

Metric properties of generalized Sierpiński graphs over stars

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Abstract

For a graph G , an infinite series of self-similar graphs is formed by the generalized Sierpiński graphs S_G^t , $t \geq 1$. In the case when G is complete we have the classical Sierpiński graphs $S_n^t = S_{K_n}^t$. In this paper the Wiener index, the Wiener complexity, and the metric dimension of their antipode family $S_{K_{1,n}}^t$ are determined. Along the way some other properties of the family are also obtained such as the number of vertex and edge orbits of the automorphism group of $S_{K_{1,n}}^t$.

Keywords: Generalized Sierpiński graph; Wiener index; Wiener dimension; metric dimension

AMS Subj. Class.: 05C76, 05C12, 05C25

1 Introduction

A common approach in studying mathematical structures is to decompose them into special sub-structures inheriting important properties. This approach is especially interesting when the considered structures have self-similarity properties, because then we typically only need to understand the substructures and the way they are linked together. An example is provided by polymer networks, where one of the models for these networks is based on generalized Sierpiński graphs, cf. [14, 35]. We also refer to [39] where (classical) Sierpiński graphs are studied as the dual Sierpiński gaskets.

In this article, we consider the classes of generalized Sierpiński graphs that are generated from stars. This yields in a way the most fragmented classes among all generalized Sierpiński graphs and can be considered as the classes opposite to (classical)

Sierpiński graphs. The latter graphs that are generated from complete graphs were introduced in [24]. The original motivation came from topological studies of universal spaces and from the Tower of Hanoi puzzle [21]. The classical Sierpiński graphs are by now quite well-understood, see the recent extensive survey [22] containing 121 references.

Sierpiński graphs were extended to generalized Sierpiński graphs in [17], where several of their properties were listed, mostly without proofs. They were further investigated with respect to the total chromatic number [15], the strong metric dimension in [12], the Roman domination number [33], and metric aspects [13]. The paper [34] studies several invariants of generalized Sierpiński graphs including the chromatic number, the vertex cover number, the clique number, and the domination number. Moreover, the paper [26] investigates the connectivity of generalized Sierpiński graphs as well as some additional properties including the existence of 1-factors and Hamiltonicity.

We proceed as follows. In the next section we formally define generalized Sierpiński graphs and give an intuitive explanation of their construction. Basic concept from metric graph theory are also recalled. In Section 3 we determine the Wiener index of the generalized Sierpiński graph over stars T_n^t . Some properties of the obtained polynomials are also deduced. In Section 4 the number of vertex and edge orbits of T_n^t is deduced and as a consequence their Wiener complexity determined. In the last section we determine the metric dimension of T_n^t .

2 Preliminaries

Let $G = (V(G), E(G))$ be a graph. If t is a positive integer, then the *generalized Sierpiński graph* S_G^t is a graph with $V(S_G^t) = V(G)^t$. We will abbreviate the notation for a vertex $u = (u_1, \dots, u_t)$ of S_G^t to $u = u_1 \dots u_t$. Vertices $u = u_1 \dots u_t$ and $v = v_1 \dots v_t$ are adjacent if and only if there exists an $i \in [t]$ such that for every $j \in [t]$ the following three conditions are fulfilled:

- (i) $u_j = v_j$, if $j < i$;
- (ii) $u_i \neq v_i$ and $u_i v_i \in E(G)$; and
- (iii) $u_j = v_i$ and $v_j = u_i$ if $j > i$.

Rephrasing the definition of $E(S_G^t)$ we can say that uv is an edge of S_G^t if and only if there exist an edge $xy \in E(G)$ and a sequence of symbols from $[t]$, say w , such that $u = wxyy \dots y$ and $v = wyxx \dots x$. The graph S_G^t can be constructed recursively from G by the following process: Let $S_G^1 = G$. For $t \geq 2$, consider $n = |V(G)|$ disjoint copies of the graph S_G^{t-1} and in the i^{th} copy, $i \in [n]$, add i at the beginning of each vertex of the copy. We will denote these n subgraphs of S_G^t by iS_G^{t-1} , $i \in [n]$. Finally, for every edge ij of G , add an edge between the vertices ij^{n-1} and ji^{n-1} .

In this paper we are primarily interested in the generalized Sierpiński graphs $S_{K_{1,n}}^t$, $n \geq 2$, $t \geq 1$. (We assume that $n \geq 2$ because T_1^t is isomorphic to the path P_{2t} .) To simplify notation we will denote $S_{K_{1,n}}^t$ by T_n^t . We will also assume throughout the

paper that $V(K_{1,n}) = \{0, 1, \dots, n\}$, where 0 is the vertex of degree n . The vertex 0^t of T_n^t will be called the *central vertex* of T_n^t . As an example consider the graph T_4^3 shown in Fig. 1.

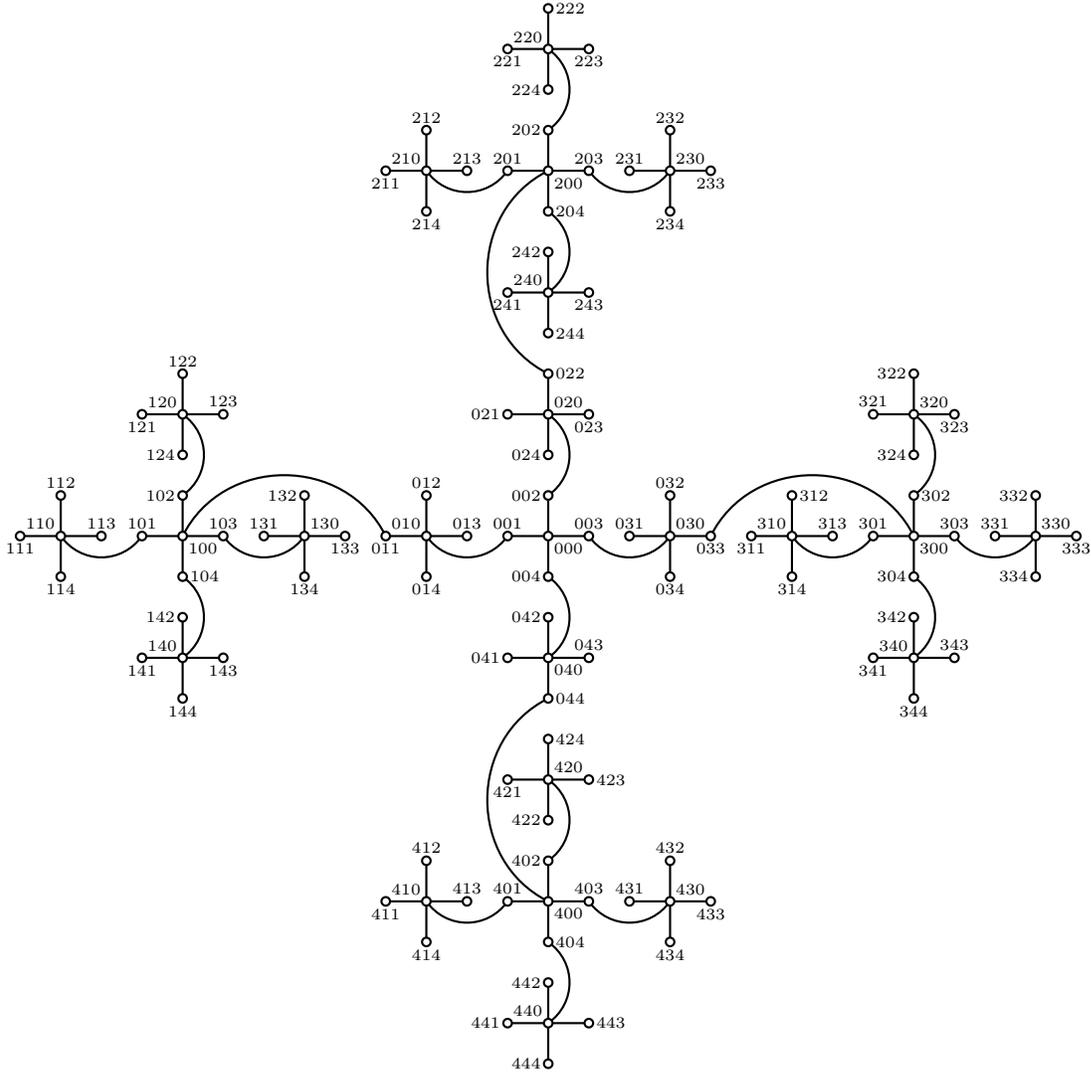


Figure 1: The generalized Sierpiński graph $T_4^3 = S_{K_{1,4}}^3$

Note that T_n^t is a tree for each $t \geq 1$. Indeed, T_n^t is connected and it easily follows by induction that $|E(T_n^t)| = |V(T_n^t)| - 1$. (More generally, if T is an arbitrary tree, then S_T^t is also a tree.) By jT_n^{t-1} , $j \in \{0, 1, \dots, n\}$, we will denote the subgraph of T_n^t induced by the vertices whose first coordinate is j . Each jT_n^{t-1} is isomorphic to T_n^{t-1} ,

and the subgraph $0T_n^{t-1}$ will be called the *central copy* of T_n^{t-1} in T_n^t .

The distance between vertices u and v of a (connected) graph G is denoted by $d_G(u, v)$ or $d(u, v)$ if G is clear from the context. The maximum distance between u and all the other vertices is the *eccentricity* $\text{ecc}(u)$ of u . The maximum and the minimum eccentricity among the vertices of G are the *diameter* $\text{diam}(G)$ and the *radius* $\text{rad}(G)$. The *transmission of a vertex* u is denoted by $\text{Tr}_G(u)$ ($\text{Tr}(u)$ for short) and defined as the sum of distances between u and all the other vertices of G [37], that is,

$$\text{Tr}_G(u) = \sum_{v \in V(G)} d(u, v).$$

The transmission of a vertex is also known as the *transmission index of a vertex* [1], or the *distance of a vertex* [10]. Different additional notations are used for the transmission of a vertex including $T_G(v)$, $d_G(v)$ [10], $d(x, V(G))$ [5], and $W_G(v)$ [27].

Finally, the automorphism group of a graph G will be denoted by $\text{Aut}(G)$. Whenever we will speak about (vertex or edge) orbits, this will refer to the orbits under the action of the group $\text{Aut}(G)$.

3 Wiener index

The *Wiener index* $W(G)$ of a graph G is defined as $W(G) = \sum d_G(u, v)$, where the summation runs over all unordered pairs of vertices of G . This index is the oldest [38] and among the most frequently studied and used topological indices in mathematical chemistry, see a selection of very recent developments [1, 7, 19, 23] and references therein. In particular, the Wiener index of trees has been extensively studied. We refer to [30] for a linear algorithm on trees, to the extensive survey [9], and to recent papers [8, 16]. In this section we add the following result to the area:

Theorem 3.1 *If $n \geq 2$ and $t \geq 1$, then*

$$W(T_n^t) = \frac{(2(2^t - 1)n^2 - n - 2^t)(n + 1)^{2t-1}}{2n + 1} + \frac{((2^{t+1} - 1)n + 2^t)(n + 1)^{t-1}}{2n + 1}. \quad (1)$$

In the rest we will frequently use the formulas $d(10^t, 0^{t+1}) = 2^t$ and $d(10^t, 1^{t+1}) = 2^t - 1$. The formulas can be obtained directly (by induction), or deduced from the more general results obtained in [13].

If H and K are disjoint subgraphs of a graph G , then let

$$W(H, K) = \sum_{x \in V(H)} \sum_{y \in V(K)} d_G(x, y).$$

By the structure of T_n^{t+1} and its symmetries, the Wiener index of T_n^{t+1} can be decomposed as follows:

$$\begin{aligned} W(T_n^{t+1}) &= \sum_{0 \leq j \leq n} W(jT_n^t) + \sum_{0 \leq r < s \leq n} W(rT_n^t, sT_n^t) \\ &= (n+1)W(T_n^t) + \binom{n}{2}W(1T_n^t, 2T_n^t) + nW(0T_n^t, 1T_n^t). \end{aligned} \quad (2)$$

To further simplify (2) we proceed with a series of lemmas.

Lemma 3.2 *If $t \geq 1$, then $W(1T_n^t, 2T_n^t) = 2(n+1)^t (\text{Tr}_{T_n^t}(0^t) + 2^t(n+1)^t)$.*

Proof. Using the fact that 10^t and 20^t are central vertices of the trees $1T_n^t$ and $2T_n^t$, respectively, we obtain

$$\begin{aligned} W(1T_n^t, 2T_n^t) &= \sum_{x \in 1T_n^t} \sum_{y \in 2T_n^t} d(x, y) \\ &= \sum_{x \in 1T_n^t} \sum_{y \in 2T_n^t} (d(x, 10^t) + d(10^t, 20^t) + d(20^t, y)) \\ &= \sum_{x \in 1T_n^t} \sum_{y \in 2T_n^t} (d(x, 10^t) + 2^{t+1} + d(y, 20^t)) \\ &= (n+1)^t \text{Tr}_{1T_n^t}(10^t) + (n+1)^{2t} 2^{t+1} + (n+1)^t \text{Tr}_{2T_n^t}(20^t) \\ &= 2(n+1)^t \text{Tr}_{T_n^t}(0^t) + (n+1)^{2t} 2^{t+1} \end{aligned}$$

as claimed. □

Lemma 3.3 *If $t \geq 1$, then $W(0T_n^t, 1T_n^t) = (n+1)^t (d_{T_n^t}(0^t) + d_{T_n^t}(1^t) + (n+1)^t)$.*

Proof. Now we compute as follows, where we again use the fact that 10^t is the central vertex of the tree $1T_n^t$:

$$\begin{aligned} W(0T_n^t, 1T_n^t) &= \sum_{y \in 0T_n^t} \sum_{x \in 1T_n^t} d(x, y) \\ &= \sum_{x \in 1T_n^t} \sum_{y \in 0T_n^t} (d(x, 10^t) + 1 + d(01^t, y)) \\ &= (n+1)^t \text{Tr}_{1T_n^t}(10^t) + (n+1)^{2t} + (n+1)^t \text{Tr}_{0T_n^t}(01^t) \\ &= (n+1)^t \text{Tr}_{T_n^t}(0^t) + (n+1)^{2t} + (n+1)^t \text{Tr}_{T_n^t}(1^t), \end{aligned}$$

and we are done. □

Lemma 3.4 *If $t \geq 1$, then $\text{Tr}_{T_n^t}(0^t) = n(n+1)^{t-1}(2^t - 1)$.*

Proof. Having in mind the symmetry of T_n^{t+1} we compute as follows:

$$\begin{aligned}
\mathrm{Tr}_{T_n^{t+1}}(0^{t+1}) &= \mathrm{Tr}_{T_n^t}(0^t) + n \cdot \sum_{x \in 1T_n^t} d(0^{t+1}, x) \\
&= \mathrm{Tr}_{T_n^t}(0^t) + n \cdot \sum_{x \in 1T_n^t} (2^t + d(10^t, x)) \\
&= \mathrm{Tr}_{T_n^t}(0^t) + n(n+1)^t 2^t + n \mathrm{Tr}_{1T_n^t}(10^t) \\
&= (n+1) \mathrm{Tr}_{T_n^t}(0^t) + n(n+1)^t 2^t.
\end{aligned}$$

Solving the above first-order inhomogeneous linear recurrence with coefficients, independent of t , for $\mathrm{Tr}_{T_n^t}(0^t)$ yields the claimed result. \square

Lemma 3.5 *If $t \geq 1$, then*

$$\mathrm{Tr}_{T_n^t}(1^t) = \frac{1}{n} \cdot ((2(2^t - 1)n^2 - n - 2^t)(n+1)^{t-1} + 2^t). \quad (3)$$

Proof. If $i, j \in [n]$, then the mapping $V(T_n^t) \rightarrow V(T_n^t)$ that exchanges the coordinates i and j , and fixes all the other coordinates, is an automorphism of T_n^t . Hence the extreme vertices j^t , $1 \leq j \leq n$, are in the same orbit. Consequently $\mathrm{Tr}_{jT_n^t}(j^t) = \mathrm{Tr}_{sT_n^t}(s^t)$ for $t, s, j \geq 1$.

$$\begin{aligned}
\mathrm{Tr}_{T_n^{t+1}}(1^{t+1}) &= \sum_{x \in 0T_n^t} d(1^{t+1}, x) + \sum_{x \in 1T_n^t} d(1^{t+1}, x) + \cdots + \sum_{x \in nT_n^t} d(1^{t+1}, x) \\
&= \sum_{x \in 0T_n^t} ((2^t - 1) + 1 + d(01^t, x)) + \mathrm{Tr}_{T_n^t}(1^t) \\
&\quad + (n-1) \sum_{x \in 2T_n^t} (2^{t+1} + 2^t - 1 + d(20^t, x)) \\
&= 2\mathrm{Tr}_{T_n^t}(1^t) + 2^t(n+1)^t + (n-1)(n+1)^t(2^{t+1} + 2^t - 1) \\
&\quad + (n-1)\mathrm{Tr}_{T_n^t}(0^t).
\end{aligned}$$

Using Lemma 3.4, we obtain the first-order inhomogeneous linear recurrence

$$\begin{aligned}
\mathrm{Tr}_{T_n^{t+1}}(1^{t+1}) &= 2\mathrm{Tr}_{T_n^t}(1^t) + 2^t(n+1)^t + (n-1)(n+1)^t(2^{t+1} + 2^t - 1) \\
&\quad + (n-1)n(n+1)^{t-1}(2^t - 1). \quad (4)
\end{aligned}$$

It is straightforward to check that $\mathrm{Tr}_{T_n^t}(1^t)$, as given in (3), satisfies both (4) and the obvious initial condition $\mathrm{Tr}_{T_n^1}(1^1) = 2n - 1$. \square

We are now ready to complete the proof of Theorem 3.1. Combining (2) with Lemmas 3.2 – 3.5 we obtain, after some calculation, that for $n \geq 2$ and $t \geq 1$, the Wiener index $W(T_n^t)$ satisfies the recurrence

$$W(T_n^{t+1}) = (n+1)W(T_n^t) + (n+1)^{2t}((2^{t+1}-1)n^2 - 2^t) + 2^t(n+1)^t. \quad (5)$$

It is now straightforward to check that $W(T_n^t)$, as given in (1), satisfies both (5) and the obvious initial condition $W(T_n^1) = n^2$, finishing the proof of Theorem 3.1. \square

The first few values of $W(T_n^t)$ are given in Tables 1 and 2.

t	$W(T_n^t)$
1	n^2
2	$3n^4 + 7n^3 + 2n^2 - 2n$
3	$7n^6 + 31n^5 + 48n^4 + 21n^3 - 13n^2 - 10n$
4	$15n^8 + 97n^7 + 255n^6 + 331n^5 + 174n^4 - 54n^3 - 104n^2 - 34n$
5	$31n^{10} + 263n^9 + 964n^8 + 1960n^7 + 2308n^6 + 1345n^5 - 116n^4 - 742n^3 - 459n^2 - 98n$

Table 1: The first five values of $W(T_n^t)$ as polynomials in n

t	$W(T_n^t)$
1	n^2
2	$n(n+1)(3n^2 + 4n - 2)$
3	$n(n+1)^2(7n^3 + 17n^2 + 7n - 10)$
4	$n(n+1)^3(15n^4 + 52n^3 + 54n^2 - 2n - 34)$
5	$n(n+1)^4(31n^5 + 139n^4 + 222n^3 + 114n^2 - 67n - 98)$
6	$n(n+1)^5(63n^6 + 346n^5 + 737n^4 + 692n^3 + 109n^2 - 324n - 258)$
7	$n(n+1)^6(127n^7 + 825n^6 + 2187n^5 + 2893n^4 + 1637n^3 - 409n^2 - 1157n - 642)$

Table 2: The first seven values of $W(T_n^t)$ in factored form

Proposition 3.6 $W(T_n^t)$ is a polynomial in n of degree $2t$ with integer coefficients, with leading coefficient $2^t - 1$, divisible by $n(n+1)^{t-1}$.

Proof. By induction on t .

$t = 1$: $W(T_n^1) = n^2$ obviously has the stated properties.

$t \rightarrow t + 1$: By induction hypothesis, the first term on the right-hand side of (5) is a polynomial in n of degree $2t + 1$ with integer coefficients, divisible by $n(n+1)^t$.

The second term on the right-hand side of (5) is a polynomial in n of degree $2t + 2$ with integer coefficients, with leading coefficient $2^{t+1} - 1$, divisible by $(n + 1)^{2t}$. The third term on the right-hand side of (5) is a polynomial in n of degree t with integer coefficients, divisible by $(n + 1)^t$. It follows that the right-hand side of (5), and hence $W(T_n^{t+1})$, is a polynomial in n of degree $2t + 2$ with integer coefficients, with leading coefficient $2^{t+1} - 1$, divisible by $(n + 1)^t$. At $n = 0$, the right-hand side of (5) evaluates to $0 - 2^t + 2^t = 0$, so $W(T_n^{t+1})$ is divisible by n as well. \square

4 Wiener complexity

The *Wiener complexity* $C_W(G)$ of G is the number of different distances of the vertices of G . This concept was introduced in [2] under the name Wiener dimension; here we follow the notation and terminology of a general approach from [3], see also [4].

For trees T of order at least 3, formulas for $\text{diam}(S_T^t)$ and $\text{rad}(S_T^t)$ were proved in [13, Theorem 14] and [13, Theorem 15], respectively. Applying these formulas to T_n^t we get

$$\text{diam}(T_n^t) = 2^{t+1} - 2 \quad (6)$$

and

$$\text{rad}(T_n^t) = 2^t - 1. \quad (7)$$

We note that the central vertex 0^t of T_n^t is the unique vertex u with $\text{ecc}(u) = 2^t - 1$.

To determine $C_W(T_n^t)$ we first give two lemmas. The first one reads as follows, where $O_v(G)$ and $O_e(G)$ denote the number of vertex orbits and the number of edge orbits under the action of $\text{Aut}(G)$, respectively.

Lemma 4.1 *If $n \geq 2$ and $t \geq 1$, then $O_v(T_n^t) = 5 \cdot 2^{t-2} - 1$ and $O_e(T_n^t) = 5 \cdot 2^{t-2} - 2$.*

Proof. It is easy to see that for $n \geq 2$ we have $O_v(T_n^1) = 2$, $O_v(T_n^2) = 4$, $O_e(T_n^1) = 1$, and $O_e(T_n^2) = 3$. Let now $t \geq 2$. In T_n^{t+1} , the subgraphs iT_n^t and jT_n^t have the same edge orbits and vertex orbits for any $i, j \in [n]$, $i \neq j$. On the other hand, the vertex and edge orbits of the subgraphs $0T_n^t$ and $1T_n^t$ are disjoint. Moreover, the edge between 01^t and 10^t yields one additional vertex orbit and two additional edge orbits. Hence we have the following recurrence relations,

$$\begin{aligned} O_v(T_n^{t+1}) &= 2O_v(T_n^t) + 1, \\ O_e(T_n^{t+1}) &= 2O_e(T_n^t) + 2. \end{aligned}$$

Solving these recurrence relations yields the result. \square

Our second lemma is a consequence of a more general result from [11] that is stated also in [9, Theorem 3]. Since its prove is short, we include it for completeness.

Lemma 4.2 *Let uv be an edge of a tree T . Let T_u and T_v be the components of $T - uv$ containing u and v , respectively. If $|V(T_u)| > |V(T_v)|$, then $\text{Tr}(v) > \text{Tr}(u)$.*

Proof. Since uv is a cut edge, we can compute as follows:

$$\begin{aligned} \text{Tr}(v) - \text{Tr}(u) &= \sum_{x \in T_u} (d(v, x) - d(u, x)) + \sum_{y \in T_v} d(v, y) - d(u, y) \\ &= \sum_{x \in T_u} 1 + \sum_{y \in T_v} -1 = |V(T_u)| - |V(T_v)| > 0 \end{aligned}$$

and we are done. \square

We are now ready for the main result of this section.

Theorem 4.3 *If $n \geq 4$, then*

$$C_W(T_n^t) = O_v(T_n^t) = 5 \cdot 2^{t-2} - 1.$$

Proof. Using again the fact that automorphisms preserve distances, the vertices from the same vertex orbit have the same distance. Hence by Lemma 4.1, $C_W(T_n^t) \leq O_v(T_n^t)$. To complete the proof we thus need to show that $C_W(T_n^t) \geq O_v(T_n^t)$. Let $P = u_1 u_2 \dots u_{2^t}$ be a path of length $2^t - 1$ starting at 0^t and ending at 1^t . Such a path exists by (7), and we have $u_1 = 0^t$ and $u_{2^t} = 1^t$. Using the notation of Lemma 4.2 for the components in edge-deleted T_n^t we infer that $|V(T_{u_i})| > |V(T_{u_{i+1}})|$, $i \in [2^t - 1]$. Hence by Lemma 4.2, $\text{Tr}(u_1) < \text{Tr}(u_2) < \dots < \text{Tr}(u_{2^t})$. Let $u = u_{2^{t-1}-1}$, $v = u_{2^{t-1}}$ and $w = u_{2^{t-1}+1}$. Then vw is the edge between $0T_n^{t-1}$ and $1T_n^{t-1}$. Let x be a pendant vertex that is adjacent to u . (For instance, in the example from Fig. 1, each of the vertices 012, 013, and 014 can be considered as x .) By Lemma 4.2, we have

$$\begin{aligned} \text{Tr}(x) - \text{Tr}(u) &= (n+1)^t - 2, \\ \text{Tr}(x) - \text{Tr}(v) &= 2(n+1)^{t-1}, \\ \text{Tr}(w) - \text{Tr}(v) &= (n+1)^t - 2(n+1)^{t-1}, \\ \text{Tr}(w) - \text{Tr}(x) &= (n+1)^t - 4(n+1)^{t-1}. \end{aligned}$$

Clearly, for $n \geq 4$ it holds that $\text{Tr}(v) < \text{Tr}(x) < \text{Tr}(w)$. Note that the vertices of the path P together with vertices such as x above form a set of representatives of the vertex orbits. Hence the vertices in different orbits have different distances, which in turn implies that $C_W(T_n^t) \geq O_v(T_n^t)$. \square

5 Metric dimension

A subset R of the vertex set $V(G)$ is a *resolving set* for the graph G , if for each pair of distinct vertices x and y there exists an $r \in R$ such that $d_G(r, x) \neq d_G(r, y)$. The

metric dimension $\mu(G)$ of G is the size of a smallest resolving set. These concepts were independently introduced in [20, 36] as an intriguing option to uniquely identify the vertices of a graph. Today, a huge bibliography on the topic exists, hence we rather only refer to the recent developments [6, 18, 28, 31] and references therein; see also [29] for an application in digital geometry.

The metric dimension of the classical Sierpiński graphs was independently determined in [32, Théorème 3.6] and in [25, Corollary 6]: If $n, t \geq 1$, then $\mu(S_{K_n}^t) = n - 1$. In this section we add the following result to the list of families for which the metric dimension is known.

Theorem 5.1 *If $n \geq 2$, then $\mu(T_n^1) = n - 1$. Moreover, if $t \geq 2$, then*

$$\mu(T_n^t) = (n + 1)^{t-2}(n^2 - n - 1) + 1.$$

Proof. Let u and v be vertices of degree 1 of a graph G , and with a common (support) neighbor. If R is a resolving set for G then $R \cap \{u, v\} \neq \emptyset$ because $d(u, x) = d(v, x)$ holds for any $x \neq u, v$. Since $T_n^1 = K_{1,n}$, this fact in particular implies that $\mu(T_n^1) = n - 1$.

Let now $t \geq 2$, and let uT_n^1 be a subgraph of T_n^t , where $u \in \{0, 1, \dots, n\}^{t-1}$. Note that uT_n^1 is isomorphic to $T_n^1 = K_{1,n}$. We say that uT_n^1 is of *type 1* if the vertices $u1, \dots, un$ are all of degree 1, and of *type 2* if among the vertices $u1, \dots, un$ all but one are of degree 1. We infer that if uT_n^1 is of type 1 or of type 2, then the vertex $u0$ has a neighbor in $u'T_n^1$ for some $u' \in \{0, 1, \dots, n\}^{t-1}$, where $u' \neq u$. We claim that T_n^t contains precisely

$$(n - 1)(n + 1)^{t-2} + 1 \tag{8}$$

subgraphs uT_n^1 that are of type 1, and precisely

$$(n + 1)^{t-2} - 1 \tag{9}$$

subgraphs uT_n^1 that are of type 2.

We first prove (8). Let a_t , $t \geq 2$, be the number of type 1 subgraphs of T_n^t . Since $1T_n^1, \dots, nT_n^1$ are type 1 subgraphs of T_n^2 , we have $a_2 = n$, hence the assertion holds for $t = 2$. We proceed inductively and assume that $t \geq 3$. Then each of the subgraphs $1T_n^{t-1}, \dots, nT_n^{t-1}$ contains precisely n type 1 subgraphs. On the other hand, in the subgraph $0T_n^{t-1}$, there are precisely n edges that connect a vertex of a type 1 subgraph of $0T_n^{t-1}$ (considered in $0T_n^{t-1}$) with a vertex in some subgraph iT_n^{t-1} . It follows that inside $0T_n^{t-1}$ the number of type 1 subgraphs (considered in T_n^t) is $a_{t-1} - n$. Consequently,

$$a_t = na_{t-1} + (a_{t-1} - n) = (n + 1)a_{t-1} - n.$$

Solving this recurrence yields (8).

Let now b_t , $t \geq 2$, be the number of type 2 subgraphs of T_n^t . There are no such subgraphs in T_n^2 , hence (9) holds for $t = 2$. For $t \geq 3$ we proceed similarly as in the above paragraph to get the recurrence

$$b_t = (n + 1)b_{t-1} + n,$$

from which (9) follows.

Using the fact from the first paragraph of the proof together with (8) and (9) we infer that if $t \geq 2$, then

$$\begin{aligned}\mu(T_n^t) &\geq (n-1)((n-1)(n+1)^{t-2} + 1) + (n-2)((n+1)^{t-2} - 1) \\ &= (n+1)^{t-2}(n^2 - n - 1) + 1.\end{aligned}$$

On the other hand, it is straightforward to verify that the set X which contains (arbitrary) $n-1$ leaves from each of the type 1 subgraphs of T_n^t , and (arbitrary) $n-2$ leaves from each of the type 2 subgraphs of T_n^t , is a resolving set for T_n^t . Consequently, $\mu(T_n^t) \leq (n+1)^{t-2}(n^2 - n - 1) + 1$ and we are done. \square

Acknowledgement

S.K. and M.P. acknowledge the financial support from the Slovenian Research Agency (research core funding No. P1-0294, P1-0297, and J1-7110).

References

- [1] A. Abiad, B. Brimkov, A. Erey, L. Leshock, X. Martínez-Rivera, S. O, S.-Y. Song, J. Williford, On the Wiener index, distance cospectrality and transmission-regular graphs, *Discrete Appl. Math.* 230 (2017) 1–10.
- [2] Y. Alizadeh, V. Andova, S. Klavžar, R. Škrekovski, Wiener dimension: fundamental properties and (5,0)-nanotubical fullerenes, *MATCH Commun. Math. Comput. Chem.* 72 (2014) 279–294.
- [3] Y. Alizadeh, S. Klavžar, Complexity of topological indices: The case of connective eccentric index, *MATCH Commun. Math. Comput. Chem.* 76 (2016) 659–667.
- [4] Y. Alizadeh, S. Klavžar, On graphs whose Wiener complexity equals their order and on Wiener index of asymmetric graphs, *Appl. Math. Comput.* 328 (2018) 113–118.
- [5] K. Balakrishnan, M. Changat, I. Peterin, S. Špacapan, P. Šparl, A. R. Subhamathi, R. Ajitha, Strongly distance-balanced graphs and graph products, *European J. Combin.* 30 (2009) 1048–1053.
- [6] R. Belmonte, F. V. Fomin, P. A. Golovach, M. S. Ramanujan, Metric dimension of bounded tree-length graphs, *SIAM J. Discrete Math.* 31 (2017) 1217–1243.
- [7] M. Črepnjak, N. Tratnik, The Szeged index and the Wiener index of partial cubes with applications to chemical graphs, *Appl. Math. Comput.* 309 (2017) 324–333.

- [8] K. Ch. Das, M. J. Nadjafi-Arani, On maximum Wiener index of trees and graphs with given radius, *J. Comb. Optim.* 34 (2017) 574–587.
- [9] A. A. Dobrynin, R. Entringer, I. Gutman, Wiener index of trees: theory and applications, *Acta Appl. Math.* 66 (2001) 211–249.
- [10] A. A. Dobrynin, I. Gutman, S. Klavžar, P. Žigert, Wiener index of hexagonal systems, *Acta Appl. Math.* 72 (2002) 247–294.
- [11] R. C. Entringer, D. J. Kleitman, L. A. Székely, A note on spanning trees with minimum average distance, *Bull. Inst. Combin. Appl.* 17 (1996) 71–78.
- [12] E. Estaji, J. A. Rodríguez-Velázquez, The strong metric dimension of generalized Sierpiński graphs with pendant vertices, *Ars Math. Contemp.* 12 (2017) 127–134.
- [13] A. Estrada-Moreno, E. D. Rodríguez-Bazan, J. A. Rodríguez-Velázquez, On distances in generalized Sierpiński graphs, *Appl. Anal. Discrete Math.* 12 (2018) 49–69.
- [14] A. Estrada-Moreno, J. A. Rodríguez-Velázquez, On the general Randić index of polymeric networks modelled by generalized Sierpiński graphs, *Discrete Appl. Math.* (2018), <https://doi.org/10.1016/j.dam.2018.03.032>.
- [15] J. Geetha, K. Somasundaram, Total coloring of generalized Sierpiński graphs, *Australas. J. Combin.* 63 (2015) 58–69.
- [16] A. Georgakopoulos, S. Wagner, Hitting times, cover cost, and the Wiener index of a tree, *J. Graph Theory* 84 (2017) 311–326.
- [17] S. Gravier, M. Kovše, A. Parreau, Generalized Sierpiński graphs, in: *Posters at EuroComb’11, Budapest*, <http://www.renyi.hu/conferences/ec11/posters/parreau.pdf> (2017–12–11)
- [18] C. Grigorious, T. Kalinowski, J. Ryan, S. Stephen, The metric dimension of the circulant graph $C(n, \pm\{1, 2, 3, 4\})$, *Australas. J. Combin.* 69 (2017) 417–441.
- [19] I. Gutman, S. Li, W. Wei, Cacti with n -vertices and t cycles having extremal Wiener index, *Discrete Appl. Math.* 232 (2017) 189–200.
- [20] F. Harary, R. A. Melter, On the metric dimension of a graph, *Ars. Combin.* 2 (1976) 191–195.
- [21] A.M. Hinz, S. Klavžar, U. Milutinović, C. Petr, *The Tower of Hanoi—Myths and Maths*, Birkhäuser/Springer, Basel, 2013.

- [22] A.M. Hinz, S. Klavžar, S. S. Zemljič, A survey and classification of Sierpiński-type graphs, *Discrete Appl. Math.* 217 (2017) 565–600.
- [23] H. Hua, B. Ning, Wiener index, Harary index and Hamiltonicity of graphs, *MATCH Commun. Math. Comput. Chem.* 78 (2017) 153–162.
- [24] S. Klavžar, U. Milutinović, Graphs $S(n, k)$ and a variant of the Tower of Hanoi problem, *Czechoslovak Math. J.* 47(122) (1997) 95–104.
- [25] S. Klavžar, S. S. Zemljič, On distances in Sierpiński graphs: almost-extreme vertices and metric dimension, *Appl. Anal. Discrete Math.* 7 (2013) 72–82.
- [26] S. Klavžar, S. S. Zemljič, Connectivity and some other properties of generalized Sierpiński graphs, *Appl. Anal. Discrete Math.* (2018), <https://doi.org/10.2298/AADM170206009K>.
- [27] M. Krnc, R. Škrekovski, Centralization of transmission in networks, *Discrete Math.* 338 (2015) 2412–2420.
- [28] D. Kuziak, J. A. Rodríguez-Velázquez, I. G. Yero, Computing the metric dimension of a graph from primary subgraphs, *Discuss. Math. Graph Theory* 37 (2017) 273–293.
- [29] R. A. Melter, I. Tomescu, Metric bases in digital geometry, *Comput. Vision Graphics Image Process.* 25 (1984) 113–121.
- [30] B. Mohar, T. Pisanski, How to compute the Wiener index of a graph, *J. Math. Chem.* 2 (1988) 267–277.
- [31] G. Moravcik, O. R. Oellermann, S. Yusim, Comparing the metric and strong dimensions of graphs, *Discrete Appl. Math.* 220 (2017) 68–79.
- [32] A. Parreau, Problèmes d’identification dans les graphes, Ph.D. Thesis, Université de Grenoble, 2012.
- [33] F. Ramezani, E. D. Rodríguez-Bazan, J. A. Rodríguez-Velázquez, On the Roman domination number of generalized Sierpiński graphs, *Filomat* 31 (2017) 6515–6528.
- [34] J. A. Rodríguez-Velázquez, E. D. Rodríguez-Bazan, A. Estrada-Moreno, On generalized Sierpiński graphs, *Discuss. Math. Graph Theory* 37 (2017) 547–560.
- [35] J. A. Rodríguez-Velázquez, J. Tomás-Andreu, On the Randić index of polymeric networks modelled by generalized Sierpiński graphs, *MATCH Commun. Math. Comput. Chem.* 74 (2015) 145–160.
- [36] P. J. Slater, Leaves of trees, in: *Proceedings of the Sixth Southeastern Conference on Combinatorics, Graph Theory, and Computing*, *Congressus Numerantium* 14 (1975) 549–559.

- [37] L. Šoltés, Transmission in graphs: a bound and vertex removing, *Math. Slovaca* 41 (1991) 11–16.
- [38] H. Wiener, Structural determination of the paraffin boiling points, *J. Amer. Chem. Soc.* 69 (1947) 17–20.
- [39] S. Wu, Z. Zhang, G. Chen, Random walks on dual Sierpinski gaskets, *European Phys. J. B* 82 (2011) 91–96.