On the signed Roman k-domination: complexity and thin torus graphs

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Abstract

A signed Roman k-dominating function on a graph $G = (V(G), E(G))$ is a function $f : V(G) \to \{-1, 1, 2\}$ such that (i) every vertex u with $f(u) = -1$ is adjacent to at least one vertex v with $f(v) = 2$ and (ii) $\sum_{x \in N[w]} f(x) \geq k$ holds for any vertex w. The weight of f is $\sum_{u \in V(G)} f(u)$, the minimum weight of a signed Roman k-dominating function is the signed Roman k-domination number $\gamma_{sR}^k(G)$ of G. It is proved that determining the signed Roman k-domination number of a graph is NP-complete for $k \in \{1, 2\}$. Using a discharging method, the values $\gamma_{sR}^2(C_3 \,\square\, C_n)$ and $\gamma_{sR}^2(C_4 \,\square\, C_n)$ are determined for all n.

Key words: Roman domination; signed Roman 2-domination; computational complexity; torus graphs; discharging

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1 Introduction

A motivation for the recently introduced signed Roman k-domination is that it combines properties of the Roman domination [2, 6, 16] and the signed domination [8, 19].

A *Roman dominating function* (RDF) on a graph G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight $\omega(f)$ of an RDF f is the value $\omega(f) = \sum_{u \in V(G)} f(u)$. The minimum weight of a Roman dominating function on a graph G is called the *Roman domination number* of G.

The recent concept of signed Roman k-domination is defined as follows. Let $G = (V(G), E(G))$ be a graph and k a positive integer. Then a function $f : V(G) \to \{-1, 1, 2\}$ is a *signed Roman k-dominating function* (SRkDF) if (i) every vertex u for which $f(u) = -1$ is adjacent to at least one vertex v for which $f(v) = 2$ and (ii) $\sum_{x \in N[w]} f(x) \geq k$ holds for any vertex w of G, where $N[w] = \{x : wx \in E(G)\} \cup \{w\}$ denotes the *closed neighborhood* of w. The *weight* of f is the value $\sum_{u \in V(G)} f(u)$, and the minimum weight of a signed Roman k-dominating function is the *signed Roman k-domination number* $\gamma^k_{sR}(G)$ of G. Let f be a SR1DF and $S \subseteq V(G)$, we denote $f(S) = \sum\limits_{v \in S} f(v)$.

For an RDF or SRkDF f of G, let $V_i = \{x : f(x) = i\}$. Then for an RDF f of G, (V_0, V_1, V_2) is the ordered partition of $V(G)$ induced by f such that $V_i = \{x : f(x) = i\}$ for $i = 0, 1, 2$; and for an SR1DF f of G, (V_{-1}, V_1, V_2) is the ordered partition of $V(G)$ induced by f such that $V_i = \{x : f(x) = i\}$ for $i = -1, 1, 2$.

The signed Roman k-domination was introduced by Henning and Volkmann in [12], generalizing the case $k = 1$ studied earlier in [1]. The paper [12] gives different bounds and exact results on $\gamma^k_{sR}(G)$. Among other results, $\gamma^2_{sR}(C_n)$ and $\gamma^k_{sR}(K_{p,p})$ $(p \geq k - 1)$ are determined. In the subsequent paper [11] (interestingly, published a year earlier!) the same authors improved a lower bound from [12] on γ^k_{sR} for trees and characterized the trees achieving equality. Volkmann [17] further extended the signed Roman k-domination to digraphs, again generalizing the case $k = 1$ that was first studied in [15]. Very recently, signed *total* Roman domination in digraphs and signed Roman *edge* k-domination were introduced and investigated in [18] and [3], respectively.

Clearly, the signed Roman k-domination number is defined only for graphs G with $\delta(G) \geq k/2 - 1$, where $\delta(G)$ is the minimum degree of G. However, as pointed out in [12], it is reasonable to assume that $\delta(G) \geq k - 1$. Since in this paper we restrict our attention to the cases $k = 1$ and $k = 2$, this assumption requires only that graphs considered have no isolated vertices.

The signed Roman k-domination problem is the following:

Signed Roman k-Domination Problem	
Input:	A graph G, and an integer ℓ.
Question:	Is there an SRkDF of G with weight at most ℓ?

Our first main result to be proved in Section 2 is:

Theorem 1.1 Signed Roman 1-Domination Problem *is NP-complete even when restricted to bipartite and planar graphs.* Signed Roman 2-Domination Problem *is NP-complete even when restricted to planar graphs.*

Recall that the *Cartesian product* $G \,\square\, H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H)$, where $(g, h)(g', h') \in E(G \,\square\, H)$ if either $gg' \in E(G)$ and $h = h'$, or $hh' \in E(H)$ and $g = g'$. The Cartesian product operation is commutative and associative, we refer to the book [10] for additional properties of this graph operation. Cartesian products of two cycles are know as *torus graphs* because of their natural embeddings into the torus. The following theorems for thin (meaning that one factor is short) torus graphs will be proved using a discharging method. While discharging is widely applied in graph coloring, cf. [5], as far as we know it has not been used earlier in domination theory.

Theorem 1.2 *If $n \geq 3$, then*

$$\gamma_{sR}^2(C_3 \,\square\, C_n) = \begin{cases} \frac{3n}{2}; & n \equiv 0 \pmod 4, \\ \lceil \frac{3n}{2} \rceil + 1; & n \equiv 1, 2, 3 \pmod 4. \end{cases}$$

Theorem 1.3 *If $n \geq 4$, then*

$$\gamma_{sR}^2(C_4 \,\square\, C_n) = \begin{cases} 10; & n = 4, \\ 11; & n = 5, \\ 2n; & n \geq 6. \end{cases}$$

Theorems 1.2 and 1.3 will be proved in Section 3.

Throughout the paper we will use the notation $[n] = \{1, \ldots, n\}$.

2 Proof of Theorem 1.1

Note first that the Signed Roman k-Domination Problem is clearly in NP.

In the rest of the section we are going to give a reduction of the NP-complete Roman domination problem, to Signed Roman 1-Domination Problem and to Signed Roman 2-Domination Problem, where the former problem is defined as follows.

Roman Domination Problem	
Input:	A graph G, and an integer ℓ.
Question:	Is there an RDF of G with weight at most ℓ?

The NP-completeness of the Roman Domination Problem is mentioned in [7]; it remains NP-complete even when restricted to split graphs, bipartite graphs, and planar graphs. All these results follow from a more general result [4, Theorem 1]. In the same paper a review of the NP-hardness of the Roman Domination Problem is also made [4, Section 4.1] and NP-harness proved when restricted to line graphs. On the other hand, the Roman domination number can be computed in linear time for several important classes of graphs including interval graphs, cographs [13], and strongly chordal graphs [14].

The reductions are presented in Subsections 2.1 and 2.2, respectively.

2.1 The case $k = 1$

In this subsection we reduce the Roman Domination Problem on planar and bipartite graphs to Signed Roman 1-Domination Problem.

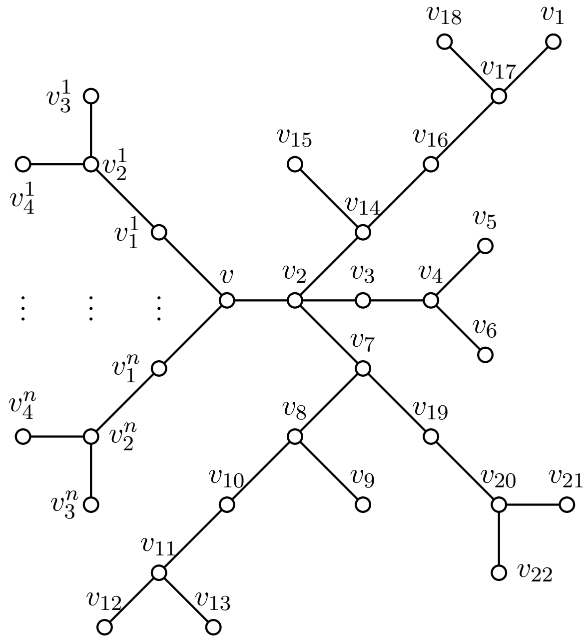

Figure 1: The tree T_v

Let G be a graph on n vertices. Then for each vertex $v \in V(G)$ let T_v be the tree as depicted in Fig. 1. Then we have $V(T_v) = S_v^1 \cup S_v^2 \cup \{v\}$, where $S_v^1 = \{v_1, \ldots, v_{22}\}$ and $S_v^2 = \bigcup_{i=1}^{n} \{v_1^i, v_2^i, v_3^i, v_4^i\}$.

Let now G' be the graph obtained from the disjoint union of the trees T_v, $v \in V(G)$, where an edge is added between $v' \in T_{v'}$ and $v'' \in T_{v''}$ if and only if $v'v'' \in E(G)$. Note that $|V(G')| = 4n^2 + 23n$ and $|E(G')| = |E(G)| + 4n^2 + 22n$. Note that if G is planar and bipartite, then G' is also such. We will show that G has an RDF f with $w(f) \leq \ell$ if and only if G' has an SR1DF g with $w(g) \leq \ell + n(n+4)$.

Suppose that f is an RDF of G with $w(f) \leq \ell$. Then define $g : V(G') \to \{-1, 1, 2\}$ as follows. For any $v \in V(G)$ set $g(v_j) = -1$ for $j \in \{1, 2, 5, 6, 7, 9, 12, 13, 15, 18, 21, 22\}$, set $g(v_j) = 1$ for $j \in \{10, 16, 19\}$, and $g(v_j) = 2$ for $j \in \{17, 4, 11, 8, 14, 20\}$. In addition, let $g(v_3^i) = g(v_4^i) = -1$, $g(v_1^i) = 1$, and $g(v_2^i) = 2$ for each $i \in [n]$. This pre-labeling of g is shown in Fig. 2.

For a fixed vertex v, the pre-labeling of g on T_v adds up to $n + 3$. For the remaining two vertices, v and v_3, we define g as follows.

- If $f(v) \in \{1, 2\}$, then we put $g(v) = f(v)$ and $g(v_3) = 1$; and
- if $f(v) = 0$, then we put $g(v) = -1$ and $g(v_3) = 2$.

Note that we set $g(v) = -1$ only in the case when $f(v) = 0$. Since f is an RDF of G, there exists a vertex v' such that $vv' \in E(G)$ and $f(v') = 2$. But then $vv' \in E(G')$ and $g(v') = 2$. We conclude that g is an SR1DF of G'.

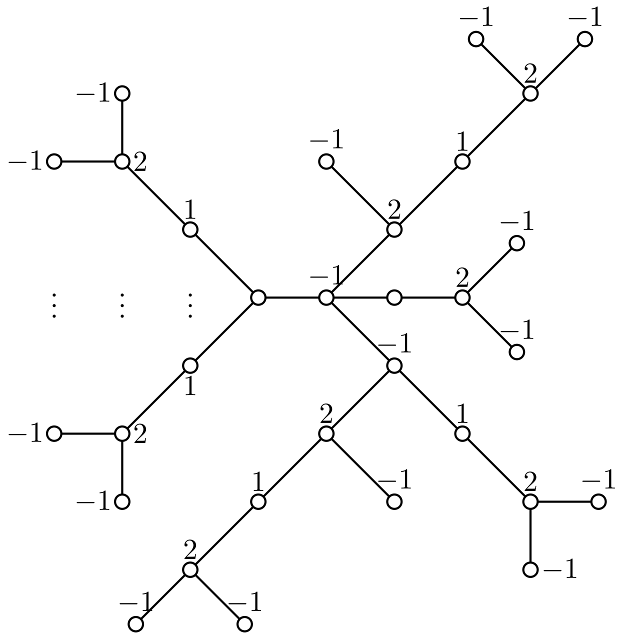

Figure 2: Pre-labeling of g

Let $w(f) = \ell' \leq \ell$. If $v \in V(G)$, then by the construction of g we have $g(v) + g(v_3) = 1 + f(v)$. If follows that $w(g) = n(\ell+3) + n + w(f) = \ell' + n(n+4)$. We have thus proved that if G has an RDF with weight at most ℓ, then G' has an SR1DF with weight at most $\ell + n(n+4)$.

To prove the converse we first show:

Lemma 2.1 *Let g be an SR1DF of G'. Then there exists an SR1DF g' of G' with $w(g') \leq w(g)$ such that for any $v \in V(G)$ the following hold.*

(i) $g'(v_3^i) = g'(v_4^i) = -1$, $g'(v_1^i) \geq 1$, *and* $g'(v_2^i) = 2$ *for* $i \in [n]$;

(ii) $g'(v_i) = -1$ *for* $i \in \{1, 5, 6, 12, 13, 18, 21, 22\}$, $g'(v_i) = 2$ *for* $i \in \{4, 11, 17, 20\}$, *and* $g'(v_i) \geq 1$ *for* $i \in \{3, 16, 19, 10\}$;

(iii) $g'(v_{15}) = g'(v_9) = -1$, $g'(v_{14}) = g'(v_8) = 2$, *and* $g'(v_{10}) = g'(v_{16}) = g'(v_{19}) = 1$.

(iv) $g'(v_7) = -1$.

Proof. (i) We may without loss of generality assume that $g(v_3^i) \leq g(v_4^i)$.

If $g(v_3^i) = 1$, then we thus have $g(v_4^i) \geq 1$. Moreover, $g(v_2^i) \geq 1$, for otherwise we would have $g(N[v_3^i]) \leq 0$. Set $g'(v_2^i) = 2$, $g'(v_3^i) = g'(v_4^i) = -1$, and $g'(x) = g(x)$ for $x \in V(G') \setminus \{v_2^i, v_3^i, v_4^i\}$. Finally if $g(v_1^i) = -1$, then put $g'(v_1^i) = 1$. Then g' is an SR1DF of G' and $w(g') \leq w(g)$, as desired.

If $g(v_3^i) = 2$, we have $g(v_4^i) = 2$. Now let $g'(v_2^i) = 2$, $g'(v_3^i) = g'(v_4^i) = -1$ and $g'(x) = g(x)$ for $x \in V(G') \setminus \{v_2^i, v_3^i, v_4^i\}$. Now if $g(v_1^i) = -1$, put $g'(v_1^i) = 1$. Then we have g' is also an SR1DF of G' and $w(g') \leq w(g)$.

If $g(v_3^i) = -1$, we have $g(v_2^i) = 2$. If $g(v_4^i) = -1$, then $g(v_1^i) \geq 1$, as desired. If $g(v_4^i) \geq 1$, then put $g'(v_4^i) = -1$ and $g'(x) = g(x)$ for $x \in V(G') \setminus \{v_2^i, v_3^i, v_4^i\}$. Now if $g(v_1^i) = -1$, put $g'(v_1^i) = 1$. Again g' is an SR1DF of G' with $w(g') \leq w(g)$.

(ii) The proof goes along the same lines as the proof of (i).

(iii) By the symmetry it suffices to show that we can extend the above g' such that $g'(v_{15}) = -1$, $g'(v_{14}) = 2$ and $g'(v_{16}) = 1$. If $g(v_{15}) = 1$, then $g(v_{14}) \geq 1$, for otherwise $g(N[v_{15}]) \leq 0$. Now let $g'(v_{15}) = -1$, $g'(v_{14}) = 2$ and $g'(x) = g(x)$ for $x \in V(G') \setminus \{v_{14}, v_{15}\}$. Then we have g' is also an SR1DF of G' and $w(g') \leq w(g)$, as desired. If $g(v_{15}) = 2$, let $g'(v_{15}) = -1$, $g'(v_{14}) = 2$ and $g'(x) = g(x)$ for $x \in V(G') \setminus \{v_{14}, v_{15}\}$. Then g' is also an SR1DF of G' with $w(g') \leq w(g)$, as desired. If $g(v_{16}) = 2$, then put $g'(v_{16}) = 1$ and $g'(x) = g(x)$ for $x \in V(G') \setminus \{v_{16}\}$. Then g' is also an SR1DF of G' with $w(g') \leq w(g)$.

(iv) Suppose $g(v_7) \geq 1$. Since $g(v_3) \geq 1$, then if $g(v_2) = -1$, we can put $g'(v_7) = -1$, $g'(v_3) = 2$ and $g'(x) = g(x)$ for $x \in V(G') \setminus \{v_3, v_7\}$. So g' is an SR1DF of G' and $w(g') \leq w(g)$. If $g(v_2) \geq 1$, we can put $g'(v_7) = -1$ and $g'(x) = g(x)$ for $x \in V(G') \setminus \{v_7\}$. So g' is an SR1DF of G' and $w(g') \leq w(g)$. □

Suppose now that G' admits an SR1DF of G' with weight at most $\ell + n(n+4)$. Then Lemma 2.1 implies that G' admits also an SR1DF g that fulfils all the assertions of Lemma 2.1 and for which $w(g) \leq \ell + n(n+4)$ holds. Recalling that $S_v^1 = \{v_1, \ldots, v_{22}\}$, $S_v^2 = \bigcup_{i=1}^{n} \{v_1^i, v_2^i, v_3^i, v_4^i\}$, and V_2 is the set of vertices x with $g(x) = 2$, we state the following facts.

Claim 2.1 *(i) If $g(v) \in \{1, 2\}$, then $g(S_v^1) \geq 4$.*

(ii) if $g(v) = -1$ and $g(v_2) = -1$, then $g(S_v^1) \geq 5$;

(iii) if $g(v) = -1$ and $g(v_2) \neq -1$, then $g(S_v^1) \geq 6$;

(iv) if $g(v) \geq 1$, then $g(S_v^1 \cup S_v^2) \geq n + 4$;

(v) if $g(v) = -1$ and $|V_2 \cap N(v) \cap (S_v^1 \cup S_v^2)| \neq 0$, then $g(S_v^1 \cup S_v^2) \geq n + 6$; and

(vi) if $g(v) = -1$ and $|V_2 \cap N(v) \cap (S_v^1 \cup S_v^2)| = 0$, then $g(S_v^1 \cup S_v^2) \geq n + 5$.

Proof. Let $T_1 = \{v_1, v_{18}, v_{16}, v_{17}, v_{14}, v_{15}\}$, $T_2 = N[v_4] = \{v_3, v_4, v_5, v_6\}$, $T_3 = N[v_{20}] = \{v_{19}, v_{20}, v_{21}, v_{22}\}$, $T_4 = N[v_8] = \{v_7, v_8, v_9, v_{10}\}$, and $T_5 = \{v_{11}, v_{12}, v_{13}\}$. By Lemma 2.1, we have $g(T_1) = 2$, $g(T_2) \geq 1$, $g(T_3) = 1$, $g(T_4) = 1$, and $g(T_5) = 0$. Then we argue as follows.

(i) If $g(v) \in \{1, 2\}$, then $g(v_2) \geq -1$ and thus $g(S_v^1) = g(v_2) + \sum_{i=1}^{5} g(T_i) \geq 4$.

(ii) If $g(v) = g(v_2) = -1$, then by Lemma 2.1 we have $g(v_7) = -1$, and thus $g(v_3) = 2$ and $g(T_2) \geq 2$. So $g(S_v^1) = g(v_2) + \sum_{i=1}^{5} g(T_i) \geq 5$.

(iii) If $g(v) = -1$ and $g(v_2) \neq -1$, then $g(v_2) \geq 1$ and thus $g(S_v^1) = g(v_2) + \sum_{i=1}^{5} g(T_i) \geq 6$.

(iv) This result follows from (i).

(v) Suppose $V_2 \cap N(v) \cap (S_v^1 \cup S_v^2) \neq \emptyset$ and assume first that $g(S_v^2) \geq n+1$. If $g(v_2) = -1$, then $g(v_3) = 2$ and hence $g(T_2) \geq 2$. And if $g(v_2) \geq 1$, then $g(v_3) \geq 1$. Therefore $g(S_v^1 \cup S_v^2) \geq n + 1 + g(v_2) + \sum_{i=1}^{5} g(T_i) \geq n+6$.

Assume next that $g(S_v^2) = n$. Then $g(v_2) = 2$, hence we also have $g(S_v^1 \cup S_v^2) \geq n + g(v_2) + \sum_{i=1}^{5} g(T_i) \geq n+6$.

(vi) In this case we have $g(S_v^1) \geq n$. Then we proceed similarly as in the proof of (v). □

Now we define a function $f : V(G) \to \{0,1,2\}$ as follows. If $g(v) \in \{1,2\}$, then $f(v) = g(v)$. By Claim 2.1, we have $g(T_v) - f(v) = g(S_v^1 \cup S_v^2) \geq n+4$. Clearly, in G, v is Roman dominated under f.

If $g(v) = -1$ and $V_2 \cap N(v) \cap (S_v^1 \cup S_v^2) = \emptyset$, then put $f(v) = 0$. By Claim 2.1(vi), we have $g(T_v) - f(v) = g(S_v^1 \cup S_v^2) - 1 \geq n+4$. Since $V_2 \cap N(v) \cap (S_v^1 \cup S_v^2) = \emptyset$,

there exists a vertex $x \in V_2$ that is in G adjacent to v. Therefore, in G, v is Roman dominated under f.

If $g(v) = -1$ and $V_2 \cap N(v) \cap (S_v^1 \cup S_v^2) \neq \emptyset$, then put $f(v) = 1$. By Claim 2.1(v) we have $g(T_v) - f(v) = g(S_v^1 \cup S_v^2) - 2 \geq n+4$. Clearly, in G, v is Roman dominated under f.

Then we have $\ell + n(n+4) \geq w(g) = \sum_{v \in V(G)} g(T_v) \geq \sum_{v \in V(G)} (n+4+f(v))$. Therefore $w(f) = \sum_{v \in V(G)} f(v) \leq \ell$ which completes the proof of Theorem 1.1.

2.2 The case $k = 2$

We next reduce the Roman Domination Problem on planar graphs to the Signed Roman 2-Domination Problem.

Let G be a graph on n vertices. For each vertex $v \in V(G)$, let H_v be the graph as depicted in Fig. 3.

Let G' be the graph obtained from the disjoint union of the graphs H_v, $v \in V(G)$, where an edge is added between $v' \in H_{v'}$ and $v'' \in H_{v''}$ if and only if $v'v'' \in E(G)$. Note that if G is planar, then G' is planar as well. Observe in addition that $|V(G')| = 6n^2 + 40n$ and $|E(G')| = |E(G)| + 8n^2 + 51n$. We will show that G has an RDF f with $w(f) \leq \ell$ if and only if G' has an SR2DF g with $w(g) \leq \ell + 3n^2 + 18n$.

Suppose that f is an RDF of G with $w(f) \leq \ell$. Then define $g : V(G') \to \{-1,1,2\}$ as follows. For any $v \in V(G)$ set $g(v_i) = 2$ for any $i \in \{2,5,11,17,23\}$, $g(v_i) = -1$ for $i \in \{1,3,6,7,12,13,18,19,24,25\}$, $g(v_i) = 1$ for $i \in \{8,9,10,14,15,16,20,21,22,26,27\}$

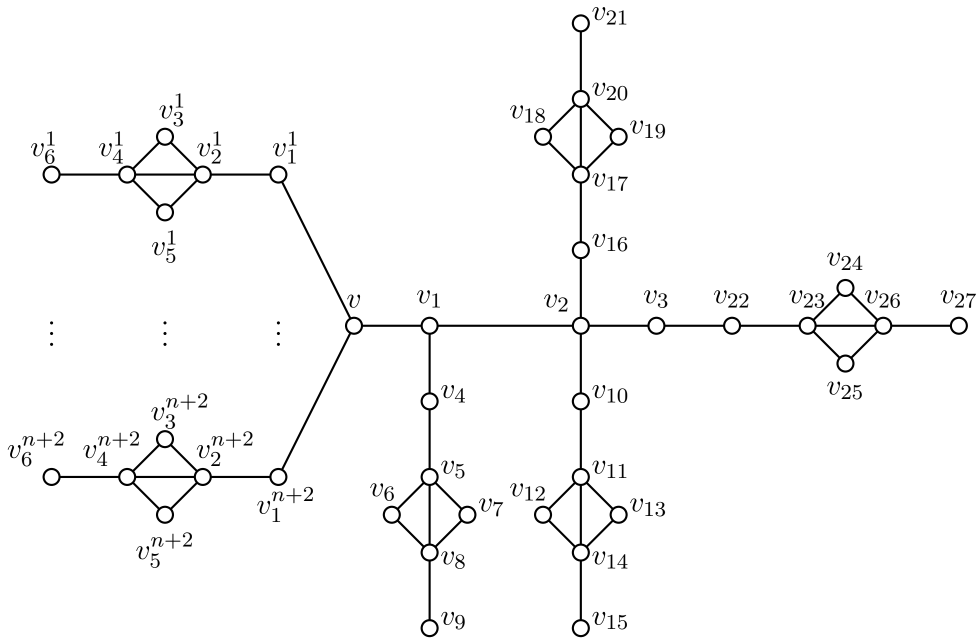

Figure 3: The graph H_v

and let $g(v_1^i) = g(v_4^i) = g(v_6^i) = 1, g(v_3^i) = g(v_5^i) = -1$, and $g(v_2^i) = 2$ for each $i \in [n+2]$. Now if $f(v) = 0$, then put $g(v) = -1$ and $g(v_4) = 2$; if $f(v) \in \{1, 2\}$, then put $g(v) = f(v)$ and $g(v_4) = 1$.

It is now straightforward to verify that g is an SR2DF. In particular, suppose that we have set $g(v) = -1$. This has happened because $f(v) = 0$. Since f is an RDF, v has a neighbor v' in G with $f(v') = 2$. But then v also has a neighbor v' in G' with $g(v') = 2$. Note next that the contribution of the vertices $\{v, v_4 : v \in V(G)\}$ to $w(g)$ is $\ell + n$. The contribution of all the other vertices is $n(3(n+2)+11)$, so that $w(g) = (\ell + n) + (3n^2 + 17n) = \ell + 3n^2 + 18n$.

To prove the converse we show that:

Lemma 2.2 *Let g be an SR2DF of G'. Then there exists an SR2DF g' of G' with $w(g') \leq w(g)$ such that for any $v \in V(G)$ the following hold.*

(i) $g'(v_4^i) = g'(v_6^i) = 1, g'(v_3^i) = g'(v_5^i) = -1$, $g'(v_1^i) \geq 1$, and $g'(v_2^i) = 2$ for each $i \in [n+2]$.

(ii) $g'(v_i) = 2$ for $i \in \{2, 5, 11, 17, 23\}$, $g'(v_i) = -1$ for $i \in \{1, 3, 6, 7, 12, 13, 18, 19, 24, 25\}$, $g'(v_i) = 1$ for $i \in \{8, 9, 14, 15, 20, 21, 26, 27\}$, and $g'(v_i) \geq 1$ for $i \in \{4, 10, 16, 22\}$.

Proof. (i) If for a given $i \in [n+2]$ we have $g(v_1^i) \geq 1$, then we set $g'(v_1^i) = g(v_1^i)$, $g'(v_4^i) = g'(v_6^i) = 1, g'(v_3^i) = g'(v_5^i) = -1$, $g'(v_2^i) = 2$, and $g'(x) = g(x)$ for every other vertex x in G'. Since $g(v_6^i) \geq 1$ and $g(N[v_2^i]) \geq 2$ we infer that $g(\{v_1^i, v_2^i, \ldots, v_6^i\}) \geq 3$. It follows that $w(g') \leq w(g)$ and hence g' is a desired SR2DF.

If $g(v_1^i) = -1$, then we set $g'(v_1^i) = 1$, $g'(v_4^i) = g'(v_6^i) = 1, g'(v_3^i) = g'(v_5^i) = -1$, $g'(v_2^i) = 2$ and $g'(x) = g(x)$ for every other vertex x in G'. Then we see as in the above paragraph that $w(g') \leq w(g)$ and g' is a desired SR2DF.

(ii) The proof is the same as in (i). □

Suppose now that G' admits an SR2DF of G' with weight at most $\ell + 3n^2 + 18n$. Then Lemma 2.2 implies that G' also admits an SR2DF g that fulfills all the assertions of Lemma 2.2 and for which $w(g) \leq \ell + 3n^2 + 18n$ holds. Let $S_v = V(H_v) \setminus \{v\}$ and let V_2 be the set of vertices x with $g(x) = 2$. Then we have the following facts.

Claim 2.2 *(i) If $g(v) \in \{1, 2\}$, then $g(S_v) \geq 3n + 18$.*

(ii) if $g(v) = -1$ and $|V_2 \cap N(v) \cap S_v| \neq 0$, then $g(S_v) \geq 3n + 20$;

(iii) if $g(v) = -1$ and $|V_2 \cap N(v) \cap S_v| = 0$, then $g(S_v) \geq 3n + 19$.

Proof. For $i \in [n+2]$ let $T_i = \{v_1^i, v_2^i, v_3^i, v_4^i, v_5^i, v_6^i\}$, and set in addition

- $M_1 = \{v_4, v_5, v_6, v_7, v_8, v_9\}$,
- $M_2 = \{v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27}\}$,
- $Q_1 = N[v_{14}] = \{v_{11}, v_{12}, v_{13}, v_{14}, v_{15}\}$, and
- $Q_2 = N[v_{20}] = \{v_{17}, v_{18}, v_{19}, v_{20}, v_{21}\}$.

By Lemma 2.2 we have $g(T_i) \geq 3$, $i \in [n+2]$, $g(M_1) \geq 3$, $g(M_2) \geq 3$, $g(Q_1) \geq 2$ and $g(Q_2) \geq 2$.

(i) We have $g(S_v) = \sum_{i=1}^{n+2} g(T_i) + g(M_1) + g(M_2) + g(Q_1) + g(Q_2) + g(N[v_2]) \geq 3(n + 2) + 3 + 3 + 2 + 2 + 2 = 3n + 18$.

(ii) Since $g(v_1) = -1$ (and $g(v) = -1$) we must have $g(v_4) = 2$, hence $g(M_1) = 4$. Therefore $g(S_v) = \sum_{i=1}^{n+2} g(T_i) + g(M_1) + g(M_2) + g(Q_1) + g(Q_2) + g(N[v_2]) \geq 3n + 20$.

(iii) Because $g(v_1) = -1$ we have $g(v_4) = 2$, hence again $g(M_1) = 4$. Therefore $g(S_v) = \sum_{i=1}^{n+2} g(T_i) + g(M_1) + g(M_2) + g(Q_1) + g(Q_2) + g(N[v_2]) \geq 3n + 19$. □

We now define a function f on $V(G)$ as follows.

- If $g(v) \in \{1, 2\}$, then set $f(v) = g(v)$.
  By Claim 2.2, we have $g(H_v) - f(v) = g(S_v) + g(v) - f(v) \geq 3n + 18$. Clearly, v is Roman dominated under f in G.
- If $g(v) = -1$ and $g(u) = 2$ for some $u \in N(v) \cap S_v$, then set $f(v) = 1$.
  By Claim 2.2(ii), we have $g(H_v) - f(v) = g(S_v) + g(v) - f(v) \geq 3n + 20 + (-1) - 1 = 3n + 18$. Clearly, v is Roman dominated under f in G.

- If $g(v) = -1$ and $g(u) \neq 2$ for every $u \in N(v) \cap S_v$, then set $f(v) = 0$.
  By Claim 2.2(iii), we have $g(H_v) - f(v) = g(S_v) + g(v) - f(v) \geq 3n + 19 + (-1) - 0 = 3n + 18$. Since $|V_2 \cap N(v) \cap (S_v)| = 0$, we have $v' \in V_2$ in $H_{v'}$ for some $v' \in V(G)$. Thus v is Roman dominated by v' in G.

The function f is thus an RDF. Since for any vertex $v \in V(G)$ we have $g(H_v) - f(v) \geq 3n + 18$, it follows that $\sum_{v \in V(G)} g(H_v) - \sum_{v \in V(G)} f(v) \geq n(3n + 18)$. Therefore,

$$\begin{aligned} w(f) &= \sum_{v \in V(G)} f(v) \\ &\leq \sum_{v \in V(G)} g(H_v) - n(3n + 18) \\ &= w(g) - n(3n + 18) \\ &\leq (\ell + 3n^2 + 18n) - n(3n + 18) = \ell\,, \end{aligned}$$

where for the last inequality we have used the assumption $w(g) \leq \ell + 3n^2 + 18n$. So f is an RDF of G with weight at most ℓ as required.

3 Signed Roman 2-domination numbers of thin torus graphs

Let us first introduce some notation. For a cycle C_ℓ let $V(C_\ell) = [\ell]$, where $kk' \in E(C_\ell)$ if $|k - k'| = 1 \pmod{\ell}$. Using this convention we have $V(C_m \,\square\, C_n) = [m] \times [n]$, and vertices (j, i) and (j', i') of $C_m \,\square\, C_n$ are adjacent if $|j - j'| + |i - i'| = 1$, where the computations are modulo m and modulo n, respectively. We will denote the set of vertices of the i^{th} C_m-layer with $C_m^{(i)}$, that is, $C_m^{(i)} = \{(j, i) : j \in [m]\}$. Two C_m-layers $C_m^{(i)}$ and $C_m^{(j)}$ of $C_m \,\square\, C_n$ are *adjacent* if $|i - j| = 1 \pmod{n}$. If f is a SR2DF of $C_m \,\square\, C_n$, then let $V_k^f = \{v \in V(C_m \,\square\, C_n) : f(v) = k\}$. When f will be clear from the context, we may simplify the notation V_k^f to V_k. Clearly, the sets V_{-1}, V_1, and V_2 partition $V(C_m \,\square\, C_n)$. For a given SR2DF f of $C_m \,\square\, C_n$, let $w_f(C_m^{(i)})$ denote the weight of the i^{th} C_m-layer, that is, $w_f(C_m^{(i)}) = \sum_{u \in V(C_m^{(i)})} f(u)$. When no confusion arises, we will use w_f^i to denote $w_f(C_m^{(i)})$ for short. Finally, for a function f and a vertex w of G we will denote with $f(N[w])$ the value $\sum_{x \in N[w]} f(x)$.

3.1 Proof of Theorem 1.2

Lemma 3.1 *Let $n \geq 4$ and let f be an SR2DF of $C_3 \,\square\, C_n$. Then the following assertions hold, where all the indices are taken modulo n.*

(i) $w_f^i \geq -1$, $i \in [n]$.

(ii) *If* $w_f^{i+1} = 0$, *then* $w_f^i \geq 3$ *and* $w_f^{i+2} \geq 3$.

(iii) *If* $w_f^{i+1} = -1$, *then* $w_f^i \geq 3$ *and* $w_f^{i+2} \geq 3$.

(iv) If $w_f^{i+1} = -1$ and $w_f^{i+2} = 3$, then $w_f^i \geq 6$ and $w_f^{i+3} \geq 1$.

(v) If $w_f^{i+1} = 0$ and $w_f^{i+2} = 3$, then $w_f^i \geq 3$ and $w_f^{i+3} \geq 1$.

(vi) If $w_f^{i+1} = 1$ and $w_f^{i+2} = 1$, then $w_f^i \geq 4$ and $w_f^{i+3} \geq 4$.

(vii) If $w_f^{i+1} = 1$ and $w_f^{i+2} = 2$, then $w_f^i \geq 2$ and $w_f^{i+3} \geq 1$.

Proof. (i) Suppose that $w_f^i \leq -2$. Then necessarily $f \equiv -1$ on $C_3^{(i)}$, but then f is not an SR2DF.

(ii) Suppose that $w_f^{i+1} = 0$. Then we may without loss of generality assume that $f(1, i+1) = -1$, $f(2, i+1) = -1$, and $f(3, i+1) = 2$. Assume now that $w_f^i \leq 2$. Then for some vertex (j, i), $j \in [3]$, we must have $f(j, i) = -1$. But then $f(N[(j, i+1)]) \leq 1$, a contradiction. Hence $w_f^i \geq 3$. Using a parallel argument we also infer that $w_f^{i+2} \geq 3$.

(iii) Suppose $w_f^{i+1} = -1$. Then we may without loss of generality assume that $f(1, i+1) = -1$, $f(2, i+1) = -1$, and $f(3, i+1) = 1$. The argument now proceeds as in (ii).

(iv) As $w_f^{i+1} = -1$ we may assume that $f(1, i+1) = -1$, $f(2, i+1) = -1$, and $f(3, i+1) = 1$. Since $w_f^{i+2} = 3$, we need to consider two cases. In the first case $f(1, i+2) = f(2, i+2) = f(3, i+2) = 1$. Then $f(1, i+3) \geq 1$ and $f(2, i+3) \geq 1$, so that $w_f^{i+3} \geq 1$. Moreover, $f \equiv 2$ must hold on on $C_3^{(i)}$ since f is an SR2DF. Therefore $w_f^i \geq 6$ (actually $w_f^i = 6$). In the second case to be considered the weights on $C_3^{(i+2)}$ are 2, 2, -1, respectively. But this case cannot happen because considering f restricted to $C_3^{(i+1)}$ we see that f is not an SR2DF.

(v) As $w_f^{i+1} = 0$ we may assume that $f(1, i+1) = -1$, $f(2, i+1) = -1$, and $f(3, i+1) = 2$. Based on the assumption that $w_f^{i+2} = 3$, we again need to consider two cases. In the first one, $f(1, i+2) = f(2, i+2) = f(3, i+2) = 1$. Then for any $j \in [3]$ we must have $f(j, i) \geq 1$, hence $w_f^i \geq 3$. Moreover, $f(1, i+3) \geq 1$ and $f(2, i+3) \geq 1$ and consequently $w_f^{i+3} \geq 1$. The second case when the weights on $C_3^{(i+2)}$ are 2, 2, -1 is again not possible because for the neighbor $(j, i+1)$ of the vertex $(j, i+2)$ with $f(j, i+2) = -1$ we would have $f(N[(j, i+1)]) \leq 1$.

(vi) Since $w_f^{i+1} = 1$, the vertices of $C_3^{(i+1)}$ have weights 1, 1, -1. The same holds for the vertices of $C_3^{(i+2)}$. Assume without loss of generality that $f(1, i+1) = f(2, i+1) = 1$ and $f(3, i+1) = -1$. If $f(1, i+2) = f(2, i+2) = 1$ and $f(3, i+2) = -1$, then $f(1, i) \geq 1$, $f(2, i) \geq 1$, and $f(3, i) = 2$. Consequently, $w_f^i \geq 4$. In addition, $f(3, i+3) = 2$, $f(1, i+3) \geq 1$, and $f(2, i+3) \geq 1$, so that $w_f^{i+3} \geq 4$. In the other case we may assume that $f(1, i+2) = -1$ and $f(2, i+2) = f(3, i+2) = +1$. Using similar arguments as above the conclusions follow.

(vii) Since $w_f^{i+1} = 1$, we may assume that $f(1, i+1) = f(2, i+1) = 1$ and $f(3, i+1) = -1$. Since $w_f^{i+2} = 2$, the vertices of $C_3^{(i+2)}$ have weights 2, 1, -1 and we need to consider several cases. Assume first that $f(1, i+2) = 2$, $f(2, i+2) = 1$, and $f(3, i+2) = -1$. Then $f(3, i) = 2$ and $f(2, i) \geq 1$ which implies that $w_f^i \geq 2$. Moreover, if $w_f^{i+3} \leq 0$ would hold, then at least two the vertices of $C_3^{(i+3)}$ would have weight -1. As f is an SR2DF this is not possible, hence this is not possible $w_f^{i+3} \geq 1$. The other cases with respect to the distribution of the weights 2, 1, and -1 on $C_3^{(i+2)}$ are treated similarly. $\square$

We will now apply Lemma 3.1 to prove the lower bound on $\gamma_{sR}^2(C_3 \,\square\, C_n)$ as stated in Lemma 3.2. For this sake we next describe the following discharging procedure. Let f be a SR2DF of $C_3 \,\square\, C_n$. Then to every C_3-layer $C_3^{(i)}$, $i \in [n]$, set the *initial charge* $s(i)$ to be equal w_f^i. The *final charge* s' is then produced using the following discharging rule.

R1: Every C_3-layer $C_3^{(i)}$ with $s(i) \geq 2$ transmits $\frac{2s(i)-3}{4}$ charge to each adjacent C_3-layer $C_3^{(j)}$ with $s(j) \leq 1$.

Lemma 3.2 *If $n \geq 3$, then $\gamma_{sR}^2(C_3 \,\square\, C_n) \geq \frac{3n}{2}$. Moveover, if the equality holds, then n is even and there exists an SR2DF f' satisfying $w_{f'}^{2\ell} = 1$ and $w_{f'}^{2\ell-1} = 2$ for all $\ell \in \left[\frac{n}{2}\right]\}$.*

Proof. Let f be an SR2DF of $C_3 \,\square\, C_n$. Set the initial charge $s(i) = w_f^i$ to every C_3-layer $C_3^{(i)}$, $i \in [n]$. Let s' be the final charge after applying the discharging rule R1 to s. We now distinguish the cases based on the value of $s(i)$.

Case 1. $s(i) \geq 2$.
Since $C_3^{(i)}$ sends out at most two $\frac{2s(i)-3}{4}$ charges, $s'(i) \geq s(i) - 2\left(\frac{2s(i)-3}{4}\right) = \frac{3}{2}$.

Case 2. $s(i) = 1$.
Note first that $s(i+1) \geq 1$ by Lemma 3.1(i), (ii) and (iii). If $s(i+1) = 1$, then $s(i-1) \geq 4$ by Lemma 3.1(vi). Then $C_3^{(i)}$ receives at least $\frac{5}{4}$ from $C_3^{(i-1)}$ and consequently $s'(i) \geq \frac{9}{4}$. Next, if $s(i+1) = 2$, then $s(i-1) \geq 2$ and $s(i+2) \geq 1$ by Lemma 3.1(vii). It follows that $C_3^{(i)}$ receives a charge at least $\frac{1}{4}$ from each of $C_3^{(i-1)}$ and $C_3^{(i+1)}$, hence $s' \geq \frac{3}{2}$. Suppose finally that $s(i+1) \geq 3$. Then $C_3^{(i)}$ receives from $C_3^{(i+1)}$ a charge at least $\frac{3}{4}$, thus $s'(i) \geq \frac{7}{4}$.

Case 3. $s(i) = 0$.
In this case Lemma 3.1(ii) asserts that $s(i-1) \geq 3$ and $s(i+1) \geq 3$. Hence, $C_3^{(i)}$ receives a charge at least $\frac{3}{4}$ from each of $C_3^{(i-1)}$ and $C_3^{(i+1)}$, thus $s'(i) \geq \frac{3}{2}$. Moreover, if $s(i+1) = 3$, then $s(i+2) \geq 1$ by Lemma 3.1(v). Hence, $s'(i+2) \geq 1 + \frac{3}{4} = \frac{7}{4}$. Otherwise, $s(i+1) \geq 4$ implies that $s'(i) \geq \frac{3}{4} + \frac{5}{4} = 2$.

Case 4. $s(i) = -1$.
If $s(i+1) = 3$, then $s(i-1) \geq 6$ by Lemma 3.1(iv). Then $s'(i) \geq \frac{3}{4} + \frac{9}{4} = 3$. Otherwise, applying Lemma 3.1(iii), we have $s(i+1) \geq 3$ and $s(i-1) \geq 4$. In this case, $s'(i) \geq \frac{3}{4} + \frac{3}{4} = \frac{3}{2}$.

In summary, $s'(i) \geq \frac{3}{2}$ holds for any $i \in [n]$. Since the charging procedure preserves the value of the initial charge, that is, $\sum_{i=1}^n s(i) = \sum_{i=1}^n s'(i)$, we conclude that $w(f) = \sum_{i=1}^n s(i) \geq \frac{3n}{2}$. Moreover, if $w(f) = \frac{3n}{2}$, then n is even and $s'(i) = \frac{3}{2}$ for all $i \in [n]$. The above analysis also implies that if $w(f) = \frac{3n}{2}$, then $s(i) \geq 1$ for all $i \in [n]$ and if $s(j) = 1$ then $s(j-1) = s(j+1) = 2$. That is to say, for any $i \in [n]$, $\{s(i), s(i+1)\} = \{1, 2\}$. By the symmetry of $C_3 \Box C_n$, there exists an SR2DF f' satisfying $w_{f'}^{2\ell} = 1$ and $w_{f'}^{2\ell-1} = 2$ for all $\ell \in \left[\frac{n}{2}\right]\}$. $\square$

Lemma 3.3 *If $C_3 \Box C_n$ has an SR2DF f such that $w(f) = \frac{3n}{2}$ and $f(p, q) = 2$ for some $p \in [3]$ and $q \in [n]$, then $f(p, q+2) \neq 2$ and $f(p, q+4) = 2$.*

Proof. By Lemma 3.2, we know that $w_f^i \in \{1, 2\}$ for any $i \in [n]$. Since f is an SR2DF and $f(p, q) = 2$, we have $w_f^q = 2$. Then again by Lemma 3.2, $w_f^{q+1} = w_f^{q+3} = 1$ and $w_f^{q+2} = w_f^{q+4} = 2$. Without loss of generality assume that $p = 1$, $f(1, q) = 2$, $f(2, q) = -1$, and $f(3, q) = 1$. Since

$$\begin{aligned} 2 &\leq \sum_{x \in N[(2,q+1)]} f(x) \\ &= f(2, q) + \sum_{i=1}^{3} f(i, q+1) + f(2, q+2) \\ &= -1 + w_f^{q+1} + f(2, q+2) = f(2, q+2)\,, \end{aligned}$$

we have $f(2, q+2) = 2$. Hence, $f(1, q+2) \neq 2$.

Similarly, since $\sum_{x \in N[(3,q+1)]} f(x) \geq 2$, we can obtain that $f(3, q+2) \geq 1$. Note that since $w_f^{q+2} = 2$, we have $f(3, q+2) = 1$ and $f(1, q+2) = -1$. Since $\sum_{x \in N[(1,q+3)]} f(x) \geq 2$, we also have $f(p, q+4) = 2$. $\square$

Theorem 3.4 *If $n \equiv 2 \pmod 4$ and $n \geq 6$, then $\gamma_{sR}^2(C_3 \Box C_n) > \frac{3n}{2}$.*

Proof. Suppose on the contrary that $\gamma_{sR}^2(C_3 \Box C_n) = \frac{3n}{2}$. By Lemma 3.2, there exists an SR2DF f satisfying $w_f^{2\ell} = 1$ and $w_f^{2\ell-1} = 2$ for all $\ell \in \left[\frac{n}{2}\right]\}$. We may without loss of generality assume that $f(1, 1) = 2$. Then $f(1, 1+4k) = 2$ and $f(1, 3+4k) \neq 2$ for all $k \in \{0, 1, \ldots, \frac{n-2}{4}\}$ by Lemma 3.3. We now construct an auxiliary cycle H_C such that $V(H_C) = \{v_0, v_1, \ldots, v_{\frac{n}{2}-1}\}$ and $v_i v_{i+1} \in E(H_C)$. We define a 2-coloring g with colors $\{1, 2\}$ of H_C such that $g(v_i) = 1$ if and only if $f(1, 2i+1) = 2$. Since H_C is an odd cycle, we have H_C is not 2-colorable, a contradiction. $\square$

Note that Lemma 3.2 takes care for the lower bound in Theorem 1.2 for the case $n \equiv 0 \pmod 4$ while Theorem 3.4 takes care for the case $n \equiv 2 \pmod 4$. In the following we proceed with the cases $n \equiv 1, 3 \pmod 4$. If $S = s(i)s(i+1)\cdots s(i+k-1)$ is a segment of the initial charge, then set

$$\delta(S) = \sum_{j=0}^{k-1} s'(i+j) - \frac{3k}{2}\,.$$

Lemma 3.5 *Let $n \equiv 1, 3 \pmod 4$. If f is an SR2DF of $C_3 \Box C_n$ with weight $\left\lceil \frac{3n}{2} \right\rceil$ and $S_0 = s(1)s(2)\cdots s(n)$ is the initial charge given by f, then $\delta(S_0) = \frac{1}{2}$.*

Proof. Since n is odd, $\frac{3n}{2} = \lceil \frac{3n}{2} \rceil - \frac{1}{2}$. Recalling that $s'(i) \geq \frac{3}{2}$ for $i \in [n]$, we have $\delta(s(i)) \geq 0$. Furthermore,

$$w(f) = \sum_{i=1}^{n} s(i) = \sum_{i=1}^{n} s'(i) = \frac{3n}{2} + \delta(S_0) = \left\lceil \frac{3n}{2} \right\rceil - \frac{1}{2} + \delta(S_0)\,.$$

Since f is an SR2DF of $C_3 \,\Box\, C_n$ with weight $\lceil \frac{3n}{2} \rceil$, we have $w(f) = \lceil \frac{3n}{2} \rceil$. We conclude that $\delta(S_0) = \frac{1}{2}$. $\square$

Lemma 3.6 *Let $n \equiv 1, 3 \pmod 4$ with $n \geq 5$. If f is an SR2DF of $C_3 \,\Box\, C_n$ with weight $\lceil \frac{3n}{2} \rceil$ and $S_0 = s(1)s(2)\cdots s(n)$ is the initial charge given by f, then*

(i) $s(i) \neq -1$, $i \in [n]$;

(ii) $(s(i), s(i+1)) \notin \{(1,1),(1,4)\}$;

(iii) $s(i) \neq 0$, $i \in [n]$;

(iv) $s(i) \leq 2$, $i \in [n]$.

Proof. (i) Assume on the contrary that $s(i) = -1$. Then by Lemma 3.1 (iii), $s(i \pm 1) \geq 3$. Moreover, combining Lemma 3.1(iv) with the fact that $s(i)$ cannot be equal to 4, we have $s(i+1)+s(i-1) \geq 9$. Then $\delta(s(i)) = \frac{1}{2}$. But we can deduce that $\delta(s(i+2))+\delta(s(i-2)) > 0$. Therefore, $\delta(S_0) > \frac{1}{2}$, contradicting Lemma 3.5.

(ii) It is clear that $\delta(s(i)s(i+1)) > \frac{1}{2}$ if $(s(i), s(i+1)) \notin \{(1,1),(1,4)\}$.

(iii) Assume $s(i) = 0$. Then $s(i \pm 1) \geq 3$ by Lemma 3.1(ii). On the other hand, since $\delta(s(i)) \leq \frac{1}{2}$, we have $s(i+1) + s(i-1) \geq 7$. If $s(i+1) + s(i-1) = 7$, we may without loss of generality assume that $s(i-1) = 3$ and $s(i+1) = 4$. Then $\delta(s(i)) = \frac{1}{2}$. But $s(i+2)$ receives a charge from $s(i+1)$ and thus $\delta(s(i+2)) > 0$. Therefore, $\delta(S_0) > \frac{1}{2}$, contradicting Lemma 3.5. If $s(i+1)+s(i-1) = 6$, then $s(i-1) = 3$ and $s(i+1) = 3$. Then it can be verified that $\delta(s(i-2)s(i-1)s(i)s(i+1)s(i+2)) > \frac{1}{2}$. Therefore, $\delta(S_0) > \frac{1}{2}$, again contradicting Lemma 3.5.

(iv) From the proofs of cases (i), (ii), and (iii) we have $1 \leq s(i) \leq 3$ for $i \in [n]$. Assume that $s(i) = 3$. Then we have $s(i-1) = 1$ or $s(i+1) = 1$. Otherwise $\delta(s(i-1)s(i)s(i+1)) > \frac{1}{2}$. We may without loss of generality assume that $s(i+1) = 1$. Since $s(i+2) \geq 3$ is impossible, we have $s(i+2) = 2$. It can be verified that $\delta(s(i-1)s(i)s(i+1)s(i+2)) > \frac{1}{2}$. Therefore, $\delta(S_0) > \frac{1}{2}$, again contradicting Lemma 3.5. $\square$

From Lemma 3.6 we deduce:

Lemma 3.7 *Let $n \equiv 1, 3 \pmod 4$ with $n \geq 5$. If $\gamma_{sR}^2(C_3 \,\Box\, C_n) = \lceil \frac{3n}{2} \rceil$, then $1 \leq s(i) \leq 2$ for any $i \in [n]$. More precisely, there exists an SR2DF f satisfying $w_f^1 = 2$, $w_f^{2\ell} = 2$ and $w_f^{2\ell+1} = 1$ for all $\ell \in \left[\frac{n-1}{2}\right]$.*

To obtain the lower bound in Theorem 1.2 also for the cases $n \equiv 1, 3 \pmod 4$, one first checks the case $n = 3$. The verification is easy and we hence omit the details. Finally, for $n \geq 5$ we obtain the lower bound from Lemma 3.7 together with a similar proof as the one of Theorem 3.4.

To complete the proof of Theorem 1.2 we need to provide the corresponding upper bounds. Consider the following matrices to be used as patterns for SR2DFs of $C_3 \,\square\, C_n$.

$$Q = \begin{bmatrix} 2 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 2 & 1 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 2 & 2 & -1 & -1 & 1 \\ 1 & -1 & 1 & 2 & -1 \\ -1 & 1 & 1 & 2 & 1 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 2 & 1 & -1 & -1 & 1 & 2 \\ 1 & 1 & 1 & 2 & -1 & -1 \\ -1 & -1 & 2 & 1 & 1 & 1 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 2 & 1 \end{bmatrix},$$

$$Q_4 = \begin{bmatrix} 2 & 2 & -1 & 1 & 2 \\ 1 & 1 & 2 & -1 & -1 \\ -1 & -1 & 1 & 1 & 1 \end{bmatrix}, \quad Q_5 = \begin{bmatrix} -1 & -1 & 1 & 2 & 2 & 1 \\ 1 & 2 & -1 & -1 & 1 & 1 \\ 2 & 1 & 1 & 1 & -1 & -1 \end{bmatrix}.$$

We will use the concatenation notation, meaning that if P and P' are $3 \times t$ and $3 \times s$ patterns, then PP' is their concatenation, that is, a $3 \times (t+s)$ pattern. By the power P^k we mean the k-tuple concatenation of P.

Now, the pattern Q^k defines an SR2DF of $C_3 \,\square\, C_{4k}$ with weight $6k$ for $k \geq 1$. The pattern $Q^k Q_1$ defines an SR2DF of $C_3 \,\square\, C_{4k+5}$ with weight $6k+9$ for $k \geq 1$. The pattern $Q^k Q_2$ defines an SR2DF of $C_3 \,\square\, C_{4k+6}$ with weight $6k+4$ for $k \geq 1$. The pattern $Q^k Q_3$ defines an SR2DF of $C_3 \,\square\, C_{4k+3}$ with weight $6k+6$ for $k \geq 1$. The pattern Q_4 defines an SR2DF of $C_3 \,\square\, C_5$ with weight 9. The pattern Q_5 defines an SR2DF of $C_3 \,\square\, C_6$ with weight 10. In this way all the upper bounds are established.

3.2 Proof of Theorem 1.3

We will now prove Theorem 1.3 along the similar lines as Theorem 1.2 was proved. In particular, the lower bound will be established using the discharging method, for which the following technical result is essential. Recall that, having in mind that we consider $C_4 \,\square\, C_n$, the notation w_f^i stands for the weight of the i^{th} C_4-layer with respect to an SR2DF f.

Lemma 3.8 *Let f be an SR2DF of $C_4 \,\square\, C_n$. Then the following assertions hold, where the indices are modulo n.*

(i) $w_f^i \geq 0$, $i \in [n]$.

(ii) *If* $w_f^{i+1} = 0$, *then* $w_f^i + w_f^{i+2} \geq 8$.

(iii) *If* $w_f^{i+1} = w_f^{i+2} = 1$, *then* $w_f^i \geq 4$ *and* $w_f^{i+3} \geq 4$.

(iv) *If* $w_f^{i+1} = 1$ *and* $w_f^{i+2} = 2$, *then* $w_f^i \geq 3$ *and* $w_f^{i+3} \geq 3$.

(v) If $w_f^{i+1} = 1$ and $w_f^{i+2} = 0$, then $w_f^i \geq 5$ and $w_f^{i+3} \geq 7$.

Proof. To shorten the presentation, the notation $w_f^i(\cdot) = (f(1,i), f(2,i), f(3,i), f(4,i))$ will be used.

(i) Suppose on the contrary that $w_f^i < 0$ for some i. Then in $C_4^{(i)}$ three vertices are assigned weight -1, hence there exists a vertex v assigned weight -1 which is adjacent to two vertices assigned weight -1. Therefore $w(f(N[v])) \leq 1$, a contradiction.

(ii) Suppose that $w_f^i = 0$. Due to the symmetry it suffices to consider the following two cases.

Case 1. $w_f^{i+1}(\cdot) = (1, 1, -1, -1)$.
Since the vertex $(1, i+1)$ needs to be signed Roman 2-dominated, we have $f(1,i)+f(1,i+2) \geq 1$. Similarly, $f(2,i)+f(2,i+2) \geq 1$, $f(3,i)+f(3,i+2) \geq 3$, and $f(4,i)+f(4,i+2) \geq 3$. Therefore, $w_f^i + w_f^{i+2} \geq 8$.

Case 2. $w_f^{i+1}(\cdot) = (1, -1, 1, -1)$.
The argument in this case is parallel with the argument in Case 1.

(iii) Suppose that $w_f^{i+1} = w_f^{i+2} = 1$. Then in $C_4^{(i+1)}$ the vertices are assigned weights 2, 1, -1, -1. The same holds for the vertices in $C_4^{(i+2)}$. Due to symmetry and the fact that $|V_{-1} \cap N[v]| \leq 2$ for any vertex v, we only need to consider the following four cases. Since the arguments are similar in all the cases, we will only prove the first one.

Case 1. $w_f^{i+1}(\cdot) = (-1, -1, 1, 2)$, $w_f^{i+2}(\cdot) = (2, 1, -1, -1)$.
Since the vertex $(1, i+1)$ needs to be signed Roman 2-dominated, we have $f(1,i)+f(1,i+2) \geq 2$ and so $f(1,i) \geq 0$. Similarly, $f(2,i) \geq 2$, $f(3,i) \geq 1$, $f(4,i) \geq 1$; $f(1,i+3) \geq 1$, $f(2,i+3) \geq 1$, $f(3,i+3) \geq 2$ and $f(4,i+3) \geq 0$. Therefore, $w_f^i \geq 4$ and $w_f^{i+3} \geq 4$.

Case 2. $w_f^{i+1}(\cdot) = (-1, -1, 1, 2)$, $w_f^{i+2}(\cdot) = (1, 2, -1, -1)$.

Case 3. $w_f^{i+1}(\cdot) = (-1, 1, -1, 2)$, $w_f^{i+2}(\cdot) = (-1, 1, -1, 2)$.

Case 4. $w_f^{i+1}(\cdot) = (-1, 1, -1, 2)$, $w_f^{i+2}(\cdot) = (-1, 2, -1, 1)$.

(iv) Assume that $w_f^{i+1} = 1$ and $w_f^{i+2} = 2$. Then in $C_4^{(i+2)}$ the vertices are either assigned weights 2, 2, -1, -1 or weights 1, 1, 1, -1. Again using the fact that $|V_{-1} \cap N[v]| \leq 2$ for any vertex v, we only need to consider the following five cases; we omit the arguments that are similar to the earlier ones.

Case 1. $w_f^{i+1}(\cdot) = (-1, -1, 1, 2)$, $w_f^{i+2}(\cdot) = (2, 2, -1, -1)$.

Case 2. $w_f^{i+1}(\cdot) = (-1, 1, -1, 2)$, $w_f^{i+2}(\cdot) = (-1, 2, -1, 2)$.

Case 3. $w_f^{i+1}(\cdot) = (-1, -1, 1, 2)$, $w_f^{i+2}(\cdot) = (1, 1, -1, 1)$.

Case 4. $w_f^{i+1}(\cdot) = (-1, -1, 1, 2)$, $w_f^{i+2}(\cdot) = (1, 1, 1, -1)$.

Case 5. $w_f^{i+1}(\cdot) = (-1, 1, -1, 2)$, $w_f^{i+2}(\cdot) = (-1, 1, 1, 1)$.

(v) The proof is again similar to the above analysis and is omitted. $\square$

Based on Lemma 3.8, we apply a discharging procedure leading to a final charge that will give us the desired lower bound. Let f be an SR2DF of $C_4 \,\square\, C_n$. We set the *initial charge* of the C_4-layer $C_4^{(i)}$ to be $s(i) = w_f^i$. We use the discharging procedure, leading to the *final charge* s', defined by the following rule:

R1: Every C_4-layer $C_4^{(i)}$ with $s(i) > 2$ transmits $\frac{s(i)-2}{2}$ charge to each adjacent C_4-layer $C_4^{(j)}$ with $s(j) < 2$.

The lower bounds 10 and 11 on $\gamma_{sR}^2(C_4 \,\square\, C_4)$ and $\gamma_{sR}^2(C_4 \,\square\, C_5)$, respectively, can be obtained by a tedious analyse hence we omit it.

Let now $n \geq 6$, let f be an SR2DF of $C_4 \,\square\, C_n$, let $s(i) = w_f^i$, $i \in [n]$, be the initial charge, and let s' be the charge obtained from s by applying the rule R1. Then we have the following.

- If $s(i) > 2$, then $C_4^{(i)}$ sends out at most two charges $\frac{s(i)-2}{2}$, hence $s'(i) \geq 2$.
- If $s(i) = 1$, then we consider the following cases.
  (a) $s(i+1) = 0$. In this case $s(i-1) \geq 5$ by Lemma 3.8(v). Then $s(i)$ receives $\frac{3}{2}$ from $s(i-1)$ and thus $s'(i) \geq \frac{5}{2}$.
  (b) $s(i+1) = 1$. In this case $s(i-1) \geq 4$ by Lemma 3.8(iii). Then $s(i)$ receives 1 from $s(i-1)$ and thus $s'(i) \geq 2$.
  (c) $s(i+1) = 2$. In this case $s(i-1) \geq 3$ and $s(i+2) \geq 3$ by Lemma 3.8(iv). Then $s(i)$ receives $\frac{1}{2}$ from both $s(i-1)$ and $s(i+2)$, thus $s'(i) \geq 2$.
  (d) $s(i \pm 1) \geq 3$. Now $s(i)$ receives at least $\frac{1}{2}$ from both $s(i-1)$ and $s(i+1)$, thus $s'(i) \geq 2$.
- If $s(i) = 0$, then we consider the following cases.
  (a) $s(i+1) = 0$. In this case $s(i-1) \geq 8$ by Lemma 3.8(ii). Then $s(i)$ receives at least 3 from both $s(i-1)$ and $s(i+1)$, thus $s'(i) \geq 3$.
  (b) $s(i+1) = 1$. This case has been already considered above.
  (c) $s(i+1) \geq 2$. It is easy to process this case by using Lemma 3.8(ii).

From the above analysis we have $s'(i) \geq 2$, $i \in [n]$. Since the charging procedure preserves the total value of the charge, that is, $\sum_{i=1}^{n} s(i) = \sum_{i=1}^{n} s'(i)$, we conclude that $w(f) = \sum_{i=1}^{n} s(i) \geq 2n$. The lower bounds are thus proved.

Now, we will show the upper bounds. Let

$$R_4 = \begin{bmatrix} -1 & 1 & -1 & 2 \\ 1 & 2 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 2 & 1 & 2 & 1 \end{bmatrix}, R_5 = \begin{bmatrix} 2 & -1 & 1 & 2 & -1 \\ 1 & 1 & -1 & 1 & 1 \\ -1 & 2 & 2 & -1 & 1 \\ 1 & -1 & 1 & -1 & 2 \end{bmatrix},$$

$$R_6 = \begin{bmatrix} 1 & 2 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 2 & -1 \\ 2 & -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 2 & -1 & 1 \end{bmatrix}.$$

The patterns R_4 and R_5 define SR2DFs of $C_4 \,\square\, C_4$ and $C_4 \,\square\, C_5$ with the desired weights, respectively. The pattern R_6^k induces an SR2DF of $C_4 \,\square\, C_{6k}$ with weight $12k$ for every $k \geq 1$. Let

$$F_1 = \begin{bmatrix} 1 & 1 & -1 & -1 & 1 & 1 & 2 \\ -1 & 1 & 2 & 1 & -1 & 2 & -1 \\ 2 & -1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 & -1 & -1 \end{bmatrix},$$

$$F_2 = \begin{bmatrix} 1 & 2 & -1 & 1 & 1 & -1 & 2 & -1 \\ -1 & 1 & 1 & -1 & 2 & -1 & 1 & 1 \\ 2 & -1 & 2 & -1 & 1 & 1 & -1 & 2 \\ 1 & -1 & 1 & 2 & -1 & 2 & 1 & -1 \end{bmatrix},$$

$$F_3 = \begin{bmatrix} 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 2 & 1 & -1 & 2 & -1 & 1 & 2 \\ 2 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 2 & 2 & -1 & 2 & 2 & -1 \end{bmatrix},$$

$$F_4 = \begin{bmatrix} 1 & 2 & -1 & -1 & 1 & 2 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 2 & -1 & 1 & 1 & 1 & 2 \\ 2 & -1 & 1 & 1 & -1 & 1 & 2 & -1 & -1 & 1 \\ 1 & -1 & 1 & 2 & -1 & 1 & -1 & 1 & 2 & -1 \end{bmatrix},$$

$$F_5 = \begin{bmatrix} 1 & 2 & -1 & -1 & 1 & 1 & 2 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 2 & -1 & -1 & 2 & -1 & 1 & 2 \\ 2 & -1 & 1 & 1 & -1 & 2 & 1 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 2 & -1 & 1 & -1 & -1 & 2 & 2 & -1 \end{bmatrix},$$

$$R_7 = \begin{bmatrix} -1 & 1 & -1 & 1 & 2 & 1 & -1 \\ 1 & 2 & -1 & 1 & -1 & -1 & 2 \\ -1 & -1 & 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & -1 & -1 & 1 & 1 \end{bmatrix},$$

$$R_8 = \begin{bmatrix} 1 & -1 & 2 & 1 & -1 & 2 & -1 & 2 \\ 2 & 1 & -1 & 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 2 & 2 & -1 & 2 & 1 \\ -1 & 2 & 1 & -1 & 1 & 1 & -1 & 1 \end{bmatrix},$$

$$R_9 = \begin{bmatrix} 1 & -1 & 2 & 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 2 & -1 & 1 & 2 & -1 \\ 1 & 2 & -1 & 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & 2 & 1 & -1 & 2 \end{bmatrix},$$

$$R_{10} = \begin{bmatrix} -1 & 2 & 1 & -1 & -1 & 2 & 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 2 & 1 & -1 & 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 1 & 1 & -1 & 2 & 1 & -1 & -1 \\ 1 & -1 & 2 & 1 & 1 & 1 & -1 & 2 & 1 & 1 \end{bmatrix}.$$

$$R_{11} = \begin{bmatrix} 2 & 2 & -1 & 2 & 2 & 1 & 2 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -1 & 1 & 2 & 1 & 2 & -1 & -1 & 2 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 2 & -1 \end{bmatrix}.$$

Then the pattern R_i, $i \in \{7, 8, 9, 10, 11\}$, induces an SR2DF of $C_4 \,\Box\, C_i$ with the weight $2i$, and the pattern $R_6^k F_i$, $i \in \{1, 2, 3, 4, 5\}$, induces an SR2DF of $C_4 \,\Box\, C_{6k+6+i}$ with the weight $12k + 12 + 2i$. Therefore, all the upper bounds are established.

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