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Packing chromatic number of base-3 Sierpiński graphs

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Abstract The packing chromatic number $\chi_\rho(G)$ of a graph G is the smallest integer k such that there exists a k-vertex coloring of G in which any two vertices receiving color i are at distance at least $i+1$. Let S^n be the base-3 Sierpiński graph of dimension n. It is proved that $\chi_\rho(S^1) = 3$, $\chi_\rho(S^2) = 5$, $\chi_\rho(S^3) = \chi_\rho(S^4) = 7$, and that $8 \le \chi_\rho(S^n) \le 9$ holds for any $n \ge 5$.

Keywords packing · packing chromatic number · Sierpiński graphs

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1 Introduction

If G is a graph and i a positive integer, then $X \subseteq V(G)$ is an *i-packing* if vertices of X are pairwise at distance more than i. The integer i is called the *width* of the packing X. The *packing chromatic number* $\chi_\rho(G)$ of G is the smallest integer k such that $V(G)$ can be partitioned into packings $X_1, \ldots, X_k$ with pairwise different widths. Since such a partition has packings of k distinct widths and because the objective is to minimize k, we can assume that X_i is an i-packing for $i = 1, \ldots, k$. Equivalently, a *k-packing coloring* of G is a function $c : V(G) \to \{1, \ldots, k\}$ such that if $c(u) = c(v) = i$, then $d(u, v) > i$, where $d(u, v)$ is the usual shortest-path distance between u and v. For a subset $X \subset V(G)$ we will denote by $c(X)$ the multiset $\{c(x) : x \in X\}$. As usual, a *k-coloring* is a partition of $V(G)$ into k 1-packings.

The concept of the packing chromatic number was introduced in [14] and given the name in [3]. For the infinite square lattice, the packing chromatic number lies between 12 and 17. The upper bound was established in [27]. The lower bound 9 from [14] was first improved to 10 (cf. [9]), and it was then reported in [5] that, using extensive computing, it can be further improved to 12. The exact packing chromatic number of the infinite hexagonal lattice is 7, the upper bound being established in [9] and the lower bound in [22]. On the other hand, the infinite triangular lattice does not have a finite packing chromatic number [10]. The packing chromatic number of hypercubes Q_d, $d \le 5$, was determined in [14], the result being extended in [29] by determining $\chi_\rho(Q_d)$ for $6 \le d \le 8$. For bounds on the packing chromatic number and some exact values on Cartesian product graphs see [20,26]. The packing chromatic number was also intensively studied on distance graphs, see [6,7,28].

In the seminal paper [14] it was proved that it is NP-complete to decide whether $\chi_\rho(G) \le 4$ holds for a graph G. Fiala and Golovach later demonstrated that the packing chromatic number is actually an intrinsically difficult graph invariant by proving that determining χ_ρ is NP-complete restricted to trees [8]. On the positive side, Argiroffo, Nasini, and Torres proved in [1] that for every fixed q, the packing coloring problem is solvable in polynomial time for the graphs in which no set of at most q vertices induces more than $q - 4$ distinct paths on four vertices. Furthermore, in [2] the same authors discovered additional classes of graphs on which the packing chromatic number can be computed in polynomial time, including caterpillars, certain superclasses of split graphs, and a certain superclass of cographs.

Hence exact values or good approximations of χ_ρ on non-trivial families of graphs are of interest, and in this paper we study the packing chromatic number on the class of base-3 Sierpiński graphs. A further motivation to study the packing chromatic number of these graphs is the fact that they are subcubic graphs. It is namely an open problem whether the packing chromatic number of cubic graphs is bounded by a constant [13]. Earlier it was asked in [14] what is the maximum of the packing chromatic number of a cubic graph of order n.

The Sierpiński graphs S_p^n were introduced in [21]. Motivations for them came from investigations of universal topological spaces (see the book [24]) and from the Tower of Hanoi problems (see the book [18]). We will formally introduce the base-3 Sierpiński graphs in the next section. Here we only mention that Sierpiński graphs form an intensively investigated class of graphs; see [4,11,16,17,23,19,25,30] for a selection of (mostly recent) publications.

The result of this paper reads as follows:

Theorem 1 *If $n \geq 1$ and S^n is the base-3 Sierpiński graph, then*

$$\chi_\rho(S^n) = \begin{cases} 3; n = 1\,, \\ 5; n = 2\,, \\ 7; n = 3, 4\,. \end{cases}$$

Moreover, if $n \geq 5$, then $8 \leq \chi_\rho(S^n) \leq 9$.

In the rest of the section Sierpiński graphs are introduced. Then, in Section 2, Theorem 1 is proved with the exception of the fact that $\chi_\rho(S^5) \geq 8$. This assertion is then proved in Section 3.

Set $[n] = \{1, \ldots, n\}$ and $[n]_0 = \{0, \ldots, n-1\}$. We will restrict our attention to the base-3 Sierpiński graphs S_3^n, hence let us simplify the notation from S_3^n to S^n in the rest of the paper. The graphs S^n are defined as follows. $S^0 = K_1$ (so that $E(S^0) = \emptyset$). For $n \geq 1$, the vertex set of S^n is $[3]_0^n$ and the edge set is defined recursively as

$$E(S^n) = \{\{is, it\} : \ i \in [3]_0\,, \{s,t\} \in E(S^{n-1})\} \cup \\ \{\{ij^{n-1}, ji^{n-1}\} \mid i, j \in [3]_0\,, i \neq j\}\,.$$

In other words, S^n can be constructed from 3 copies of S^{n-1} as follows. For each $j \in [3]_0$ concatenate j to the left of the vertices in a copy of S^{n-1} and denote the obtained graph with jS^{n-1}. Then for each $i \neq j$ join copies iS^{n-1} and jS^{n-1} by the single edge $e_{ij}^{(n)} = \{ij^{n-1}, ji^{n-1}\}$. In Fig. 1 the construction of S^3 is illustrated.

If $1 \leq d < n$ and $\underline{s} \in [3]_0^d$, then the subgraph of S^n induced by the vertices whose labels begin with $\underline{s}$ is isomorphic to S^{n-d}. It is denoted with $\underline{s}S^{n-d}$ in accordance with the above notation jS^{n-1}. Note that S^n contains 3^d pairwise disjoint subgraphs $\underline{s}S^{n-d}$, $\underline{s} \in [3]_0^d$. The vertices i^n, $i \in [3]_0$, of S^n are called *extreme vertices* (of S^n). The triangle in which an extreme vertex lies is an *extreme triangle*. We will also use this notation for the subgraphs $\underline{s}S^{n-d}$. For instance, consider S^4 and the vertex 12^3. Although this vertex is not an extreme vertex of S^4, it is an extreme vertex of $1S^3$.

2 Proof of Theorem 1 except that $\chi_\rho(S^5) \geq 8$

If G is a graph of order n and $S \subset [n]$, then let $\alpha_S(G)$ denote the size of a largest set $X \subseteq V(G)$, such that $X = \cup_{i \in S} X_i$, where each X_i is an i-packing. Using this notation we can redefine the packing chromatic number as $\chi_\rho(G) = \min\{k : \ \alpha_{[k]}(G) = n\}$. Suppose that a color $k \geq \mathrm{diam}(G)$ is used in a packing coloring of G. The color k is used exactly once. From this

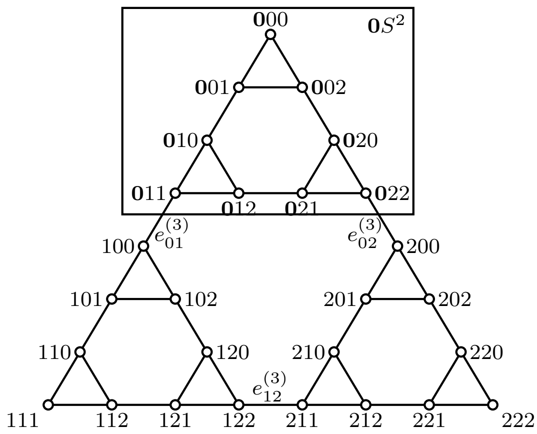

Fig. 1 The Sierpiński graph S^3

observation, the following result from [1, Lemma 3] (cf. also [29, Lemma 2.3]) readily follows.

Lemma 1 *If G is a connected graph of order n and diameter d, then $\chi_\rho(G) \leq n + (d-1) - \alpha_{[d-1]}(G)$. Moreover, if $\alpha_{[d-1]}(G) < n$, then the equality holds.*

The following lemma will be very useful in our arguments, its proof is straightforward.

Lemma 2 *Let c be a partial packing coloring of S^3, in which the extreme triangles are colored with colors different from 4. If $|c^{-1}(4)| = 3$, then $\{v : c(v) = 4\}$ is one of the two sets shown in Fig. 2 with filled dots and squares, respectively.*

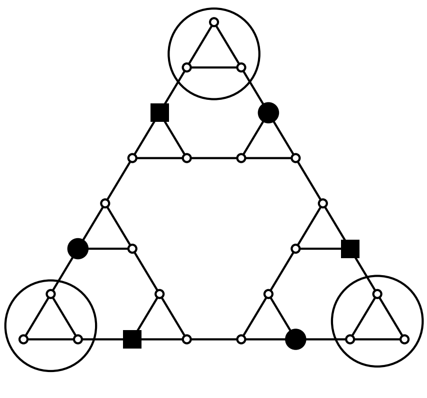

Fig. 2 Partial packing coloring with $|c^{-1}(4)| = 3$

Since $S^1 = K_3$ we clearly have $\chi_\rho(S^1) = 3$.

In any subgraph $iS^1(= K_3)$ of S^2, a packing coloring can use at most one color 1 and at most one color 2. Therefore, $\alpha_{[2]}(S^2) \leq 6 < |V(S^2)|$. Moreover, coloring the extreme vertices with color 2 it easily follows that $\alpha_{[2]}(S^2) = 6$. Hence by Lemma 1, $\chi_\rho(S^2) = 9 + (3-1) - 6 = 5$.

Consider next S^3. We are going to prove that $\chi_\rho(S^3) \geq 7$. Suppose on the contrary that $c : V(S^3) \to [6]$ is a packing coloring. Since S^3 partitions into 9 triangles, $|c^{-1}(1)| \leq 9$. If color 2 appears three times in some subgraph iS^2, then the only possibility is that the extreme vertices of iS^2 receive color 2. That is, $c(i^3) = c(ijj) = c(ikk) = 2$, where $\{i,j,k\} = [3]_0$. But then $|c^{-1}(2) \cap jS^2| \leq 2$ and $|c^{-1}(2) \cap kS^2| \leq 2$. It follows that $|c^{-1}(2)| \leq 7$. Since $\mathrm{diam}(iS^2) = 3$, $|c^{-1}(\ell)| \leq 3$ holds for $\ell = 3,4,5,6$. The maximum possible frequencies of all colors from [6] thus sum up to 28. As $|V(S^3)| = 27$ we can conclude that exactly one of $|c^{-1}(\ell)|$, $1 \leq \ell \leq 6$, does not reach the established upper bound and misses the bound by exactly 1.

Case 1. $|c^{-1}(2)| = 7$.
In this case there exists an i such that $|c^{-1}(2) \cap iS^2| = 3$. We may assume without loss of generality that $i = 0$ and hence $c(000) = c(011) = c(022) = 2$. Because $|c^{-1}(2) \cap 1S^2| = |c^{-1}(2) \cap 2S^2| = 2$, we must be in the position shown on the left side of Fig. 3, where $\times$ next to a vertex s means that $c(s) \neq 2$. Note that $|c^{-1}(6)| = 3$ is only possible if all the extreme vertices are colored with 6. Hence $|c^{-1}(6)| = 2$, and 6 is the only color that does not attain the above frequency bound. In particular, $|c^{-1}(5)| = 3$. Then color 5 must appear in each of the extreme triangles. Using the fact that $|c^{-1}(1)| = 9$, each of these triangles contains colors 1, 2, and 5. We may without loss of generality assume that $c(001) = 5$. Then the coloring looks like as shown in the middle of Fig. 3. Since $|c^{-1}(4)| = 3$, it follows from Lemma 2 that c is as shown on the right-hand side of Fig. 3. But now the uncolored vertices do not contain a diametrical pair which in turn implies that we cannot color two vertices with 6.

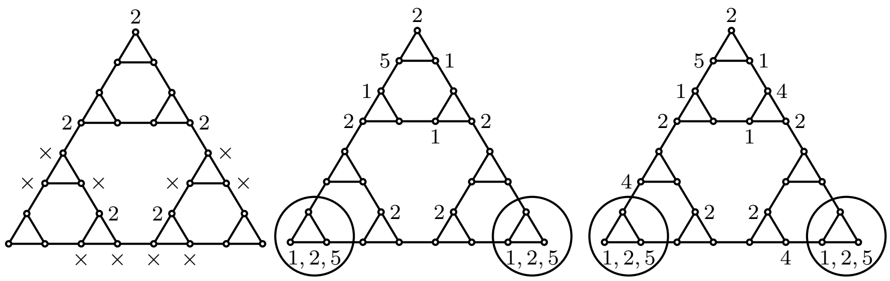

Fig. 3 Partial 6-packing coloring(s) of S^3 when $|c^{-1}(2)| = 7$

Case 2. $|c^{-1}(2)| = 6$.
Then $|c^{-1}(6)| = 3$ which is only possible when $c(000) = c(111) = c(222) = 6$. Further, because $|c^{-1}(5)| = 3$ and $|c^{-1}(1)| = 9$, each of the extremal triangles is colored with colors 1, 5, and 6. Assuming without loss of generality that $c(001) = 5$, these triangles are uniquely colored. See the partial coloring presented on the left-hand side of Fig. 4. Since $|c^{-1}(4)| = 3$, Lemma 2 implies that there are exactly two possibilities for assigning color 4. They are shown in the middle and on the right of Fig. 4. In the first case, since $|c^{-1}(2) \cap 0S^2| = 2$ must hold, we get $c(011) = 2$. But then $|c^{-1}(2) \cap 1S^2| \leq 1$, a contradiction. Similarly, in the second case we are forced to have $c(022) = 2$ and we get a contradiction because then $|c^{-1}(2) \cap 2S^2| \leq 1$ would hold.

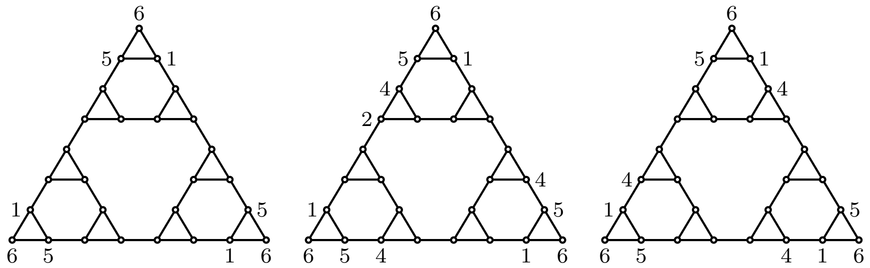

Fig. 4 Partial 6-packing coloring(s) of S^3 when $|c^{-1}(2)| = 6$

We have thus proved that $\chi_\rho(S^3) \geq 7$. Consequently, $\chi_\rho(S^n) \geq 7$ holds for any $n \geq 3$. On the other hand, the 7-packing coloring from Fig. 5 (in which the colors 8 and 9 should be ignored) demonstrates that actually $\chi_\rho(S^4) = 7$ holds (and consequently also $\chi_\rho(S^3) = 7$).

We next show that $\chi_\rho(S^n) \leq 9$ holds for any $n \geq 5$. Recall that S^n contains 3^{n-4} disjoint copies of subgraphs isomorphic to S^4, that is, the subgraphs $\underline{s}S^4$, where $\underline{s} \in [3]_0^{n-4}$. First color each of these subgraphs using the packing coloring from Fig. 5 using 9 colors. More precisely, we color each vertex for which two colors are listed with the higher color (either 8 or 9). Suppose now that the subgraphs $\underline{t}S^4$ and $\underline{u}S^4$, where $\underline{t} \neq \underline{u}$, are connected by an edge and let $x \in \underline{t}S^4$ and $y \in \underline{u}S^4$ be its endvertices. Vertices x and y are extreme vertices of $\underline{t}S^4$ and of $\underline{u}S^4$, respectively. Now modify the above coloring such that in the extreme triangle of $\underline{t}S^4$ that contains x, colors 8 and 9 are changed to 2 and 6, respectively. In this way each of the subgraphs $\underline{s}S^4$ is 9-packing colored. Moreover, since the distance between any vertices of S^n colored with 8 (or colored with 9) is at least 10 (actually, at least 14), the whole S^n is 9-packing colored as well.

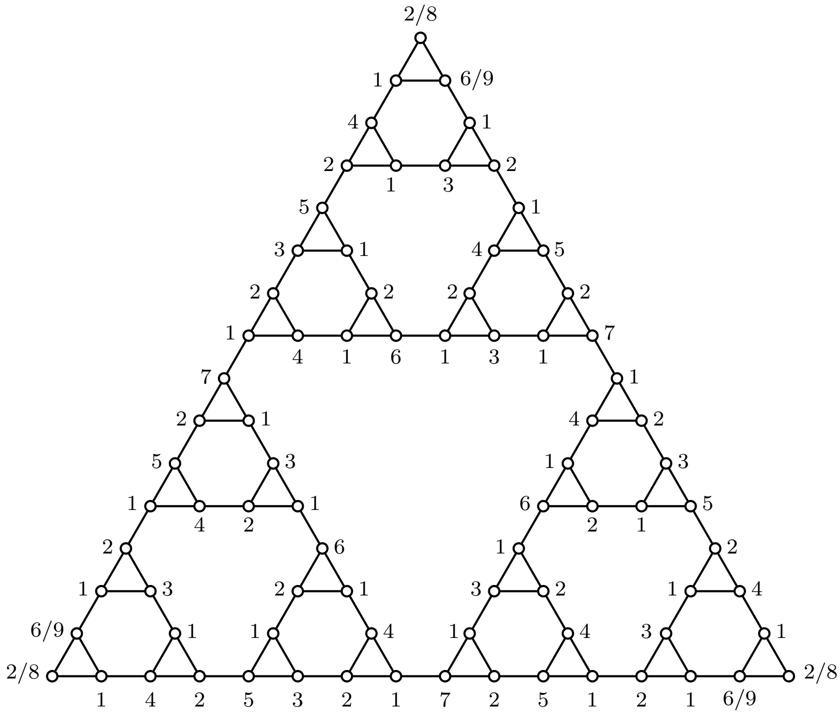

Fig. 5 Packing coloring(s) of S^4

3 Proof of $\chi_\rho(S^5) \geq 8$

Let $c : V(S^3) \to [7]$ be any 7-packing coloring of S^3. We start by recalling that the possible sizes of color classes with respect to c enjoy the upper bounds:

$$|c^{-1}(1)| \leq 9; |c^{-1}(2)| \leq 7; |c^{-1}(i)| \leq 3, 3 \leq i \leq 6; |c^{-1}(7)| \leq 1\,. \qquad (1)$$

Since the sum of the bounds in (1) is 29, and $|V(S^3)| = 27$, at most two of $|c^{-1}(\ell)|$, $1 \leq \ell \leq 6$, do not reach the established upper bound (we know already from the previous section that $\chi_\rho(S^3) = 7$, and so $|c^{-1}(7)| = 1$).

Now, for $s, t \in [6]$, we say that a packing coloring c of S^3 is of *type* $C_{s,t}$ if $|c^{-1}(s)|$ and $|c^{-1}(t)|$ do not achieve the bounds in (1). For instance, in the coloring of type $C_{5,2}$ we have

$$|c^{-1}(1)| = 9; |c^{-1}(2)| = 6; |c^{-1}(i)| = 3, i \in \{3, 4, 6\}; |c^{-1}(5)| = 2; |c^{-1}(7)| = 1.$$

Note that we allow $s = t$, which implies that $|c^{-1}(s)|$ misses the corresponding bound in (1) by 2 and all the other bounds must then be achieved.

Not all $C_{s,t}$ are possible. It is clear that whenever $|c^{-1}(6)| = 3$, the only possibility for three vertices to receive color 6 is that they are the three extreme vertices of S^3. On the other hand, if $|c^{-1}(2)| = 7$, then in one of the jS^2 three vertices are colored 2, which is only possible if these three vertices are the extreme vertices of the jS^2, one of them being the extreme vertex of S^3. Hence, if $\{2,6\} \cap \{s,t\} = \emptyset$, then a packing coloring of type $C_{s,t}$ does not exist.

Now suppose that $\{5,6\} \cap \{s,t\} = \emptyset$. This implies that the three vertices colored 5 lie in the extreme triangles of S^3 and the three vertices colored 6 are the extreme vertices of S^3. It now follows as before that $|c^{-1}(2)| \leq 6$. If $|c^{-1}(1)| = 9$ and $|c^{-1}(2)| = 6$, then the colors in each extreme triangle are $1,5,6$, and it is easy to see that $c^{-1}(2) = \{010, 020, 101, 121, 202, 212\}$. The only vertices left uncolored belong to a 12-cycle, and half of the vertices on this cycle must be colored 1. However, it is now impossible to color the remaining six vertices on this 12-cycle with either the multiset $\{3,3,4,4,4,7\}$ or the multiset $\{3,3,3,4,4,7\}$. Hence, packing colorings of type $C_{3,2}$ and $C_{4,2}$ do not exist.

If $|c^{-1}(1)| = 9$ and $|c^{-1}(2)| = 5$, then again the three extremal triangles are colored with colors $1,5,6$, and, by Lemma 2, either $c^{-1}(4) = \{010, 121, 202\}$ or $c^{-1}(4) = \{020, 212, 101\}$. In both of these cases it is straightforward to argue that at most four vertices can be colored 2, a contradiction. Hence, a type $C_{2,2}$ packing coloring does not exist.

Next, suppose that $|c^{-1}(1)| = 8$ and $|c^{-1}(2)| = 6$. All three extreme vertices are colored 6, and at least two of the three extreme triangles have colors $1,5,6$. Without loss of generality we assume the vertices in the extreme triangles $00S^1$ and $22S^1$ have colors $1,5,6$. In addition, there are two vertices colored 2 in each of $0S^2$, $1S^2$, and $2S^2$. None of $011, 100, 122, 211, 200, 022$ is colored 2, which in turn implies that $c^{-1}(2) = \{010, 020, 101, 121, 212, 202\}$. The only vertices left uncolored are twelve on a 12-cycle and another vertex v at distance 2 from the 12-cycle. Six of these are colored 1, three are colored 3, three are colored 4 and one is colored 7. Consequently, $c(v) = 4$ and half of the vertices on the 12-cycle must be colored 1. Regardless of which six vertices are colored 1, it is not possible to color two of the remaining vertices 4 and three of them 3. Therefore, a type $C_{2,1}$ packing coloring does not exist.

Finally, assume that c is of type $C_{6,3}$. This implies that the vertices in each of the three extreme triangles of S^3 are colored by colors $1, 2$, and 5. We assume without loss of generality that three vertices in $0S^2$ are colored 2 and that $c(001) = 1$ and $c(002) = 5$. By Lemma 2 either $c^{-1}(4) = \{010, 202, 121\}$ or $c^{-1}(4) = \{020, 212, 101\}$. It is not possible for $c^{-1}(4) = \{020, 212, 101\}$ since three vertices in $0S^2$ are colored 1. Hence $c^{-1}(4) = \{010, 202, 121\}$. It now follows that $c(012) = c(020) = 1$ and consequently $6 \notin c(0S^2)$. We can conclude that exactly one of 200 or 100 is colored 6 by c. Each of these possibilities would imply that a vertex in one of the extreme triangles receives color 6. This is a contradiction, and thus a 7-packing coloring of type $C_{6,3}$ does not exist.

This proves the following lemma.

Lemma 3 *If $c : V(S^3) \to [7]$ is a 7-packing coloring of S^3, then c is of one of the types $C_{6,6}, C_{6,5}, C_{6,4}, C_{6,2}, C_{6,1}, C_{5,2}$.*

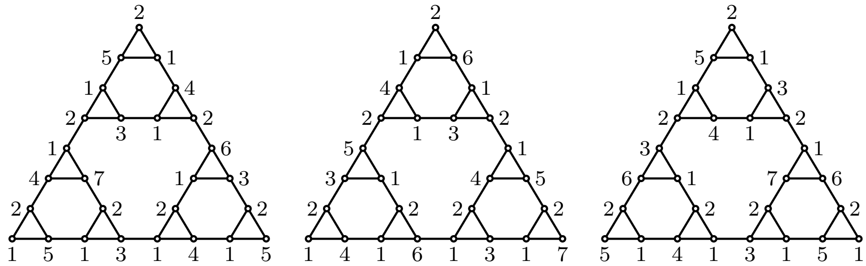

Fig. 6 Colorings of types $C_{6,6}$, $C_{6,5}$, and $C_{6,4}$

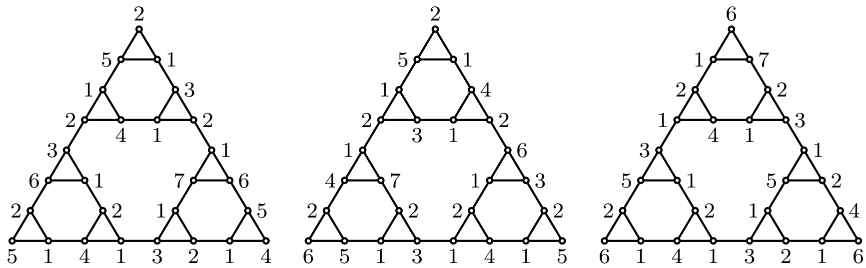

Fig. 7 Colorings of types $C_{6,2}$, $C_{6,1}$, and $C_{5,2}$

Suppose that $c : V(S^4) \to [7]$ is a 7-packing coloring of S^4. Clearly, the restriction of c to each jS^3 in S^4 is then a 7-packing coloring of jS^3, hence it must be of one of the types described in Lemma 3. We will prove that in fact the only possibility, in which the 7-packing colorings of $0S^3$, $1S^3$, and $2S^3$ are combined to form a packing coloring of S^4, is that all three restricted colorings of c to jS^3 are of type $C_{6,5}$.

In the following five claims we will use the following construction. Let $2\times S^3$ be the graph obtained from two copies of S^3 by adding an edge between two of its extreme vertices. We will denote the first copy of S^3 by S and use the standard labels for its vertices, while for the second copy of S^3 denoted by S' we will use labels $i'j'k'$, where $i, j, k \in [3]_0$. To define the edges in the second copy just ignore the prime symbols. We may without loss of generality assume that 000 is adjacent to $0'0'0'$.

We start with the following claim, which follows from the fact that in a coloring c of type $C_{s,t}$ with $5 \notin \{s, t\}$ the set $c^{-1}(5)$ consists of three vertices, each belonging to a distinct extreme triangle of S^3.

Claim 1 *Let c be a 7-coloring of $2 \times S^3$ such that c restricted to S is of type $C_{s,t}$ and c restricted to S' is of type $C_{s',t'}$. If $5 \notin \{s, t, s', t'\}$, then c is not a 7-packing coloring of $2\times S^3$. In particular, this holds when $\{s, t\} = \{s', t'\}$ and $\{s, t\}$ is one of the pairs $\{6, 6\}$, $\{6, 4\}$, $\{6, 2\}$, $\{6, 1\}$.*

By a similar argument we prove the following assertion.

Claim 2 *If c is a 7-coloring of $2 \times S^3$ such that c restricted to S and to S' is of type $C_{5,2}$, then c is not a 7-packing coloring of $2 \times S^3$.*

Claim 2 follows from the fact that $6 \notin \{s,t\}$ implies that in a packing coloring c of type $C_{s,t}$ we have $c(j^3) = 6$ for all $j \in [3]_0$.

Claim 3 *If c is a 7-coloring of $2 \times S^3$ such that c restricted to S is of type $C_{6,6}$ or $C_{6,1}$ and c restricted to S' is of type $C_{6,5}$, then c is not a 7-packing coloring of $2 \times S^3$.*

Proof Since c is a 7-coloring of $2 \times S^3$, which restricted to S is of type $C_{6,6}$ or $C_{6,1}$, it implies that each of the extreme triangles of S contains a vertex colored 5. In particular, this holds for $00S^1$, which implies that in $0'S'^2$ no vertex may be colored 5 if we want c to be a packing coloring (which we assume for the purposes of obtaining a contradiction). In addition, the extreme vertex $0'0'0'$ must not be colored 2, because otherwise it is not possible to use color 2 seven times on S. It follows that in $0'S'^2$ only two vertices are colored 2. Moreover, as c restricted to S' is of type $C_{6,5}$, we infer that $c(0'S'^2) = \{1,1,1,2,2,3,4,6,7\}$. Hence no vertex in $0S^2$ may be colored 7. But then, as at most two vertices can be colored 2 in $0S^2$ (the extreme vertex 000 must not be colored 2 by the same reason we used for $0'0'0'$), we get $c(0S^2) = \{1,1,1,2,2,3,4,5,6\}$. Clearly, the vertices colored 6 in $0S^2$ and $0S'^2$ must be (at least) 7 apart, and so we may assume without loss of generality that $c(0'1'1') = 6$. This readily implies that the extreme triangle $2'2'S'^1$ contains colors $1,2$, and 6, which, in addition, implies the colors that must be used in $2'S'^2$. (Note that the positions of the color 2 in S' are as depicted in the left graph of Fig. 3 with $c(1'1'1') = 2$.) In particular, $c(2'0'1') = 2$, hence the only place for the color 5 in $2'S'^2$ is in one of the vertices $2'0'0', 2'0'2'$, which also implies $c(2'1'S'^1) = \{1,3,4\}$. Now, not both vertices $1'2'0'$ and $1'2'1'$ can receive color 1, but from the location of the colors in $2'S'^2$ they also cannot receive colors $3,4,5$, and 6. Since $c(1'2'2') = 2$ and since $0'S'^2$ contains color 7, we also derive that $1'2'0'$ and $1'2'1'$ cannot receive these two colors, which is the final contradiction. Thus c is not a packing coloring. □

Claim 4 *If c is a 7-coloring of $2 \times S^3$ such that c restricted to S is of type $C_{6,5}$ and c restricted to S' is of type $C_{6,4}$, then c is not a 7-packing coloring of $2 \times S^3$.*

Proof Let c be as in the statement of the claim, and assume that it is a packing coloring. Clearly, in each of $0S^2$ and $0'S'^2$ only two vertices are colored 2, and since the extreme triangle $0'0'S^1$ contains a vertex colored 5, we infer that $0S^2$ does not contain a vertex colored 5. Hence $c(0S^2) = \{1,1,1,2,2,3,4,6,7\}$. Thus 7 does not appear in $0'S'^2$, which implies that $c(0'S'^2) = \{1,1,1,2,2,3,4,5,6\}$, and the vertex colored 6 must be at distance 3 from $0'0'0'$. Without loss of generality we may assume that $c(0'1'1') = 6$. But then the extreme triangle $2'2'S'^1$ must also contain color 6, and we know from the structure of the coloring of type $C_{6,4}$ that $c(2'2'S'^1) = \{1,2,5\}$, which is clearly a contradiction. □

Claim 5 *If c is a 7-coloring of $2 \times S^3$ such that c restricted to S is of type $C_{6,4}$ or $C_{6,1}$ and c restricted to S' is of type $C_{5,2}$, then c is not a 7-packing coloring of $2 \times S^3$.*

Proof Assume that c is a packing coloring of $2 \times S^3$. Since c restricted to S' is of type $C_{5,2}$, we infer that $c(0'0'0') = 6$. In addition, there are (at most) two vertices colored 2 in $0'S'^2$, and there is no vertex colored 5 in $0'S'^2$ (because the extreme triangles of S contain 5), hence $c(0'S'^2) = \{1, 1, 1, 2, 2, 3, 4, 6, 7\}$. This implies that 7 does not appear in $0S^2$, and so $c(0S^2) = \{1, 1, 1, 2, 2, 2, 3, 4, 5\}$. In addition, two vertices with color 6 must appear in S at distance at least 6 from the vertex 000. This can only happen if the vertices colored by 6 are in the extreme triangles $11S^1$ and $22S^1$. But then $c(11S^1) = c(22S^1) = \{2, 5, 6\}$, which is a contradiction because color 1 must also be present in these triangles. We conclude that c is not a packing coloring. □

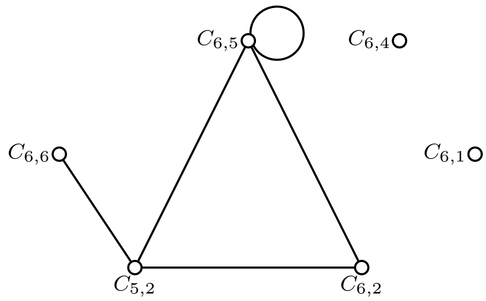

Fig. 8 The relation graph of the remaining compatible types of 7-packing colorings of S^3

Claims 1–5 largely restrict the possibilities of two copies of S^3 with a prescribed 7-packing coloring to be combined with a 7-packing coloring of the (colored) graph, obtained by adding an edge between two extreme vertices, one from each copy. In Fig. 8 the edges represent the relation between two types of colorings of S^3, which have not been excluded in the preceding claims. The loop at $C_{6,5}$ indicates that two copies of an S^3, each possessing a 7-packing coloring of type $C_{6,5}$, can be combined to a 7-packing coloring of $2 \times S^3$. In fact, as we know, even three copies of S^3, each of type $C_{6,5}$, can create 7-packing coloring of S^4. Note that altogether there are four possibilities for three copies of S^3 to be combined to form a 7-packing coloring of S^4, and they are represented by the following types:

- $C_{6,5}, C_{6,5}$, and $C_{6,5}$,
- $C_{6,5}, C_{6,2}$, and $C_{5,2}$,
- $C_{6,5}, C_{6,5}$, and $C_{5,2}$,
- $C_{6,5}, C_{6,5}$, and $C_{6,2}$.

We will now show that the triples of graphs S^3 possessing 7-packing colorings of the latter three types described above do not yield a 7-packing coloring of S^4 (i.e. the only way to obtain a 7-packing coloring of S^4 is to combine three copies of 7-packing colorings of S^3 all of type $C_{6,5}$). This will suffice to complete the proof of the theorem.

Claim 6 *If $c : V(S^4) \to [7]$ is a 7-coloring of S^4 such that the restriction of c to $0S^3$ is of type $C_{5,2}$, the restriction of c to $1S^3$ is of type $C_{6,5}$, and the restriction of c to $2S^3$ is of type $C_{6,2}$, then c is not a 7-packing coloring of S^4.*

Proof Suppose on the contrary that c is a 7-packing coloring. Since the coloring c restricted to $0S^3$ is of type $C_{5,2}$, we have $c(0j^3) = 6$ for all $j \in [3]_0$. Because in $1S^3$ two vertices receive color 6, the only places to put these two colors into $1S^3$ are the triangles 1^3S^1 and $122S^1$, respectively. From the same reason we find that each of the triangles 2^3S^1 and $211S^1$ contains a vertex colored 6. Since the vertices from $122S^1$ and $211S^1$ are pairwise at distance at most 3, the coloring c is not a 7-packing coloring. □

Using arguments parallel to those from the proof of Claim 6 we get:

Claim 7 *If $c : V(S^4) \to [7]$ is a 7-coloring of S^4 such that the restriction of c to $0S^3$ is of type $C_{5,2}$ and the restriction of c to each of $1S^3$ and $2S^3$ is of type $C_{6,5}$, then c is not a 7-packing coloring of S^4.*

For the last claim we need two lemmas.

Lemma 4 *Let $c : V(S^3) \to [7]$ be a 7-packing coloring of S^3 of type $C_{6,2}$. If $c(211) = 6$, then $2 \in c(22S^1)$; and by symmetry, if $c(122) = 6$, then $2 \in c(11S^1)$.*

Proof Suppose that the conditions of the statement hold, and for the purposes of getting a contradiction, assume that in the extreme triangle $22S^1$ there is no vertex colored by 2. Since $c(211) = 6$, and c is of type $C_{6,2}$, the colors that appear in the extreme triangle $00S^1$ are $1, 5$, and 6. It is clear that there is no place to color three vertices by 2 in any of the subgraphs jS^2, $j \in [3]_0$ (if this happened in $1S^2$, then only one 2 could be placed in $0S^2$ and only one 2 in $2S^2$). Hence in each of the subgraphs jS^2 there are exactly two vertices colored by 2. In particular this implies that $c(212) = c(202) = 2$, and this further implies that $c(210) = c(200) = 1$. We then also infer that $c(020) = 2$ and $c(021) = 1$. Now, the only possible color that can be assigned to 022 is 3, which then implies that the color of 201 cannot be 3, and so $c(201) = 4$. This in turn implies that $c(010) = 4$, and the positions of both vertices colored 4 further imply that $c(11S^1) = \{1, 4, 5\}$. Since $c(011) = 2$, we have $c(012) = 1$, but 012 is adjacent to 021, which is also colored 1, the final contradiction. □

Lemma 5 *Let $c : V(S^3) \to [7]$ be a 7-packing coloring of S^3 of type $C_{6,5}$. If no vertex of the subgraph $0S^2$ is colored 5 or 6, then $c(100) = 6$ and $c(112) = 5$, or $c(200) = 6$ and $c(211) = 5$.*

Proof As no vertex is colored 5 or 6 in $0S^2$, we get $c(0S^2) = \{1,1,1,2,2,2,3,4,7\}$. Let us assume that a vertex colored 4 in $0S^2$ is closer to the vertex 1^3 than to 2^3. Since $c(000) = 2$, this implies that $4 \notin c(10S^1)$.

Now, suppose that there is a vertex colored 4 in the extreme triangle $11S^1$. Since vertices colored by 5 and by 6 cannot lie in the triangle $12S^1$ (because there must be vertices colored by 5 and 6 also in $2S^2$), we infer that the triangle $10S^1$ contains both colors 5 and 6. From this we derive that either $c(121) = 3$ or $c(122) = 3$ (note that $c(120) = 2$ is fixed because seven vertices are colored 2). In either case this implies that the triangle $21S^1$ must not have a vertex colored 3, but also not 5 nor 6. Since beside colors 1 and 2 we must have another color in this triangle, we readily infer that $c(212) = 4$. This in turn implies $c(111) = 4$, $c(211) = 1$, and in turn we find that $c(112) = 1$. But now each of the remaining possibilities, namely $c(121) = 1$ and $c(122) = 1$ leads to a contradiction. This implies that $4 \notin c(11S^1)$.

The only remaining possibility is that $c(121) = 4$. This implies $c(122) = 1$, $c(212) = 1$, and $c(\{220, 222\}) = \{1, 2\}$. We then also find that one of the vertices 200 or 202 must be colored 4, but because of either 001 or 010 colored 4, we conclude that $c(202) = 4$. Now, it is clear that the only place for color 6 is in the vertex 200, which only leaves space in the vertex 221 for color 5.

Note that the choice of a vertex colored 4 in $0S^2$ being closer to the vertex 222 than to 111 yields $c(100) = 6$ and $c(112) = 5$. □

Claim 8 *If $c : V(S^4) \to [7]$ is a 7-coloring of S^4 such that the restriction of c to $0S^3$ is of type $C_{6,2}$ and the restriction of c to $1S^3$ and to $2S^3$ is of type $C_{6,5}$, then c is not a 7-packing coloring of S^4.*

Proof Suppose on the contrary that c is a 7-packing coloring. Since the coloring c restricted to $0S^3$ is of type $C_{6,2}$, each of the respective extreme triangles of the subgraph $0S^3$ contains a vertex colored 5, which implies that the subgraphs $10S^2$ and $20S^2$ do not have a vertex colored 5. Since there are two vertices colored 6 in the subgraph $0S^3$, at least one of the subgraphs $01S^2, 02S^2$ must contain a vertex colored 6. We assume without loss of generality that there is a vertex colored 6 in $01S^2$, and distinguish three cases with respect to the position of the vertex in $01S^2$, which is colored 6.

Case 1. $c(0100) = 6$.

In this case the extreme triangle $022S^1$ of $0S^3$ contains color 6. This in turn implies that colors 5 and 6 are not allowed in $20S^2$ and so the restriction of c to $2S^3$, which is of type $C_{6,5}$, fulfills the conditions of Lemma 5. Hence either $c(2200) = 6$ or $c(2100) = 6$. This is in contradiction with $c(022j) = 6$, where $j \in \{0, 2\}$, since the distance between $2i00$ and $022j$ is at most 6.

Case 2. $c(0122) = 6$.

Since the restriction of c to $0S^3$ is of type $C_{6,2}$, we can use Lemma 4, which (by symmetry) implies that the triangle $011S^1$ contains a vertex colored 2 (thus the triangle contains colors $1, 2$, and 5). Then the subgraph $10S^2$ contains only two vertices colored 2, and we know from above that it contains no vertex colored 5. Thus $10S^2$ has to contain a vertex colored 6, and there remain only two vertices, which can be colored 6 in $10S^2$, that is 1011 and 1022. Indeed, if 1012 or 1021 were colored 6, then only one vertex from $1S^3$ would receive color 6.

Case 2.1. $c(1011) = 6$. This implies that $c(122S^1) = \{1, 2, 6\}$. For color 2 this follows from the fact that either $11S^2$ or $12S^2$ contains three vertices colored 2. Hence the subgraph $21S^2$ contains no vertex colored 6, which implies that a vertex colored 6 must lie in $20S^2$ and a vertex colored 6 must lie in $22S^2$. Moreover, since $c(0122) = 6$, we infer that it must be $c(2010) = 6$. Next, since there is no vertex with color 6 in the subgraph $02S^2$ and there are at most two vertices in this subgraph colored 2, there must be a vertex with color 7 in $02S^2$. Thus $20S^2$ has no vertex colored 7 (and no vertex colored 5), which readily implies that $c(20S^2) = \{1, 1, 1, 2, 2, 2, 3, 4, 6\}$, and the position of vertices colored by 2 in $20S^2$ is fixed. Moreover, we derive that $c(2012) = 1$, and so $c(2020) = c(2001) = 1$. From the remaining positions for the vertex with color 4 in $20S^2$ (notably, 2002 and 2021) and having in mind that three vertices from $2S^3$ receive color 4, we derive that a vertex colored 4 in $22S^2$ can only be one of $2220, 2212, 2221, 2222$. In addition, since $c(2010) = 6$, we know that color 6 must be in one of the vertices $2212, 2221, 2222$. Since color 2 is either in 2220 or 2222 and color 1 must be present in $222S^1$, we conclude that $c(\{2220, 2222, 2221, 2212\}) = \{1, 2, 4, 6\}$. Hence $c(2212) \neq 1$. Since $c(2210) = c(2120) = 2$, it follows that $c(2211) = 1$ and in turn also $c(2121) = c(2111) = 1$. But then $c(1222) \notin \{1, 2, 6\}$, which is a contradiction.

Case 2.2. $c(1022) = 6$. Since there is no vertex colored 5 and there are only two vertices colored 2 in the subgraph $10S^2$, there must be a vertex colored 7 in $10S^2$. On the other hand, position of color 6 in 1022 implies that there is no vertex colored 6 in the subgraph $12S^2$ (and again there are only two vertices colored 2 in $12S^2$, because of the edge between 1222 and 2111). This implies that there must be a vertex colored 7 in the subgraph $12S^2$, which is in contradiction with color 7 in the subgraph $10S^2$.

Case 3. The vertex colored 6 in $01S^2$ is at most 2 apart from the vertex 0111.

This implies that no vertex is colored 6 in $10S^2$. Since there is also no vertex colored 5 in $10S^2$, we derive that there must be three vertices colored 2 in $10S^2$, in particular $c(1000) = 2$. Hence $2 \notin c(011S^1)$, which also implies that each of the subgraphs $00S^2, 01S^2$, and $02S^2$ must have exactly two vertices colored 2.

Moreover, c restricted to $1S^3$ is of type $C_{6,5}$, and by Lemma 5 either $c(1100) = 6$ or $c(1200) = 6$. Suppose that $c(1200) = 6$. Then by Lemma 5 we have $c(1221) = 5$, and so there is no vertex colored 5 in $21S^2$, which is a contradiction, because there is no vertex colored 5 also in $20S^2$. Hence $c(1100) = 6$, which implies that either $c(0101) = 6$ or $c(0121) = 6$. We deal with each of these two possibilities separately.

Case 3.1. $c(0121) = 6$. Note that the only remaining possibility to put two colors 2 in $01S^2$ is if $c(0122) = 2$. Since $c(1100) = 6$, a vertex in the triangle $122S^1$ is colored 6, which implies that there must be a vertex colored 6 in $20S^2$ and $22S^2$. Thus $c(22S^2) = \{1, 1, 1, 2, 2, 3, 4, 5, 6\}$. It is also clear that

$c(2111) \neq 2$, which implies $c(2000) = 2$. But then there is a place for only one vertex to be colored 2 in the subgraph $02S^2$, a contradiction.

Case 3.2. $c(0101) = 6$. This implies that a vertex colored 6 lies in $02S^2$, and it is at most 2 apart from 0222. Hence $6 \notin c(20S^2)$. As there is also no color 5 in the subgraph $20S^2$, we infer by Lemma 5 (because $2S^3$ is of type $C_{6,5}$) that either $c(2100) = 6$ or $c(2200) = 6$. Now, $c(2100) = 6$ is not possible since there is a vertex x colored 6 in the triangle $122S^1$. But $c(2200) = 6$ implies that there is a vertex y colored 6 in the triangle $211S^1$. Since y is at most 3 apart from x this is the final contradiction. □

We can now finish the argument that $\chi_\rho(S^5) \geq 8$. By Claims 6, 7, and 8, the only possibility for a 7-coloring $c : V(S^4) \to [7]$ to be a 7-packing coloring of S^4 is that the restrictions of c to iS^3 for all $i \in [3]_0$ are 7-packing colorings of type $C_{6,5}$. This readily implies that in any 7-packing coloring c of S^4, we have $c(i^4) = 2$ for all $i \in [3]_0$.

Now, suppose that there exists a 7-packing coloring of S^5. Then, for each $j \in [3]_0$ the coloring c restricted to jS^4 is a packing coloring of the corresponding subgraph, isomorphic to S^4. By the above, $c(ji^4) = 2$ for all $i, j \in [3]_0$, and we derive a contradiction with c being a packing coloring. Hence $\chi_\rho(S^5) > 7$.

4 Concluding remarks

In this paper we determined the exact values of the packing chromatic number of the graphs S_3^n, $n \leq 4$, and established that $\chi_\rho(S_3^n) \in \{8, 9\}$ when $n \geq 5$. It would certainly be of interest to decide, which of $\chi_\rho(S^n)$ equal 8 and which equal 9, for $n \geq 5$.

There are other possible directions to continue the investigation of this paper. Firstly, it would be worth exploring the packing chromatic number of Sierpiński graphs S_p^n, for base p greater than 3. A generalization in a different direction is that of studying the S-packing coloring number of Sierpiński graphs. The concept of S-packing coloring is a generalization of packing coloring. It was introduced in [14] and further investigated in [8, 12, 15].

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