On semicube graphs

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Abstract

Eppstein [6] introduced semicube graphs as the key tool for efficient computation of the lattice dimension of a graph. In this paper it is shown that, roughly speaking, every graph can be realized as the semicube graph of some partial cube. Semicube graphs of trees are studied in detail. In particular the chromatic number, the independence number and the domination number of semicube graphs of trees are determined in terms of related invariants of trees.

1 Introduction

The lattice dimension of a graph G is the smallest d such that G embeds isometrically into the d-dimensional integer lattice $\mathbb{Z}^d$. To determine the lattice dimension of a graph G, Eppstein [6] introduced the semicube graph $\mathrm{Sc}(G)$ and proved that the lattice dimension of G is equal to $k-|M|$, where k is the isometric dimension of G and M a maximum matching of $\mathrm{Sc}(G)$. He further suggested that it would be of interest to investigate more carefully the combinatorial properties of the semicube graph. Motivated by this suggestion we proceed as follows.

In the rest of this section we give concepts and definitions needed in the paper, introduce the semicube graphs, and observe some preliminary facts about them. In Section 2 we prove that every graph can be realized as the semicube graph of some partial cube, in fact, of a median graph. More precisely, if G is a graph on n vertices, then the semicube graph of the simplex graph of the complement of G is the disjoint union of G and n isolated vertices. In the final section we have a closer look at the semicube graphs of trees and determine their chromatic, independence,

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and domination number. For instance, the domination number of the semicube graph of a tree T is the number of vertices in the union of closed neighborhoods of the leaves of T.

Let u and v be vertices of a connected graph G. Then $d_G(u,v)$, or $d(u,v)$ for short, denotes the length of a shortest u,v-path in G. A subgraph H of G is *isometric* if $d_H(u,v) = d_G(u,v)$ for all $u,v \in V(H)$. The *interval* $I(u,v)$ between u and v is the set of all vertices on shortest u,v-paths. A connected graph is a *median graph* if for every triple u,v,w of its vertices $|I(u,v) \cap I(u,w) \cap I(v,w)| = 1$.

A graph G is a *partial cube* if G is an isometric subgraph of some hypercube. Recall that median graphs are partial cubes [12] as well as are trees and many other important classes of graphs, see [4, 7, 8]. Edges $e = xy$ and $f = uv$ of G are in relation Θ if $d(x,u) + d(y,v) \neq d(x,v) + d(y,u)$, see [5, 14]. Θ is reflexive and symmetric. On partial cubes it is also transitive [14] and hence an equivalence relation. Let $[G]_\Theta$ denote the set of the Θ-classes of a partial cube G. The number of these classes, $|[G]_\Theta|$, is known as the *isometric dimension* of G. It is equal to the smallest k such that G embeds isometrically into the k-cube Q_k.

Let G be a graph with finite lattice dimension. This means that G is isometrically embeddable into the Cartesian product of paths. Since Cartesian products of partial cubes are partial cubes, we infer that G is a partial cube. Conversely, given a partial cube G isometrically embedded into Q_k, it is clear that G has finite dimension because Q_k is the Cartesian product of k paths K_2. Therefore, graphs with finite lattice dimension are partial cubes. (See [6].) We add that the lattice dimension of combinohedrons has been determined in [13] while in [9] the lattice dimension of benzenoid systems has been studied. We also refer to the book [4] for the lattice dimension of several infinite partial cubes and to the recent book [7].

For a connected graph $G = (V,E)$ and an edge ab of G let

$$W_{ab} = \{w \in V \mid d(a,w) < d(b,w)\}\,.$$

Following [6] we will call the sets W_{ab} *semicubes* of G. The *semicube graph* $\mathrm{Sc}(G)$ of a partial cube G is the graph whose vertices are the semicubes of G, semicubes W_{ab} and W_{cd} being adjacent if

$$W_{ab} \cup W_{cd} = V \quad \text{and} \quad W_{ab} \cap W_{cd} \neq \emptyset\,.$$

Note that these conditions are equivalent to

$$W_{dc} \subsetneq W_{ab} \quad \text{and} \quad W_{ba} \subsetneq W_{cd}\,.$$

For instance, $\mathrm{Sc}(Q_n)$ consists of $2n$ isolated vertices, and $\mathrm{Sc}(P_3 \,\square\, P_3)$ (the Cartesian product of the path on 3 vertices with itself) is the disjoint union of four copies of K_1 and two copies of K_2. More generally, it is not difficult to see that for any G and any H,

$$\mathrm{Sc}(G \,\square\, H) = \mathrm{Sc}(G) + \mathrm{Sc}(H)\,,$$

where $\mathrm{Sc}(G) + \mathrm{Sc}(H)$ denotes the disjoint union of $\mathrm{Sc}(G)$ and $\mathrm{Sc}(H)$.

Let ab be an edge of a connected graph G. In addition to the previously introduced set of vertices W_{ab} we also set

$$F_{ab} = \{e \in E(G) | \ e \text{ is an edge between } W_{ab} \text{ and } W_{ba}\}\,.$$

In a partial cube G the Θ-class containing ab coincides with the set F_{ab}, see [8]. Hence we may write $F_{ab} \in [G]_\Theta$.

The following remarks will be (implicitly) used throughout the paper. Let ab and cd be edges of a partial cube G such that $F_{ab} \neq F_{cd}$. Clearly, W_{ab} is in $\mathrm{Sc}(G)$ not adjacent to W_{ba}. Furthermore, W_{ab} is adjacent to W_{cd} if and only if W_{dc} is properly contained in W_{ab}. It follows that the subgraph of $\mathrm{Sc}(G)$ induced by W_{ab}, W_{ba}, W_{cd}, and W_{dc} contains at most one edge.

By a *terminal semicube* we mean a semicube that is minimal with respect to inclusion. More precisely, W_{ab} is a terminal semicube if $W_{cd} \subseteq W_{ab}$ implies $W_{cd} = W_{ab}$. We state the following fact for further use. Its proof follows immediately from definitions.

Lemma 1.1 *Let ab be an edge of a partial cube G. Then W_{ab} is an isolated vertex of $Sc(G)$ if and only if W_{ab} is a terminal semicube.*

2 Semicube graphs are universal

In this section we show that every graph can be found in the semicube graph of some median graph. More precisely, for any graph G there exists a median graph H such that $\mathrm{Sc}(H)$ is the disjoint union of G and some isolated vertices. This implies that Eppstein's algorithm needs to use (the best) algorithm for finding a maximum matching in a general graph.

For our purposes crossing graphs and simplex graphs are of utmost help. We first introduce the crossing graphs.

Let $F_{ab}, F_{cd} \in [G]_\Theta$. Then F_{ab} and F_{cd} *cross*, $F_{ab} \#_G F_{cd}$, if

$$W_{ab} \cap W_{cd} \neq \emptyset, W_{ab} \cap W_{dc} \neq \emptyset, W_{ba} \cap W_{cd} \neq \emptyset, \text{ and } W_{ba} \cap W_{dc} \neq \emptyset\,.$$

The crossing graph $G^\#$ of a partial cube G has elements of $[G]_\Theta$ as vertices, where $F_{ab}, F_{cd} \in [G]_\Theta$ are adjacent if $F_{ab} \#_G F_{cd}$, see [11]. For instance, $Q_n^\# = K_n$.

Let ab and cd be edges of a partial cube G. Then note that the subgraph of $\mathrm{Sc}(G)$ induced by W_{ab}, W_{ba}, W_{cd}, and W_{dc} contains an edge if and only if $\neg(F_{ab} \# F_{cd})$.

Let $\mathrm{Con}(\mathrm{Sc}(G))$ be the graph obtained from $\mathrm{Sc}(G)$ by identifying vertices W_{ab} and W_{ba} for any Θ-class F_{ab}. Note that $\mathrm{Con}(\mathrm{Sc}(G))$ has no loops and no multiple edges. Let $\overline{G}$ denote the complement of a graph G. To prove the main result of this section the following lemma is needed.

Lemma 2.1 *Let G be a partial cube. Then $G^{\#} \cong \overline{\mathrm{Con}(\mathrm{Sc}(G))}$.*

Proof. For an edge ab of G, let w_{ab} be the vertex of $\mathrm{Con}(\mathrm{Sc}(G))$ obtained by contracting W_{ab} and W_{ba}. Let the mapping $g : V(\mathrm{Con}(\mathrm{Sc}(G))) \to V(\overline{G^{\#}})$ be defined with $g(w_{ab}) = F_{ab}$.

We show that g is an isomorphism. Clearly, g is a bijection. Suppose that w_{ab} is adjacent to w_{cd} in $\mathrm{Con}(\mathrm{Sc}(G))$. Then we may without loss of generality assume that W_{dc} is properly contained in W_{ab}. It follows that Θ-classes F_{ab} and F_{cd} do not cross, hence F_{ab} is adjacent to F_{cd} in $\overline{G^{\#}}$. Assume next that w_{ab} is not adjacent to w_{cd} in $\mathrm{Con}(\mathrm{Sc}(G))$. Then none of W_{cd} and W_{dc} is properly contained in W_{ab} and W_{ba}. But then F_{ab} and F_{cd} cross, hence F_{ab} is not adjacent to F_{cd} in $\overline{G^{\#}}$ and the lemma is proved. □

We next introduce the second crucial concept. The *simplex graph* $S(G)$ of a graph G is the graph whose vertices are the complete subgraphs of G including the empty graph, two vertices of $S(G)$ being adjacent if, as complete subgraphs of G, they differ in exactly one vertex. The simplex graphs have been introduced in [2], where it has in particular been shown that they are median graphs. In fact, simplex graphs can be characterized as the median graphs with a vertex which is common to all maximal cubes [1, Proposition 2.3].

Theorem 2.2 *Let G be a graph on n vertices. Then $Sc(S(G)) = \overline{G} + nK_1$.*

Proof. Let the vertex set of G be $\{1, \ldots, n\}$. Then all Θ-classes of the simplex graph $S(G)$ are of the form $F_{\emptyset i}$, $i = 1, \ldots, n$. In particular, $\mathrm{Sc}(S(G))$ contains $2n$ vertices.

Hence all Θ-classes of $S(G)$ meet the vertex $\emptyset$ of $S(G)$. Therefore, for any $i \in \{1, 2, \ldots, n\}$ the semicube $W_{i\emptyset}$ is terminal, for otherwise we would have a Θ-class that would be properly contained in the semicube $W_{i\emptyset}$, contradicting the fact that in the simplex graph all Θ-classes meet the vertex $\emptyset$. Hence by Lemma 1.1, these semicubes induce n isolated vertices, nK_1, in $\mathrm{Sc}(S(G))$.

Consider next the semicubes $W_{\emptyset i}$, $i = 1, \ldots, n$. Two such semicubes are adjacent in $\mathrm{Sc}(S(G))$ if and only if the corresponding Θ-classes do not cross. We now recall from [11, Theorem 3.1] that for every graph G we have $G = S(G)^{\#}$. Lemma 2.1 completes the proof. □

Let G be an arbitrary graph on n vertices. Then Theorem 2.2 implies that $\mathrm{Sc}(S(\overline{G})) = G + nK_1$ which is the announced representation.

Two results of similar nature as Theorem 2.2 are known. In [11, Theorem 3.1] it was proved that every graph G can be represented as a crossing graph of its simplex graph $S(G)$, while in [10, Theorem 2.3] it was shown that every graph G can be represented as a τ-graph of a simplex graph $S(\overline{G})$.

3 Semicube graphs of trees

Since every graph can be realized as a semicube graph of some graph it seems interesting to consider semicube graphs of specific families of graphs. (The situation is similar as is with intersection graphs, where every graph is an intersection graph of some set system. Restricting to some specific family of sets interesting classes of graphs are obtained, for instance interval graphs and chordal graphs.) In this section we restrict to trees and their semicube graphs.

Each Θ-class of a tree consists of a single edge. Let ab be an edge of a tree T. Then the semicube W_{ab} is adjacent to all semicubes W_{cd}, where $c, d \in W_{ab}$ and $a, b \in W_{cd}$. Therefore $\deg(W_{ab}) = |W_{ab}| - 1$. Furthermore, the terminal semicubes of T are of the form $W_{ab} = \{a\}$, where a is a leaf of T. Hence, using Lemma 1.1, isolated vertices of $\mathrm{Sc}(T)$ are precisely the leaves of T. The remaining semicubes of T form a nontrivial connected component.

Consider the following special cases. If $T = K_{1,n}$, then any two semicubes different from leaves of T are adjacent in $\mathrm{Sc}(T)$, therefore $\mathrm{Sc}(K_{1,n}) = K_n + \overline{K_n}$. Let $v_1, \ldots, v_n$ be the vertices of P_n adjacent in the natural way. Then the sets of semicubes

$$A = \{W_{v_i v_{i+1}} \mid 1 \le i < n\} \quad \text{and} \quad B = \{W_{v_{i+1} v_i} \mid 1 \le i < n\}$$

form a bipartition of $\mathrm{Sc}(P_n)$. A semicube $W_{v_i v_{i+1}}$ from A is adjacent to a semicube $W_{v_{j+1} v_j}$ from B if and only if $i > j$. For another example consider Fig. 1. In the figure we have shortened W_{ij} to ij.

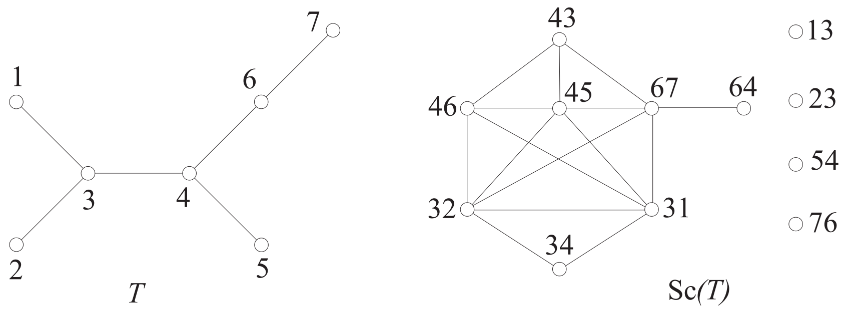

Figure 1: A tree and its semicube graph

In the rest of the section we study different properties of the semicube graphs of trees. For a tree T, let $\ell(T)$ denote the number of the leaves of T. As usually, $\chi(G)$ and $\omega(G)$ denote the chromatic number and the clique number of G, respectively.

Theorem 3.1 *Let T be a tree with at least two edges. Then $\chi(Sc(T)) = \omega(Sc(T)) = \ell(T)$.*

Proof. Let $v_1, \ldots, v_\ell$ be the leaves of T and let $v'_1, \ldots, v'_\ell$ be their neighbors in T. Then the semicubes $W_{v'_i v_i}$, $i = 1, \ldots, \ell(T)$, induce a complete subgraph of $\mathrm{Sc}(T)$. Therefore $\chi(\mathrm{Sc}(T)) \geq \ell(T)$.

To construct an $\ell(T)$-coloring we inductively construct paths $L_1, \ldots, L_{\ell(T)}$ as follows. Let v be a vertex of T of degree at least 2 and let L_1 be the v_1, v-path in T. Suppose $L_1, \ldots, L_k$ are already constructed, where $1 \leq k < \ell$. Then let L_{k+1} be the path between v_{k+1} and the first vertex from $\cup_{i=1}^{k} L_i$. It follows that paths $L_1, \ldots, L_{\ell(T)}$ are internally disjoint and they cover the vertices of T. Note that by the construction, if $i \neq j$ and $L_i \cap L_j = \{u\}$, then u is an endvertex for at least one of L_i and L_j.

We partition the set of all semicubes of T into sets $C = \{W_{ab} \mid v \in W_{ab}\}$ and $D = \{W_{ab} \mid v \notin W_{ab}\}$. For any L_i, $1 \leq i \leq \ell(T)$, the set of semicubes $L_i \cap C$ forms a chain with respect to inclusion and is therefore an independent set of semicubes. We color $W_{ab} \in C$ with i, where $ab \in L_i$.

The set D is an independent set of $\mathrm{Sc}(T)$ since no semicube from D contains the vertex v. Now color $W_{ab} \in D$ with color k, where v_k is an arbitrary leaf not in W_{ab}. Such a leaf exists because v is not a leaf. By the construction, W_{ab} is in $\mathrm{Sc}(T)$ not adjacent to any of the semicubes from $C \cap L_k$, because they also do not contain v_k. We conclude that $\chi(\mathrm{Sc}(T)) \leq \ell(T)$. Since in general $\chi(\mathrm{Sc}(T)) \geq \omega(\mathrm{Sc}(T))$ the proof is complete. $\square$

We next determine the independence number α.

Theorem 3.2 *Let T be a tree with at least one edge. Then $\alpha(Sc(T)) = |V(T)|$. The maximum independent sets are $\{W_{ab}, W_{ba}\} \cup \{W_{cd} \mid W_{cd} \subset W_{ab}$ or $W_{cd} \subset W_{ba}\}$, where $ab \in E(G)$.*

Proof. Let $\mathcal{M}$ be the set of all maximum independent sets of $\mathrm{Sc}(T)$. Consider an arbitrary edge $ab \in E(T)$. First we show that in every maximum independent set of $\mathrm{Sc}(T)$ at least one of the semicubes W_{ab} and W_{ba} is included. Suppose on the contrary that there exists $M \in \mathcal{M}$ such that none of the sets W_{ab} and W_{ba} is in M. By the maximality of M there exist semicubes $W_{cd}, W_{ef} \in M$ such that W_{cd} is adjacent to W_{ab} and W_{ef} is adjacent to W_{ba}. By the definition of the semicube graph this is equivalent to $W_{cd} \cap W_{ab} \neq \emptyset$, $W_{cd} \cup W_{ab} = V(T)$ and $W_{ef} \cap W_{ba} \neq \emptyset$, $W_{ef} \cup W_{ba} = V(T)$. Hence also $W_{cd} \cap W_{ef} \neq \emptyset$, $W_{cd} \cup W_{ef} = V(T)$, therefore W_{cd} and W_{ef} are adjacent in $\mathrm{Sc}(T)$ which is not possible since M is independent.

If W_{ab} is included in a maximum independent set M then also all semicubes such that $W_{cd} \subseteq W_{ab}$ are in M. Moreover maximal (with respect to the inclusion) semicubes in M are complementary. Also only one pair of complementary semicubes can be included in M. Therefore $\alpha(T) = |V(T)|$ and every independent set M is as claimed. In addition, any edge of T can be chosen such that its corresponding semicubes form maximal semicubes of an independent set. Therefore $\mathrm{Sc}(T)$ has $|V(T)| - 1$ different maximum independent sets. $\square$

Theorem 3.2 and its proof are illustrated in Fig. 2, where the maximum independent set is determined with respect to the edge 34.

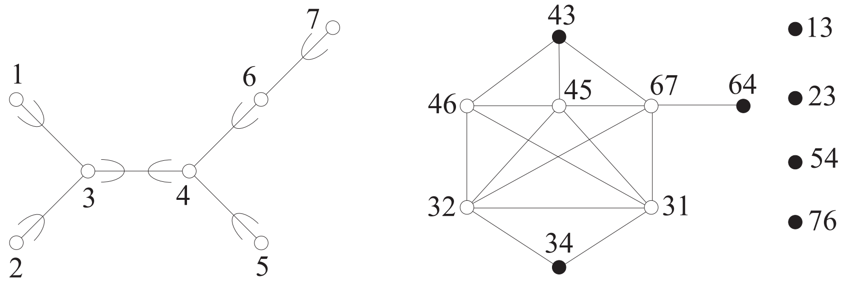

Figure 2: A tree and a maximum independence set in its semicube graph

For the domination number γ we need the following notation. For a tree T, let T^- be the tree obtained from T by removing all its leaves.

Theorem 3.3 *Let T be a tree with at least one edge. Then $\gamma(Sc(T)) = \ell(T) + \ell(T^-)$.*

Proof. Throughout the proof the notation W_{uw} will be used for the semicube W_{uw} considered in T.

The statement is clear for K_2, hence assume in the rest that T has at least two edges. By Lemma 1.1, $\mathrm{Sc}(T)$ contains $\ell(T)$ isolated vertices. We need all of them to dominate themselves. Consider the tree T^-. If $T^- = K_1$ then T is a star and the statement of the theorem clearly holds. Suppose in the following that T^- contains at least one edge. Then any leaf of T^- has at least one neighbor in T^-. Let u be an arbitrary leaf of T^- and w its neighbor in T^-. Consider W_{uw} and let $u_1, \ldots, u_r$ be the leaves of T adjacent to u. Then in $\mathrm{Sc}(T)$ the semicube W_{uw} has degree r; it is adjacent to the semicubes $W_{uu_1}, \ldots, W_{uu_r}$. Hence in order to dominate W_{uw}, one of the vertices from $D(u) = \{W_{uw}, W_{uu_1}, \ldots, W_{uu_r}\}$ must be selected. If x is another leaf of T^- then we infer that $D(x) \cap D(u) = \emptyset$. We conclude that $\gamma(\mathrm{Sc}(T)) \geq \ell(T) + \ell(T^-)$.

Let $v_1, \ldots, v_d$ be the leaves of T^- and let v_i' be a leaf of T adjacent to v_i, $1 \leq i \leq d$. We claim that $D = \cup_{i=1}^{d} W_{v_i v_i'}$ dominates the nontrivial connected component of $\mathrm{Sc}(T)$. Let ab be an arbitrary edge of T. We consider two cases.

Case 1. At least one of a and b is a leaf of T^-.
Suppose $a = v_i$ for some $1 \leq i \leq \ell(T)$. Then b is either a leaf of T (possibly equal to v_i') or an inner vertex of T. In both cases $W_{v_i v_i'}$ dominates W_{ab}.

Case 2. None of a, b is a leaf of T^-.
In this case both a and b have degrees at least two in T^-. Then ab lies on some

path between two leaves v_i and v_j of T^-, $i \neq j$, where $d(a, v_i) = d(b, v_i) - 1$ and $d(b, v_j) = d(a, v_j) - 1$. Then W_{ab} is adjacent to $W_{v_i v_i'}$ and W_{ba} is adjacent to $W_{v_j v_j'}$ in $\mathrm{Sc}(T)$ and both are therefore dominated by D. $\square$

For an illustration of Theorem 3.3 and its proof see Fig. 3, where the tree T' is obtained from T by removing its leaves.

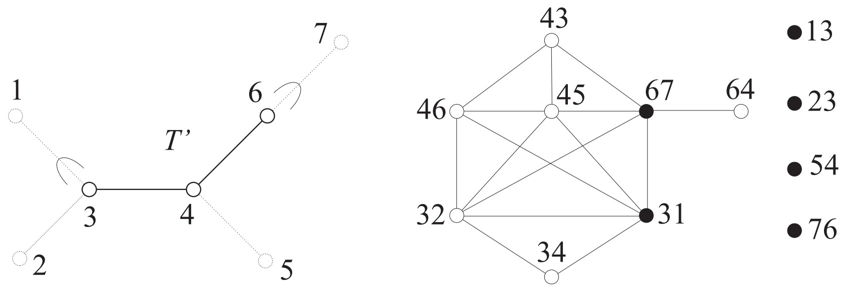

Figure 3: A tree and a minimum dominating set in its semicube graph

Note that Theorem 3.3 can be rephrased by saying that $\gamma(\mathrm{Sc}(T))$ is the number of vertices in the union of the closed neighborhoods of the leaves of T.

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