, which in turn implies $g(G) = 2$. $\square$

6 Recognition of median graphs with geodetic number two

As already mentioned, median graphs are isometric subgraphs of hypercubes (*partial cubes* for short), and the recognition complexity for such graphs is $O(mn)$. In other words, there exists an algorithm that recognizes whether any given graph G with n vertices and m edges is a partial cube in $O(mn)$ time. The algorithm also provides an embedding of G. In the rest of this section n and m will denote the number of vertices and edges of a given graph.

However, if it is known that a graph G is a median graph, then G can be embedded isometrically into a hypercube in $O(m \log n)$ time. This discrepancy between the embedding complexity and the recognition complexity was a strong motivation to find better recognition algorithms for median graphs. The algorithm of Hagauer, Imrich and Klavžar [11] with complexity $O(m\sqrt{n})$ was the first of this kind. Later Imrich [13, Theorem 7.27] derived the asymptotically better result $O((m \log n)^{1.41})$. Here the exponent 1.41 actually is $2\omega/(\omega+1)$, where ω is the exponent of matrix multiplication with its current value 2.376. By a result of Imrich, Klavžar and Mulder [14] this recognition complexity is closely related with the recognition complexity of triangle-free graphs. Hence improvements of the recognition complexity of median graphs seem to be very difficult.

Nonetheless, some classes of median graphs can be recognized much faster. This includes planar median graphs [14], which can be recognized in linear time and acyclic cubical complexes [12], which can be recognized in $O(m \log n)$ time. Here we show that median graphs with geodetic number two can also be recognized in $O(m \log n)$ time. This is possible because of a bound on the maximum degree of a median graph with geodetic number two and the fact that every peripheral subgraph meets geodetic set, see Brešar and Tepeh Horvat [6].

We begin with the bound on the maximum degree $\Delta(G)$ of a median graph G with $g(G) = 2$.

Lemma 6.1 *Let G be a median graph with $g(G) = 2$. Then $\Delta(G) \leq 2\log_2 n$.*

Proof. Suppose $G = I_G(v, w)$ and let $L_0, L_1, \ldots, L_r$ be the levels of the BFS-ordering of the vertices of G with respect to a root v; see e.g. [13, p. 41]. Let $x \in L_i$ and $xy \in E(G)$. Since G is bipartite $y \notin L_i$. If $y \in L_{i-1}$ we call the edge xy a *down-edge* and otherwise an *up-edge*. Clearly y is closer to v than x if xy is a down-edge, and closer to w if xy is an up-edge. In other words, the up-edges with respect to v are the down-edges with respect to w. By [13, Lemma 3.35] the number of down-edges of every vertex x in a median graph is bounded by $\log_2 n$. Clearly the number of up-edges satisfies the same bound, hence $d(v) \leq 2\log_2 n$ for all $v \in V(G)$. $\square$

Next we show how to check efficiently whether a given induced subgraph of a graph G is also a convex subgraph. For a subgraph H of a graph G let ∂H be the set of edges with one endvertex in H and the other in $G \setminus H$.

Lemma 6.2 *Let H be an induced connected subgraph of a partial cube for which the Θ-classes are already known. Then the complexity of recognizing whether H is a convex subgraph of G is $O(|E(H)| + |\partial H|)$.*

Proof. By the convexity lemma [13, Lemma 2.7] it suffices to show that no edge of ∂H is in the relation Θ with an edge of H. In other words, we have to show that the list of Θ-classes that meet $E(H)$ is disjoint from the list of Θ-classes that meet ∂H.

Let $E_1, \ldots, E_k$, where $k < n$, be the Θ-classes of G and $\mathbf{v}_H$ the 0,1-vector of length k with $\mathbf{v}_H(i) = 0$ if $E_i \cap E(H) = \emptyset$ and $\mathbf{v}_H(i) = 1$ otherwise. Since the Θ-classes are known, we can assume that there exists a function $c : E(G) \to \{1, \ldots, k\}$ that computes the index i for which $e \in E_i$ in constant time. With a well known trick, see [9, Exercise 12.1-4], the vector $\mathbf{v}_H$ can be determined in $O(|E(H)|)$ time, even if $|E(H)|$ is much less than k, by scanning all edges of H (with the trick we avoid the initialization and the scan of the entire vector $\mathbf{v}_H$ which could be more costly). Moreover we scan all edges of ∂H. If $e \in E_i$ and $\mathbf{v}_H(i) = 1$, then H is not convex. We thus have to check whether $\mathbf{v}_H(c(e)) = 0$ for all $e \in \partial H$. Clearly this can be done in $O(|\partial H|)$ time. □

Next we show how to efficiently check the convexity of U-sets.

Corollary 6.3 *Let H be a partial cube for which the Θ-classes are already known, and Δ the maximum degree of vertices in G. Then one can check in $O(m\Delta + m \log n)$ time whether all U-sets are convex.*

Proof. First note that the total size of U-sets (i.e. the sum of the orders of all U-sets) in G is $2m$. Indeed, every vertex from a U-set corresponds uniquely to an edge, and each such edge appears exactly twice when checking vertices of all U-sets. Furthermore $|E(U_{ab})| < |U_{ab}| \log_2 |U_{ab}|$ by Graham's density lemma [13, Proposition 1.24]. Hence, for the total number of edges in the U-sets we have the following inequality

$$(\sum |U_{ab}|) \max(\log_2 |U_{ab}|) \leq 2m \log_2 n\,.$$

Let $\mathbf{v}_{U_{ab}}$ be defined as in Lemma 6.2. Then it is clear that the set of vectors $\mathbf{v}_{U_{ab}}$ can be determined in $O(m \log n)$ time. Since the total size of the sets ∂U over all U-sets is bounded by $m\Delta$ the corollary follows. □

Proposition 6.4 *Let G be a graph with $\Delta(G) \leq 2 \log_2 n$. Then one can check in $O(m \log n)$ time whether G is a median graph, determine all Θ-classes and all U-sets.*

Proof. By [13, Lemma 7.15] one can check in $O(m \log n)$ time whether G is a partial cube, determine all Θ-classes and all U-sets. By [13, Corollary 2.27] a partial cube is a median graph if and only if all U-sets are convex. Now the proof is completed by the observation that the convexity of the U-sets of a given partial cube can be checked in $O(m \log n)$ by Corollary 6.3. □

Next we describe a procedure which can be used to construct all median graphs. For a connected graph H and its convex subgraph P *the peripheral expansion of H*

along P is the graph G obtained as follows. Let P' be an isomorphic copy of P and α a corresponding isomorphism. Take the disjoint union $H+P'$ and join each vertex $v \in P$ by an edge with $\alpha(v) \in P'$. We call the new graph a *peripheral expansion* of H along P and denote it by $G = pe(H;P)$. Mulder [20] proved that a graph is a median graph if and only if it can be obtained from K_1 by a sequence of peripheral expansions.

We still have to find a geodetic set consisting of two elements. In order to accomplish this, we will use this sequence of peripheral expansions to determine all geodetic sets. We begin with a relationship between the geodetic sets of a median graph H and the graph $G = pe(H,P)$.

Lemma 6.5 *Let $G = pe(H;P)$ be a median graph and $\{x,y\}$ a geodetic set of H, where $y \in P$. Then the set $\{x,z\}$, where z is the neighbor of y in $G \setminus H$ is a geodetic set in G. Moreover, all minimum geodetic sets of G are of this form.*

Proof. We have to show that every vertex w of G is on a shortest xz-path. Suppose first $w \in H$. Then, clearly w is on a xy-geodesic, since $\{x,y\}$ is a geodetic set in H. Thus w is also on xz-geodesic going through y. Suppose next $w \in G \setminus H$ and let w' be a neighbor of w, where $w' \in H$. Then w' lies on xy-geodesic. Let L_1 denote the yw'-geodesic and let L_2 denote the $w'x$-geodesic. Since P is a convex subgraph of H (and therefore also of G) L_1 is completely contained in P. Recall that in median graph for any edge ab we have $U_{ab} \cong U_{ba}$ and that the isomorphism is induced by the edges between U_{ab} and U_{ba}. Let L_1' be the projection of L_1 into P' by this isomorphism. Then $L_1' \cup ww' \cup L_2$ is a zx-geodesic in G containing w. Conversely if $\{x,z\}$ is a geodetic set in $G = pe(H;P)$ then by [6, Lemma 2] x or z must be in P'. Suppose z is in P'. Then we can use the same arguments as above to see that $\{x,y\}$ is a geodetic set in H, where y is a neighbor of z in H. $\square$

If $\{x,y\}$ is a geodetic set in G then this is the only minimum geodetic set containing x, since by Lemma 6.5 x is uniquely determined by y and vice versa.

Corollary 6.6 *Let $G = pe(H;P)$ be a median graph with $g(G) = 2$. Then all minimum geodetic sets of G can be obtained from the minimum geodetic sets of H in $O(|P|)$ time.*

Proof. Let $P' = U_{ab}$, where $a \in G \setminus H$. To find the geodetic sets of G we scan all vertices z of U_{ab}. If the neighbor y of z in U_{ba} is in the geodetic set $\{y,x\}$ of H, then by Lemma 6.5 $\{z,x\}$ is a geodetic set of G. Clearly the complexity of this task is $O(|U_{ab}|)$. $\square$

Corollary 6.7 *Let G be a median graph with $g(G) = 2$. If the representation of G as a series of peripheral expansions, starting from K_1, is known, then all minimum geodetic sets of G can be obtained in $O(n)$ time.*

Proof. At every expansion step $|U_{ab}|$ vertices are added at a total cost of $O(|U_{ab}|)$. The observation that $n-1$ vertices are added altogether completes the proof. $\square$

We are thus left with the task of representing G by a series of peripheral expansions.

Theorem 6.8 *Let G be a median graph with $\Delta(G) \leq 2\log_2 n$. Then a representation of G by a series of peripheral expansions can be found in $O(m\log n)$ time.*

Proof. By [13, Lemma 7.15] and Proposition 6.4 we know that one can recognize G as a median graph, partition its edge set into Θ-classes, and determine all U-sets in $O(m\log n)$ time. We show now that we can determine all peripheral U-sets within the same time complexity. We first observe that the peripheral U-sets are characterized by the fact that ∂U consist of $|U|$ independent edges that meet every vertex of a U-set. In other words U_{ab} is peripheral if

$$\deg_G(v) = \deg_{U_{ab}}(v) + 1\,,$$

for every $v \in U_{ab}$. Clearly $\deg_{U_{ab}}(v) + 1 \leq \deg_G(v)$ for $v \in G$. Thus, setting

$$ex_{U_{ab}}(v) = \deg_G(v) - \deg_{U_{ab}}(v) - 1$$

it is clear that U_{ab} is peripheral if and only if

$$ex(U_{ab}) = \sum_{v\in U_{ab}} ex_{U_{ab}}(v) = 0\,.$$

Intuitively, $ex(v)$ is the *excess* of the degree of v above its minimum.

We thus need the degrees of every vertex in its U-sets and in G. The degrees of all vertices from a given U-set U_{xy} can be determined in $|E(U_{xy})|$ time and the degrees of all vertices in G in $O(m)$ time. Since the total number of edges in the U-sets is $O(m\log n)$ (see the proof of Corollary 6.3) we can thus determine all degrees in $O(m\log n)$ time.

In a second run, scanning all vertices in the U-sets, we determine excesses of all vertices of G and calculate the sum of all corresponding excesses of vertices from some U-set. Since the total number of vertices in the U-sets is $O(m)$, this can be done in the required time too.

In this process we keep a record of all these numbers and consider the first peripheral set we find, say U_{ab}.

We now show that we can remove U_{ab} from G and determine for $H = G\setminus U_{ab}$ the same data structure we had for G. In other words, we can determine the adjacency list of all new U-sets in the graph H, all degrees and the new values of the excess numbers for all vertices in H and all the new U-sets in $O(|U_{ab}|\log n)$ time.

We first find the new adjacency list of the new U-sets of H. We first recall that the removal of a vertex v and all incident edges from a graph is of complexity $O(\deg(v))$ if the graph is represented by an extended adjacency list or the adjacency matrix; see pp. 37 in [13]. In G every vertex v is also a vertex of every U_{vw}, where w is a neighbor of v in G. Thus every $v \in U_{ab}$ is in at most $O(\log n)$ sets U_{vw}. The degree of the vertex v in such a U_{vw} is $\deg_G(v) - 1 = \deg_{U_{ab}}(v)$. The cost of removing v from all U_{vw} is thus $O(\deg_{U_{ab}}(v) \log n)$. For all $v \in U_{ab}$ this amounts to a total of $O(|E(U_{ab})| \log n)$.

We also have to determine all new degrees and the new excess numbers. This concerns all vertices of U_{ab}. Every such vertex is contained in at most $2 \log n$ graphs U^H_{xy}. Hence all these numbers can be computed in $O(\log n |U_{ab}|)$ time if all vertices of U_{ab} are removed. In other words, the data structure of $H = G \backslash U_{ab}$ can be determined from that of G in $O(\log n |U_{ab}|)$ time, including all degrees, excess numbers etc. (In the course of the action we take note of the first peripheral U-sets we encounter.)

We now repeat this process by removing peripheral U-sets until we reach K_1. The total complexity is then $O(\log n \sum |U_{ab}|) = O(m \log n)$. □

Now all prerequisites are ready for the following algorithm that recognizes whether a given graph G is a median graph with $g(G) = 2$. (Note that the *isometric dimension* $\mathrm{idim}(G)$ of a partial cube G coincides with the number of its Θ-classes).

Algorithm 1

Input: *The adjacency list of a graph G.*

Output: *YES and a list of all geodetic pairs if G is a median graph with $g(G) = 2$. NO otherwise.*

Step 0: *If $\Delta(G) > 2 \log_2 n$, reject.*
If G is not a median graph, reject.
Otherwise determine all Θ-classes and the adjacency lists of all U-sets.
Set $i = k$, where $k = \mathrm{idim}(G)$, and $G_k = G$.

Step 1: *Compute the excess for all vertices in the U-sets and of all U-sets.*

Step 2: *Find a peripheral U_{ab} as in Theorem 6.8.*

Step 3: *Remove U_{ab} to obtain G_{i-1}.*

Step 4: *Repeat Step 2 and 3 (sequence of contractions) until $G_0 = K_1$.*

Step 5: *For $i = 0$ to $k - 1$ do:*
Find all geodetic pairs of G_i and determine those of G_{i+1} with the aid of Corollary 6.7.

Step 6: *If there are no such sets, return NO.*
Otherwise return YES and the list of all geodetic pairs.

Theorem 6.9 *Let G be a graph G. Then Algorithm 1 correctly recognizes whether G is a median graph with $g(G) = 2$. It can be implemented to run in $O(m \log n)$ time.*

Proof. Combining Lemma 6.1 and Proposition 6.4 we infer that Step 0 can be implemented in $O(m \log n)$ time. Steps 1–4 are an algorithmic interpretation of the proof of Theorem 6.8. As stated in Theorem 6.8, one can perfom these steps in $O(m \log n)$ time. From Corollary 6.7 we find that Step 5 can also be performed in the desired time. $\square$

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