A survey and classification of Sierpiński-type graphs

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Abstract

The purpose of this survey is to bring some order into the growing literature on a type of graphs which emerged in the past couple of decades under a wealth of names and in various disguises in different fields of mathematics and its applications. The central role is played by Sierpiński graphs, but we will also shed some light on variants of these graphs and in particular propose their classification. Concentrating on Sierpiński graphs proper we present results on their metric aspects, domination-type invariants with an emphasis on perfect codes, different colorings, and embeddings into other graphs.

Keywords: Sierpiński triangle; Sierpiński graphs; Hanoi graphs; graph distance; domination in graphs; graph colorings

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Contents

0 Introduction and classification of Sierpiński-type graphs

Graphs[1] whose drawings can be viewed as approximations to the famous *Sierpiński triangle* have been studied intensely in the past 25 years. The interest for these graphs comes from many different sources such as games like the *Chinese rings* or the *Tower of Hanoi*, topology, physics, the study of interconnection networks,

[1] Throughout we will use standard notation from graph theory, where we will mainly follow [114] and [59]. All graphs considered will be simple and connected, unless otherwise stated.

and elsewhere. Therefore it is not surprising that different names have been used for the same object and the same name for different objects. We think it is time to bring some order into this zoo of *Sierpiński-like graphs* and to summarize the properties of the most important species, namely the *Sierpiński graphs* proper. We will therefore start with defining the latter first in order to be able to point out similarities and differences for more general Sierpiński-type graphs.

0.1 Hanoi and Sierpiński graphs

This was precisely the title of [65, Section 2], where the isomorphism between *Hanoi graphs* and a sequence of graphs obtained from approximations to the *Sierpiński triangle* was constructed. Hanoi graphs H_p^n are the state graphs of the *Tower of Hanoi* (TH) game with $p \in \mathbb{N}_3$ pegs and $n \in \mathbb{N}_0$ discs.[2] For a formal definition and for properties of these graphs, see [59, Sections 2.3 and 5.5]. Suffice it here to say that their vertex sets are $V(H_p^n) = [p]_0^n$, where $[p]_0 := \{0, \ldots, p-1\}$ is the p-element segment of $\mathbb{N}_0$, for which we will also write P in the sequel:

$$P = \{0, \ldots, p-1\}\,.$$

Obviously $|H_p^n| = p^n$ so that we will call p the *base* and n the *exponent* of H_p^n. Elements of $[p]_0^n$ will be written in the form $s = s_n \ldots s_1$, where s_d is the label of the peg disc d is lying on in the state represented by s; here discs get labels from $[n] := \{1, \ldots, n\}$ according to increasing size.[3] A vertex $s_n \ldots s_1$ of H_p^n is called *perfect* if $s_1 = \cdots = s_n$. Edges stand for the legal moves of individual discs. The classical case is with base 3; a drawing of the corresponding graph can be found already in [104].

In [77] Klavžar and Milutinović introduced a variant of the TH, namely the *Switching Tower of Hanoi* (STH) for p pegs and n discs, and (an isomorphic image of[4]) its state graph S_p^n which we will call a *(proper) Sierpiński graph*[5] here. Unlike in its famous archetype, where only one disc may be moved at a time, a move of the Switching Tower of Hanoi consists of the exchange of a topmost disc on one peg with the subtower of all smaller discs on top of another peg, including the case where the single disc is the smallest one and the corresponding subtower therefore empty. Such a move of disc 1 only will be called of *type 0*, whereas

[2]We write $\mathbb{N}_k$ for the set of natural numbers greater than or equal to k. The set $\mathbb{N}_1$ is denoted by $\mathbb{N}$, because this is the most natural set of natural numbers.

[3]Consecutive equal entries in a string will be abbreviated with powers; for example $11102222 = 1^3 0 2^4$. Note that i^0 is the empty string.

[4]Cf. [77, Theorem 1].

[5]The name itself was given to S_p^n only later in [78].

any move involving other discs as well will be of *type 1*. Sierpiński graphs were also motivated by investigations of a type of universal topological spaces [93] (see the book of Lipscomb [91] for more information about these spaces) and can be defined for all $p \in \mathbb{N}$ in the following way.

Definition 0.1 *For $p \in \mathbb{N}$ and $n \in \mathbb{N}_0$ the* Sierpiński graph S_p^n *is given by*

$$V(S_p^n) = P^n,\ E(S_p^n) = \left\{\{\underline{s}ij^{d-1}, \underline{s}ji^{d-1}\} \mid i,j \in P,\ i \neq j;\ d \in [n];\ \underline{s} \in P^{n-d}\right\}.$$

Every edge represents one move of the STH, namely the switch of disc d on peg i with the subtower consisting of all $d-1$ smaller discs on peg j, while all discs larger than d remain in their positions, subsumed in $\underline{s}$. A type 0 move corresponds to $d = 1$ (i.e. $d-1=0$); if $d \neq 1$ in an edge, then the corresponding move is of type 1.

In the trivial cases $n = 0$ or $p = 1$, there is only one vertex and no edge, i.e. $S_p^0 \cong K_1 \cong S_1^n$; moreover, $S_p^1 \cong K_p$ with K_p denoting the complete graph of order p. Although it cannot be the function of this survey to collect the facts known about these graphs, we include them for comparison.

The first interesting case is $p = 2$, where $S_2^n \cong P_{2^n}$, the path graph of order 2^n, with the bit strings $s \in [2]_0^n$ arranged in the natural order of their number values in the binary system. Of course, path graphs are also well understood such that, e.g., $\operatorname{diam}(S_2^n) = 2^n - 1$. But it is worth mentioning that $S_2^n \cong R^n$, the state graph of the Chinese rings game and that the isomorphism from S_2^n to R^n is given by the *Gray code*. See [59, Chapter 1] for a comprehensive discussion of the mathematical theory of the Chinese rings.

The first non-trivial result about Sierpiński graphs is [77, Theorem 2], namely $S_3^n \cong H_3^n$ for any n. In other words, both graphs, even living on the same vertex set, can be represented by the same drawing but with different labelings.[6] This difference allows for an alternative approach to the Tower of Hanoi. A beautiful example for using the Sierpiński labeling is due to Romik [102], who solved the so-called *P*2*-decision problem* for the TH by solving it on S_3^n; cf. [59, Section 2.4.3]. He also gave an alternative constuction of the isomorphism between H_3^n and S_3^n in form of an automaton; this is explicated in [59, p. 143–145]. Already before [77] this isomorphism had been anticipated in [104, p. 98] and [65, Lemma 2]. Drawings of Sierpiński graphs S_3^3 and also S_4^2 are shown in Figure 1.

Sierpiński graphs S_3^n being isomorphic to Hanoi graphs H_3^n and therefore helpful in the study of the classical TH tasks, one might wonder whether S_p^n can be employed in the mathematical theory of the TH with $p \geq 4$ pegs.

[6] A (vertex-)**labeling** is a specification of a representative from a class of isomorphic graphs, i.e. an individual graph of that class.

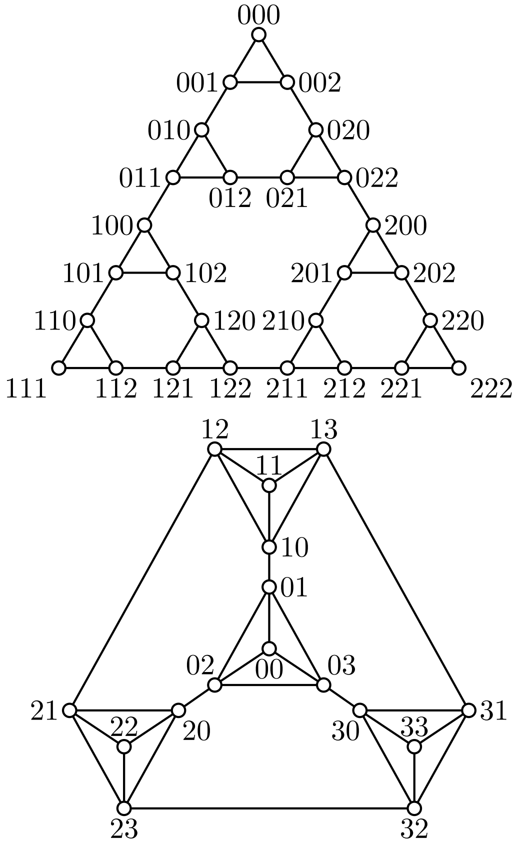

Figure 1: Examples of Sierpiński graphs: S_3^3 (top) and S_4^2 (bottom)

It was Henry Ernest Dudeney who in his book [29] from 1907 proposed the extension of the problem to more than 3 pegs.[7] His game, *The Reve's puzzle*, included 4 stools instead of pegs, and loaves of cheese instead of inedible discs, but the glove was thrown.[8] The extension of the original game to $p \geq 4$ pegs is the most intriguing generalization of the original TH. For 3 pegs, many aspects of the puzzle have been studied, starting with the minimal number of moves to transfer the entire stack of discs to another peg. For a comprehensive summary of known results see [59, Chapter 2]. When we introduce the 4th peg, or even more, to the classical problem, we enter completely unfamiliar territory. The

[7]Édouard Lucas, the inventor of the original TH, marketed a 5-peg version for the Paris World's Fair in 1889, but strangely enough posed other tasks than the classical one.

[8]Dudeney, under the pseudonym "Sphinx", had posed the 4-pegs, 10-discs problem already in 1896 in *The Weekly Dispatch*. (Historical documents from P. K. Stockmeyer.)

first mathematicians to boldly cross its borders after Dudeney were Frame and Stewart, who, in 1941 ([35] and [105], respectively), independently came up with solutions to get from one *perfect state* s, i.e. with all discs on the same peg, say, $s = 0^n$, to another one, $(p-1)^n$, say. They used different approaches to arrive at similar conclusions for the minimal number of moves to solve the task, now jointly called the *Frame-Stewart Conjecture* (FSC) because they missed to prove optimality. See [59, Sections 5.1 and 5.4] for an extended discussion. Recently, the case $p = 4$, i.e. The Reve's puzzle, has been solved by Bousch [12], but to date the FSC for larger p is still open!

Alas! Although the graphs S_p^n and H_p^n are defined on the same vertex set, they cannot be isomorphic anymore for $p > 3$ and $n > 1$. This follows, for instance, from the fact, proved below, that for these values of the parameters $\|S_p^n\| < \|H_p^n\|$, where $\|G\|$ denotes the size of a graph G. (Obviously, $H_p^0 \cong K_1 \cong S_p^0$ and $H_p^1 \cong K_p \cong S_p^1$ for all $p \geq 3$.) Another big difference is the fact that subgraphs iH_p^n and jH_p^n are linked by $(p-2)^n$ edges in H_p^{1+n} (see [59, (5.7)]) and not just by one as in Sierpiński graphs. Therefore there is a greater choice for paths in Hanoi graphs, leading to smaller diameters than the p-independent $\mathrm{diam}(S_p^n) = 2^n - 1$, which is always attained by a pair of *extreme vertices* $k \ldots k = k^n$ (cf. Proposition 2.12 below). From the theoretical point of view it is, however, somewhat disturbing that no better upper bound for the diameter of Hanoi graphs has been found so far and that the only known lower bound comes from the trivial fact that in a typical perfect-to-perfect task each disc, except the largest one, has to move at least twice, once away from atop the largest and once back onto it (cf. [59, Section 5.6]): $2n - 1 \leq \mathrm{diam}(H_p^n) \leq 2^n - 1$.

Clearly, S_p^n contains p extreme vertices and they are of degree $p-1$; all the other vertices are of degree p. The degree sequence of H_p^n is more complex; see [59, Proposition 5.24].

Many results on Sierpiński graphs are proved making use of their recursive structure with respect to $n \in \mathbb{N}_0$ when $p \in \mathbb{N}$ is fixed. It becomes visible in an alternative definition of the edge sets (recall that $E(S_p^0) = \emptyset$):

Proposition 0.2 *For every $p \in \mathbb{N}$ and $n \in \mathbb{N}_0$:*

$$E(S_p^{1+n}) = \{\{is, it\} \mid i \in P\,, \{s,t\} \in E(S_p^n)\} \cup \{\{ij^n, ji^n\} \mid i, j \in P\,, i \neq j\}\,.$$

That means, to get S_p^{1+n}, we take p copies of S_p^n, concatenate a $j \in P$ to the left of the labels of the vertices in each copy to obtain jS_p^n,[9] respectively, and join copies iS_p^n and jS_p^n, $i \neq j$, by the single edge $\{ij^n, ji^n\}$, which will be denoted by

[9]The subgraph of S_p^{1+n} induced by those vertices, which have the prefix $\underline{s} \in P^{1+n-\nu}$, $\nu \in [n]$, in common, is denoted by $\underline{s}S_p^\nu$ and is isomorphic to S_p^ν.

$e_{ij}^{(n+1)}$; note that these edges, representing moves involving disc $n+1$, are pairwise disjoint, i.e. non-adjacent, if $n \geq 1$. We can generalize this concept by considering the edge $\underline{s}e_{ij}^{(d)}$, $d \in [n+1]$, between subgraphs $\underline{s}iS_p^{d-1}$ and $\underline{s}jS_p^{d-1}$, $\underline{s} \in P^{1+n-d}$. For $p \geq 3$, edges of the form $\underline{s}e_{ij}^{(d)}$, $d > 1$, will be called *non-clique edges*, since they are not included in any of the p-cliques.[10] They correspond to the moves of type 1 in the STH. Accordingly, all other edges, the *clique edges*, correspond to the moves of type 0 in the STH.

The case $p = 2$ is a bit special: all non-trivial cliques are 2-cliques. (For $p = 1$ there are no non-trivial cliques, of course.) For consistency, we will nevertheless only call $\{\underline{s}0, \underline{s}1\}$, $\underline{s} \in [2]_0^n$, *clique edges* of S_2^{1+n}.

Obviously, for all $p \geq 2$, there are $p^n \binom{p}{2}$ clique edges in S_p^{1+n}, groups of $\binom{p}{2}$ of them forming a p-clique $\underline{s}S_p^1$. By induction on n one can prove the following result.

Proposition 0.3 [59, Theorem 4.3] *If $p \in \mathbb{N}_2$ and $n \in \mathbb{N}_0$, then the only maximal cliques (with respect to inclusion) in S_p^{1+n} are the p-cliques $\underline{s}S_p^1$ with $\underline{s} \in P^n$ and 2-cliques induced by the non-clique edges. In particular, the* clique number *of S_p^{1+n}, i.e. the order of a largest clique in the graph, is $\omega(S_p^{1+n}) = p$.*

The analogous result for Hanoi graphs reads as follows.

Proposition 0.4 *If $p \in \mathbb{N}_3$ and $n \in \mathbb{N}_0$, then every non-trivial clique of H_p^{1+n} is induced by edges corresponding to moves of one and the same disc. The only p-cliques of H_p^{1+n} are of the form $\underline{s}H_p^1$ with $\underline{s} \in P^n$, and $\omega(H_p^{1+n}) = p$.*

Proof. Take any vertex s joined to two vertices s' and s'' by edges corresponding to the moves of two different discs. Then the positions of these discs differ in s' and s''. Since vertices in H_p^{1+n} can only be adjacent if they differ in precisely one coordinate, s' and s'' cannot be adjacent. This proves the first assertion. Any state s is contained in the p-clique induced by s and those states which differ from s only by the position of the smallest disc. On the other hand, a disc $d \neq 1$ can be transferred to at most $p - 2$ pegs, namely those not occupied by disc 1, so that no clique larger than p exists. □

These immediate results on Sierpiński graphs will be complemented by further basic properties such as planarity, hamiltonicity, connectivity, and complexity in Chapter 1.

[10] A (q-)*clique* of a graph G is a complete subgraph of G (of order q); 0- and 1-cliques will be called *trivial*.

In the most extensive part of this survey paper, Chapter 2, we will address metric properties of Sierpiński graphs. Two of the most important results are definitely the distance lemma [77, Lemma 4] and [77, Theorem 5], which already in 1997 started the chase for metric properties. The first ones to join the competition, almost a decade later, were Romik [102] with the afore mentioned decision automaton for shortest paths in the classical case (i.e., for $p = 3$), and Parisse [94], with numerous results such as diameter, eccentricity and other distance-related outcomes. For instance, Wiesenberger pitched in with the average distance on Sierpiński graphs in his diploma thesis [115] in 2010. The latest contribution is the generalization of Romik's automaton to an arbitrary p by Hinz and Holz auf der Heide [57]. Their algorithm almost compensates the fact that no explicit formula exists for the distance between two arbitrary vertices of Sierpiński graphs due to the phenomenon that the largest disc may move more than once in an optimal solution. While there can be as many as $p - 1$ so-called *Largest Disc Moves* (LDMs) in the Hanoi case H_p^n (cf. [4]), there are only at most two, and consequently at most two optimal paths, in S_p^n [77, Theorem 6].

In Chapter 3 we turn to various domination-type problems of Sierpiński graphs. The first such problem studied was the existence of perfect codes in S_3^n [23] and more generally in S_p^n [78]. It turned out that such codes always exist which in turn immediately yields the domination number of Sierpiński graphs. Related topics that will be addressed include (a, b)-codes, identifying codes, $L(2, 1)$-labelings, total domination number, generalized power domination, global strong defensive alliance, and hub number.

We then continue with a chapter in which results on coloring aspects of Sierpiński graphs are collected. In the first part of the chapter standard colorings are considered: vertex-, edge-, and total-colorings. In addition to them, b-colorings, $\{P_r\}$-free colorings, path t-colorings, linear t-colorings, the edge ranking number, and the packing chromatic number of Sierpiński graphs were all investigated. The main results concerned are listed in the second part of Chapter 4.

The subsequent Chapter 5 is devoted to various types of embeddings of Sierpiński graphs. Although in general Hanoi and Sierpiński graphs are not isomorphic for $n > 1$, we may nevertheless connect the newly deduced metric properties of S_p^n with Hanoi graphs. In Section 5.1 we will study embeddings of Sierpiński graphs into Hanoi graphs. In particular, we will deal with the question whether S_p^n is a spanning subgraph of H_p^n. In the subsequent section we will turn our attention to embeddings of Sierpiński graphs into Cartesian product graphs. We will explicitly determine the so-called canonical metric representation of Sierpiński graphs and investigate their Hamming dimension. We will also describe an embedding of Sierpiński graphs into the Cartesian product of

Sierpiński triangle graphs.

We will conclude this survey in Chapter 6 with some open problems and new developments such as *subdivided-line graphs* and graphs recursively constructed from general graphs. But before we concentrate on the species "Sierpiński graph", we will give an overview over the genus "Sierpiński-type graph".

0.2 Sierpiński-type graphs

In this section we will standardize and harmonize the terms of Sierpiński graphs, Sierpiński triangle graphs etc., which we now all address as **Sierpiński-type graphs**. We can characterize a representative of Sierpiński-type graphs as a graph which is derived from or leads to the Sierpiński triangle, one of the most popular fractals. The main constituting classes of Sierpiński-type graphs are shown in Figure 2.

The first row of the diagram in Figure 2 is composed of the origins of Sierpiński-type graphs, starting with the classical Hanoi graphs H_3^n. In 1990, the graphs S_3^n were introduced and employed to determine the average distance on the Sierpiński triangle by Hinz and Schief [65]. There the name "Sierpiński graphs" was used for the first time. In [65] the authors also proved that $S_3^n \cong H_3^n$, a fact represented in Figure 2 by an arrow between H_3^n and S_3^n in both directions. There is also such an arrow between S_3^n and Sierpiński triangle graphs $\widehat{S}_3^n$. The reason for the direction $S \to \widehat{S}$ is the way we will define Sierpiński triangle graphs in Definition 0.5 below. The other direction can be derived by interpreting each (clique) triangle of $\widehat{S}_3^n$ as a new vertex and connecting two of these vertices by an edge if the corresponding triangles share a vertex in $\widehat{S}_3^n$. Note that this does *not* imply isomorphy between S_3^n and $\widehat{S}_3^n$.

The name "Sierpiński graphs" was given by some authors to the graphs which we now call Sierpiński triangle graphs $\widehat{S}_3^n$. The list of names for them is hereby far from over. Often they were called *Sierpiński gasket graphs* or either *Sierpiński sieve graphs*. Some authors even call graphs $\widehat{S}_3^n$ (or either H_3^n; cf. [1]) just *Sierpiński gasket*, which is actually one of the names of the Sierpiński triangle fractal and is therefore even more confusing.

Let us move to the second row of the diagram in Figure 2. As $S_3^n \cong H_3^n$, the idea arose in 1997 to introduce the family of Sierpiński graphs $S(n,k)$ (in our notation S_p^n, where we replaced k by p for "pegs") as the state graphs of the Switching Tower of Hanoi [77]. So the graphs S_3^n were generalized to S_p^n, where $p \in \mathbb{N}$. In a similar way how Sierpiński triangle graphs $\widehat{S}_3^n$ are constructed from graphs S_3^n, we can perform this for any p; see Definition 0.6 below. The family of *generalized Sierpiński triangle graphs* was first introduced by Jakovac in [71].

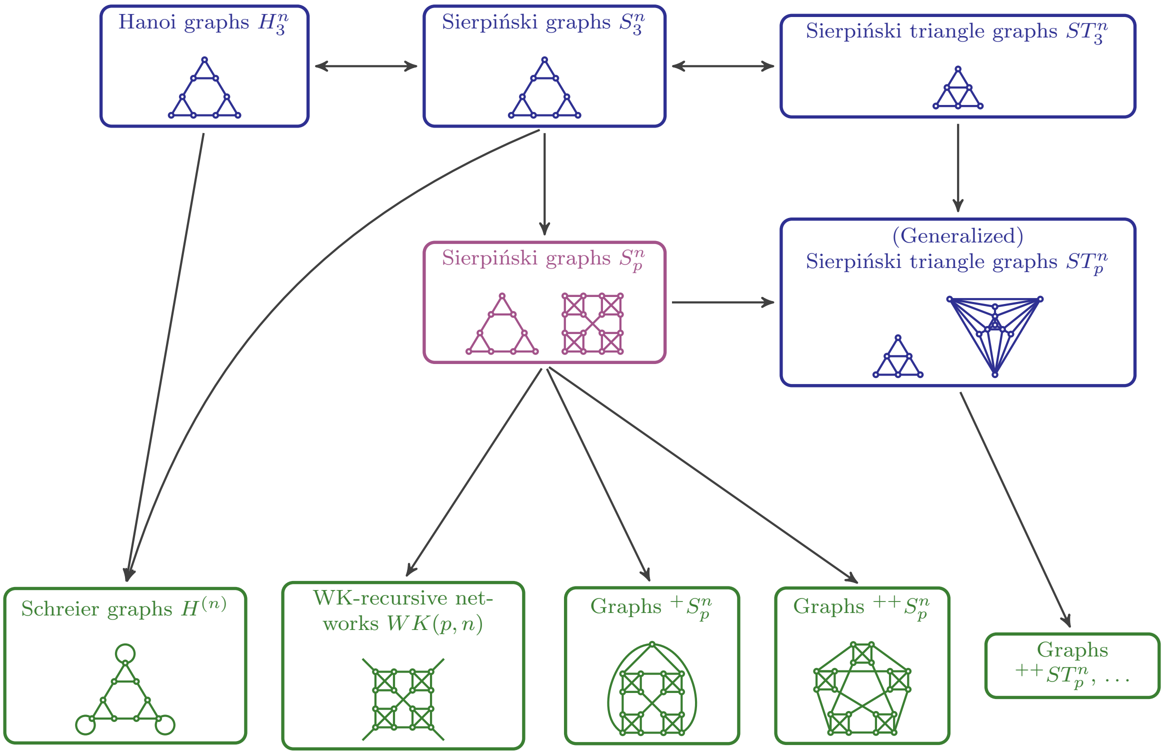

Figure 2: A diagram of Sierpiński-type graphs

He used the notation $S[n,k]$ for the graphs which we now denote by $\widehat{S}_p^n$ (with k again replaced by p) and called them *generalized Sierpiński gasket graphs*. Later we decided to call them generalized Sierpiński triangle graphs, but we first used the notation $\widehat{S_k^n}$ in [82] with the hat covering everything. Having overcome this tentative notation, we finally arrived at the present notation $\widehat{S}_p^n$.

Proceeding further down in Figure 2, there are a couple of ways to regularize Sierpiński (triangle) graphs into the graphs ${}^+S_p^n$ and ${}^{++}S_p^n$ (${}^+\widehat{S}_p^n$ and ${}^{++}\widehat{S}_p^n$). Other similar families are the *WK-networks* and *Schreier graphs* (for $p = 3$), which have some additional open edges or loops, respectively. All these families will be collected in the class of **Sierpiński-like graphs**. They are similar to Sierpiński graphs, but not isomorphic to them and could also be addressed as *regularizations of Sierpiński graphs*, see Section 0.2.2. The rightmost family represented in the bottom row of Figure 2 is a regularization of Sierpiński triangle graphs ${}^{++}\widehat{S}_p^n$ and has not been introduced before. This regularization can be achieved in a similar way as in the case of the graphs ${}^{++}S_p^n$.

To summarize the above discussion we group *Sierpiński-type graphs* into three classes:

- (proper) Sierpiński graphs,
- (generalized) Sierpiński triangle graphs,
- Sierpiński-like graphs.

In the first group we find, apart from the classical definition of Section 0.1, two approaches making use of iteration. The first are truncations of maps, studied by Pisanski and Tucker [97]. The *truncation* $\tau(G)$ of a graph G is obtained from G by replacing each vertex v with a clique Q_v whose order is equal to the degree of v. In addition, if $uv \in E(G)$, then one vertex of Q_v is adjacent to one vertex of Q_u and no vertex of Q_v is adjacent to more than one vertex outside Q_v, cf. [3, Definition 1.2]. Then $\tau(S_3^1) \cong S_3^2$. Repeating this, we get $\tau^n(S_3^1) \cong S_3^{n+1}$.

The recursive definition of Sierpiński graphs also gives rise to the interpretation as *iterated complete graphs*, sometimes denoted by K_p^n.[11] See, for instance, the article by Cull, Merrill, and Van [22] or papers of Cull's students like, e.g., [74, 113], which were compiled during *Summer Research Experiences for Undergraduates Program in Mathematics* at Oregon State University. They considered these graphs (isomorphic to Sierpiński graphs) in relation to codes. For odd p the graphs K_p^n were interpreted as state graphs of some artificial variants of the Tower of Hanoi game. For even values of p they generalized the idea of the *spin-out puzzle*, which is the same as Keister's *Locking disc puzzle* and therefore essentially the same as the Chinese rings; cf. [59, p. 65]. So the K_p^n can be viewed as siblings of S_p^n.

0.2.1 Sierpiński triangle graphs

A class of graphs that often has been mistaken for and also been called Sierpiński graphs can be obtained from the latter by simply contracting all non-clique edges. We will call them *Sierpiński triangle graphs*, because for $p = 3$ their drawings in the plane represent approximations of the *Sierpiński triangle* (ST) fractal; see [59, Figure 0.14 and Section 4.3]. Actually, many papers referring to these cousins of Sierpiński graphs are only using their drawings and do not consider graph properties at all. For instance, the study of a diffusion process on ST arising as the limit of random walks on Sierpiński triangle graphs in [8] opened a new field of analysis on fractals; cf. [106]. Sierpiński triangle graphs were also used mistakenly

[11]This must not be mixed up with the Cartesian product of n copies of K_p.

as "problem spaces" of the TH game in psychological tests; for a discussions of this issue, see [61, 56].

Sierpiński triangle graphs can be defined in various ways, but basically all of these definitions originate in the Sierpiński triangle fractal. We will use the notation $\widehat{S}_3^n$ for Sierpiński triangle graphs, which will make sense when generalizing them to $\widehat{S}_p^n$ for arbitrary $p \in \mathbb{N}$.

Definition 0.5 *Let* $n \in \mathbb{N}_0$. *Then the* class of the Sierpiński triangle graph $\widehat{S}_3^n$ *is obtained from* S_3^{n+1} *by contracting all non-clique edges.*

Apart from the (generic) Sierpiński graphs S_p^n, these graphs have been most commonly studied in literature. Here we will first give two different representations of their vertex sets.

One way to label Sierpiński triangle graphs is by iteration. We start with a complete graph on 3 vertices, $\widehat{S}_3^0 \cong K_3$, and label it with $V(\widehat{S}_3^0) = \widehat{T} := \{\hat{0}, \hat{1}, \hat{2}\}$. These labels will be regarded as being of *length* 0. Now assume we have constructed $\widehat{S}_3^n$. To obtain $\widehat{S}_3^{n+1}$ we subdivide each edge of every triangle of $\widehat{S}_3^n$ and connect any two of the three new vertices of a triangle by a new edge. An easy way to explain how we label them is with the help of Sierpiński graphs. We inscribe S_3^{n+1} into the partially labeled graph and mirror the labels of the Sierpiński graph S_3^{n+1} on every unlabeled triangle. An example is shown in Figure 3 with the underlying Sierpiński triangle graph $\widehat{S}_3^3$ and the Sierpiński graph S_3^3.

By this construction we get

$$V(\widehat{S}_3^n) = \widehat{T} \cup \{\ s \in T^\nu \mid \nu \in [n]\}\ .$$

For reasons stemming from the Tower of Hanoi game, we will call this labeling the *idle peg labeling* of $\widehat{S}_3^n$. (This will make sense later when we describe the connection between both discussed labelings.) Obviously,

$$V(\widehat{S}_3^{n+1}) = V(\widehat{S}_3^n)\ \cup\ V(S_3^{n+1})\,,$$

and the edge set can be described explicitly as

$$\begin{aligned} E(\widehat{S}_3^{n+1}) \ = \ & \Big\{\{\hat{k}, k^n j\} \mid k \in T,\ j \in T \setminus \{k\}\Big\} \ \cup \\ & \big\{\{\underline{s}k, \underline{s}j\} \mid \underline{s} \in T^n,\ \{j,k\} \in \tbinom{T}{2}\big\} \ \cup \\ & \big\{\{\underline{s}(3-i-j)i^{n-\nu}k, \underline{s}j\} \mid \underline{s} \in T^{\nu-1},\ \nu \in [n],\ i \in T,\ j,k \in T \setminus \{i\}\big\}\ . \end{aligned} \tag{1}$$

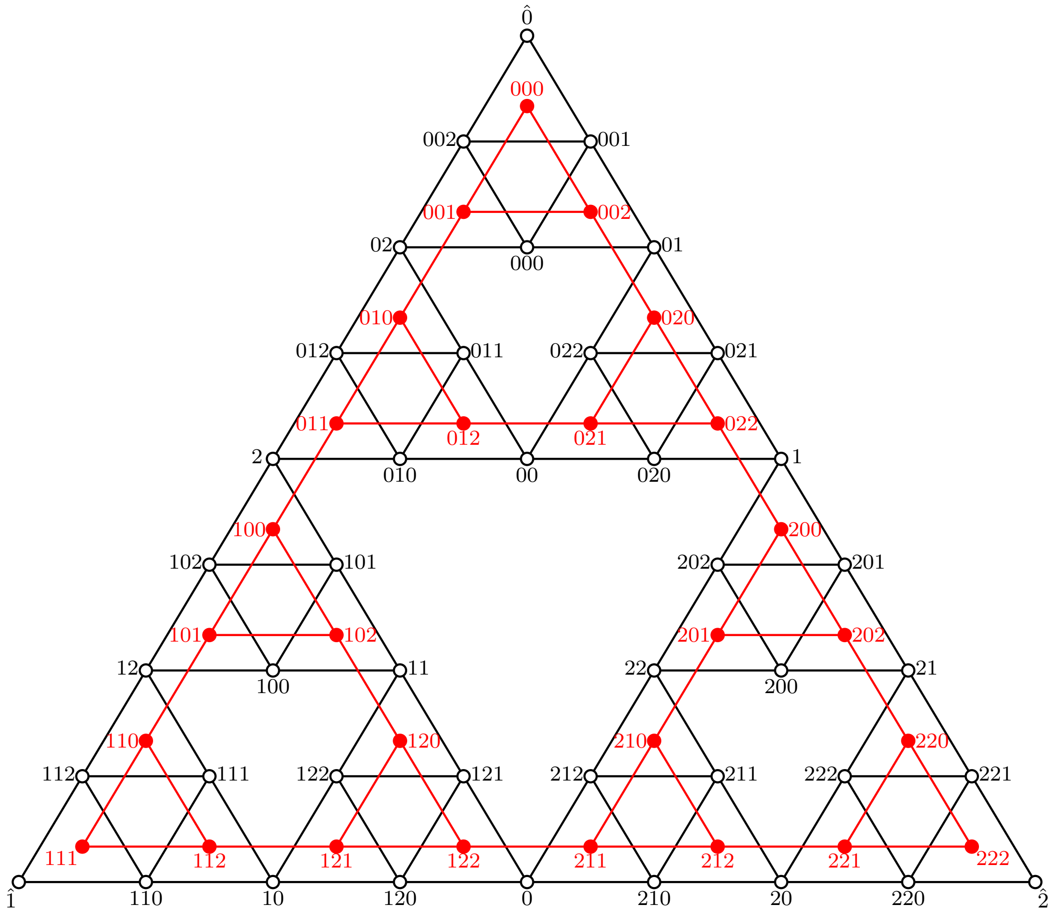

Figure 3: Combined Sierpiński triangle graph $\widehat{S}_3^3$ (black) and Sierpiński graph S_3^3 (red)

From the definition of the graphs $\widehat{S}_3^n$ we can derive another family of labeled Sierpiński triangle graphs. Denote the vertex obtained by contracting the edge $\{\underline{s}ij^{n-\nu+1}, \underline{s}ji^{n-\nu+1}\}$ by $\underline{s}\{i,j\}$. Then the vertex set can be written as

$$V(\widehat{S}_3^n) = \widehat{T} \cup \left\{\underline{s}\{i,j\} \mid \underline{s} \in T^{\nu-1},\ \nu \in [n],\ \{i,j\} \in \tbinom{T}{2}\right\} .$$

Let us call this labeling the *contraction labeling* of $\widehat{S}_3^n$. Note that the two definitions of Sierpiński triangle graphs give us labels of different lengths. It is, of course, possible to pass from one labeling to the other. Let $\widehat{S}_3^n$ be labeled with the contraction labeling. The *idle peg* for i and j is defined as $k := 3 - i - j$ (see [59, p. 74]). To obtain the idle peg labeling of $\widehat{S}_3^n$ we simply replace each vertex $\underline{s}\{i,j\}$ by $\underline{s}k$.

Here we have just briefly explained both labelings. Teguia and Godbole [107] studied the basic properties of (base-3-)Sierpiński triangle graphs. They proved that their chromatic number is 3, and that the graphs $\widehat{S}_3^n$ are hamiltonian and pancyclic (i.e. they contain cycles of all lengths from 3 to $|\widehat{S}_3^n| = \frac{3}{2}(3^n+1)$). In the same paper they also determined the domination numbers $\gamma(\widehat{S}_3^n) = 3^{n-1}$, $n \in \mathbb{N}_2$, and $\gamma(\widehat{S}_3^1) = 2$. (Obviously, $\gamma(\widehat{S}_3^0) = 1$.)

The definition of $\widehat{S}_3^n$ by contraction can easily be generalized:

Definition 0.6 *Let $p \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then the* class of the (generalized) Sierpiński triangle graph $\widehat{S}_p^n$ *is obtained by contracting all non-clique*[12] *edges from the Sierpiński graph S_p^{n+1}.*

The vertex set of the graph $\widehat{S}_p^n$ can be written similarly as in the case $n = 3$. Again, we denote the vertex obtained from $\{\underline{s}ij^{n-\nu+1}, \underline{s}ji^{n-\nu+1}\} \in E(S_p^{n+1})$ by $\underline{s}\{i,j\}$. Then (with $\widehat{P} = \{\hat{k} \mid k \in P\}$)

$$V(\widehat{S}_p^n) = \widehat{P} \cup \left\{\underline{s}\{i,j\} \mid \underline{s} \in P^{\nu-1},\ \nu \in [n],\ \{i,j\} \in \tbinom{P}{2}\right\} .$$

Writing the vertex set this way allows us to represent the edge set explicitly in a similar fashion as described for $p = 3$:

$$\begin{aligned} E(\widehat{S}_p^{n+1}) = {} & \left\{\{\hat{k}, k^n\{j,k\}\} \mid k \in P,\ j \in P \setminus \{k\}\right\} \cup \\ & \left\{\{\underline{s}\{i,j\}, \underline{s}\{i,k\}\} \mid \underline{s} \in P^n,\ i \in P,\ \{j,k\} \in \tbinom{P\setminus\{i\}}{2}\right\} \cup \\ & \left\{\{\underline{s}ki^{n-\nu}\{i,j\}, \underline{s}\{i,k\}\} \mid \underline{s} \in P^{\nu-1},\ \nu \in [n],\ i \in P,\ j,k \in P \setminus \{i\}\right\} . \end{aligned} \tag{2}$$

The (generalized) Sierpiński triangle graph $\widehat{S}_4^1$ is shown in Figure 4.

Note that for $p = 3$ the characterization of the edge set in formula (2) is compatible with the one in (1).

By the definition of their vertex sets, we can deduce the order of Sierpiński triangle graphs immediately, while their size follows directly from their construction, since we glue together complete graphs of order p.

Proposition 0.7 [71, Proposition 2.3] *If $p \in \mathbb{N}$ and $n \in \mathbb{N}_0$, then*

$$|\widehat{S}_p^n| = \frac{p}{2}(p^n+1)\,, \quad \textit{and} \quad \|\widehat{S}_p^n\| = \frac{p-1}{2}p^{n+1}\,.$$

[12] Recall that non-clique edges of S_p^{n+1} have the unique form, $\{ij^\ell, ji^\ell\}$, for distinct $i,j \in P$, and $\ell \in [n]$ and correspond to the move of type 1 in the Switching Tower of Hanoi.

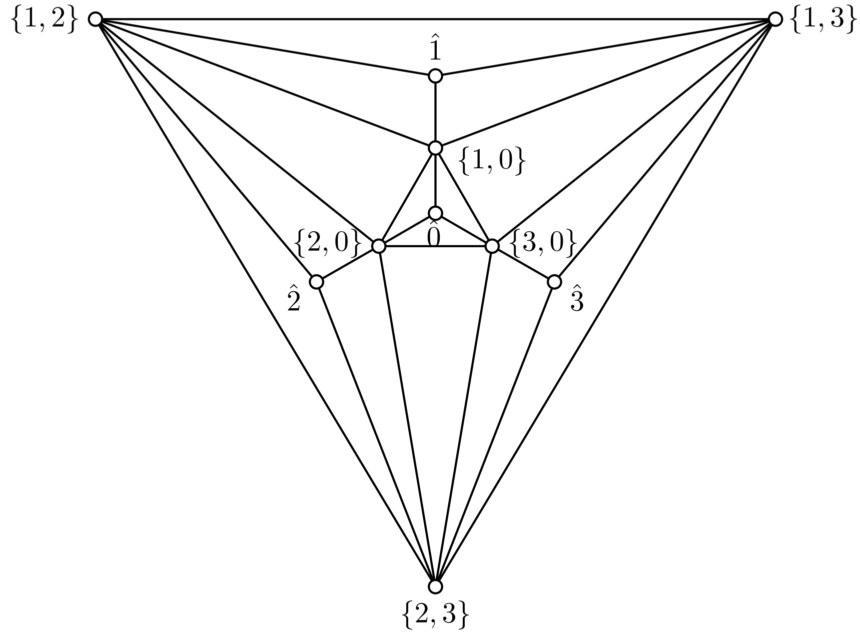

Figure 4: Sierpiński triangle graph $\widehat{S}_4^1$

From Definition 0.6 we can determine the degrees of the vertices in $\widehat{S}_p^n$. An extreme vertex obviously has the same degree as the extreme vertex of S_p^{n+1}, that is $p-1$. All the other vertices have, by contraction, degree $2(p-1)$. Some other features of the graphs $\widehat{S}_p^n$ were studied by Jakovac [71]. He proved that Sierpiński triangle graphs are hamiltonian (for $p \geq 3$) and that their chromatic number equals p. Metric properties of Sierpiński triangle graphs will be investigated in [66]. For instance, the diameter is $\mathrm{diam}(\widehat{S}_p^n) = 2^n$.

0.2.2 Sierpiński-like graphs

Sierpiński-like graphs are obtained from Sierpiński (triangle) graphs with some small modifications. These include regularizations of Sierpiński (triangle) graphs.

Schreier graphs

A family of graphs similar to those constructed by truncation are the *Schreier graphs* of *Hanoi Towers groups*; see [48] and [47]. As opposed to the truncated triangle, the Schreier graphs are not completely isomorphic to the graphs S_3^n because to each extreme vertex a loop is attached. Schreier graphs were introduced in relation to Hanoi Towers groups by Grigorchuk and Šunić [48] and are more closely related to Hanoi graphs.

WK-recursive networks

Another structure very similar to Sierpiński graphs is the class of *WK-recursive networks*. It was introduced by Della Vecchia and Sanges [25] in 1988 as a model for interconnection networks. In fact, $WK(p,n)$ is almost isomorphic to S_p^n: both graphs are defined on the same vertex set $V(WK(p,n)) = P^n = V(S_p^n)$, and the edges are also the same, with the only exception that $WK(p,n)$ has p additional *open edges* (or *links*) incident to the extreme vertices. The open edges serve for further extensions. In this context various properties of these networks have been studied.

In [25], a VLSI implementation of these networks was designed as well as a corresponding message exchange algorithm. Subsequently several structural properties of these networks were derived by Chen and Duh [19]. Among other results they determined the diameter [19, Theorem 2.1] (see Proposition 2.12 below), the connectivity [19, Theorem 2.3] (see Proposition 1.1), and the hamiltonicity [19, Theorem 2.2] (see Theorem 1.7) of these networks. They also considered the routing problem (that is, the problem to transmit a message from a vertex to another vertex) and the broadcasting problem (that is, the problem to transmit a message from a vertex to all the other vertices). More than a decade later Fang et al. [32] follow with a simple broadcasting algorithm. In [38, 39, 67] several results dealing with hamiltonian problems on WK-recursive networks have been obtained; we will present them in Chapter 1. Finally, very recently Savitha and Vijayakumar [103] studied additional invariants on Sierpiński graphs that are important in communication networks—forwarding indices and bisection width.

Regularizations

For $p, n \in \mathbb{N}_2$, all non-extreme vertices of the Sierpiński graph S_p^n have degree p and extreme vertices have degree $p-1$. So Sierpiński graphs are almost regular. (Of course, S_1^n, S_p^0 and S_p^1 *are* regular.) This was the motivation to define two new families of Sierpiński-like graphs. As there are p vertices of degree $p-1$ in S_p^n, there are two natural ways to regularize them: we either add another vertex to S_p^n and connect it with all the extreme vertices or we add another copy of S_p^{n-1} and connect the extreme vertices of S_p^n with the extreme vertices of that S_p^{n-1}, respectively. To understand these two constructions better, see Figure 5 for the case $p = 4$ and $n = 2$. The first possibility gives us the graph ${}^+S_p^n$, where the additional vertex w is called the *special vertex* of ${}^+S_p^n$. More formally:

Definition 0.8 *Let $p, n \in \mathbb{N}$. Then the graph ${}^{+}S_p^n$ is defined by*

$$V({}^{+}S_p^n) = P^n \cup \{w\}\,,$$
$$E({}^{+}S_p^n) = E(S_p^n) \cup \{\{w, i^n\} \mid i \in P\}\,.$$

Directly from the definition of ${}^{+}S_p^n$ and the size of S_p^n, we get

Proposition 0.9 *If $p, n \in \mathbb{N}$, then*

$$|{}^{+}S_p^n| = p^n + 1 \quad \textit{and} \quad \|{}^{+}S_p^n\| = \frac{p}{2}(p^n + 1)\,.$$

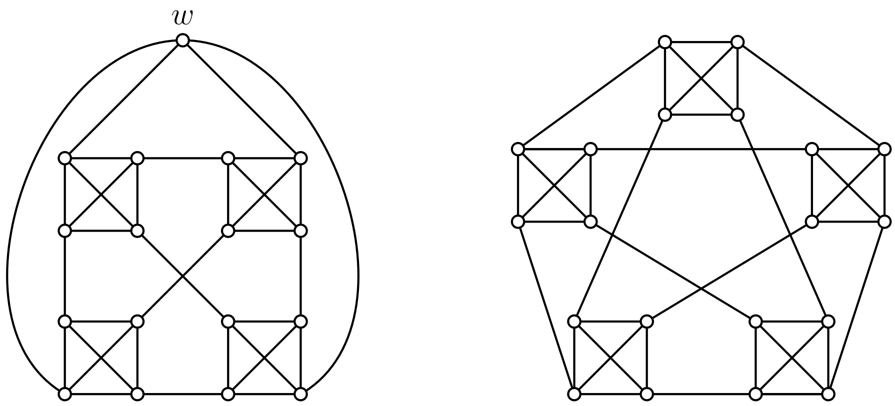

Figure 5: Regularizations ${}^{+}S_4^2$ (left) and ${}^{++}S_4^2$ (right)

The other regularization, i.e. adding another copy of S_p^{n-1} to S_p^n, is denoted by ${}^{++}S_p^n$. It can also be characterized as taking $p+1$ copies of S_p^{n-1} (when building a Sierpiński graph S_p^n we take only p such copies) and joining their extreme vertices in the sense of the complete graph K_{p+1}. On the right-hand side of Figure 5 there are 5 copies of K_4 joined together as K_5: we may think of the K_4s as the vertices of K_5. This construction is similar to the construction of a Sierpiński graph, but with complete graphs of different orders. Here is a formal definition:

Definition 0.10 *Let $p, n \in \mathbb{N}$. Then the graph ${}^{++}S_p^n$ is defined by*

$$V({}^{++}S_p^n) = P^n \cup \{p\bar{s} \mid \bar{s} \in P^{n-1}\}\,, \tag{3}$$
$$E({}^{++}S_p^n) = E(S_p^n) \cup \left\{\{p\bar{s}, p\bar{t}\} \mid \{\bar{s}, \bar{t}\} \in \binom{P^{n-1}}{2}\right\} \cup \left\{\{pi^{n-1}, i^n\} \mid i \in P\right\}\,. \tag{4}$$

Again we can immediately deduce the order and the size of the graphs ${}^{++}S_p^n$:

Proposition 0.11 *If $p, n \in \mathbb{N}$, then*

$$|{}^{++}S_p^n| = (p+1)p^{n-1}\,, \quad \textit{and} \quad \|{}^{++}S_p^n\| = \frac{p+1}{2}p^n\,.$$

For an advanced property of regularized Sierpiński graphs see Theorem 1.5.

1 Basic properties of Sierpiński graphs

In this chapter we present fundamental structural properties of Sierpiński graphs: their order and size, connectivity and edge connectivity, planarity and crossing number, automorphism group, hamiltonicity, and number of spanning trees and matchings.

Order and size

Recall that from the definition of $V(S_p^n) = P^n$, we have $|S_p^n| = p^n$. Moreover, by Proposition 0.2, the number of edges of S_p^n fulfils the recurrence

$$\|S_p^0\| = \emptyset,\ \forall\, n \in \mathbb{N}_0:\ ||S_p^{1+n}|| = p \cdot \|S_p^n\| + \tbinom{p}{2}\,,$$

which solves to

$$\|S_p^n\| = \binom{p}{2}\sum_{d=1}^{n} p^{n-d} = \frac{p}{2}(p^n - 1)\,.$$

For $p = 3$ this applies to Hanoi graphs H_3^n by isomorphy. But when building a Hanoi graph H_p^{1+n} for $p > 3$, we add more edges between each of the p copies of H_p^n than we did with Sierpiński graphs. These edges correspond to the moves of the largest disc. So when moving disc $n+1$ from peg i to peg j, the discs 1 to n are neither on peg i nor on peg j. This means that there are $(p-2)^n$ edges between iH_p^n and jH_p^n, whereas in the case of Sierpiński graphs the edge between iS_p^n and jS_p^n is unique. This leads to the recurrence

$$\|H_p^0\| = \emptyset,\ \forall\, n \in \mathbb{N}_0:\ ||H_p^{1+n}|| = p \cdot \|H_p^n\| + \tbinom{p}{2}(p-2)^n\,,$$

which gives us

$$\|H_p^n\| = \frac{p(p-1)}{4}(p^n - (p-2)^n)$$

and, moreover, $\|S_p^n\| < \|H_p^n\|$ for $p \in \mathbb{N}_4$ and $n \in \mathbb{N}_2$.

Connectivity and edge-connectivity

The extreme vertices of S_p^n are of degree $p-1$, all the other vertices are of degree p. Hence the minimum degree is $\delta(S_p^n) = p-1$ and the maximum degree is $\Delta(S_p^n) = p$, if $p \in \mathbb{N}_2$. Sierpiński graphs are connected which can be shown by a simple induction argument. Actually, its connectivity κ and its edge-connectivity κ' are as large as possible, that is, $\kappa(S_p^n) = \kappa'(S_p^n) = \delta(S_p^n)$ as the next result asserts.

Proposition 1.1 [19, Theorem 2.3], [59, Exercise 4.7] *If $p, n \in \mathbb{N}$, then $\kappa(S_p^n) = \kappa'(S_p^n) = p-1$.*

Proposition 1.1 can be proved by first showing by induction that removing arbitrary $p-2$ vertices does not disconnect S_p^n. All the rest then follows from the fact that $\kappa(G) \le \kappa'(G) \le \delta(G)$ holds for any graph G.

Planarity

To determine which Sierpiński graphs are planar, recall first that $S_1^n \cong K_1$ and $S_2^n \cong P_{2^n}$, so that S_p^n is planar for $p = 1, 2$ and $n \in \mathbb{N}_0$. In addition, the standard drawing of the graphs S_3^n demonstrates that all base-3-Sierpiński graphs are planar. The graphs $S_4^1 \cong K_4$ and S_4^2 are planar, but S_4^3 is not planar: a planar drawing of S_4^2 is shown in Figure 1 and a K_5 subdivision in S_4^3 is indicated by filled vertices in the drawing of Figure 6.

Since S_4^3 is contained in any S_4^n for $n \ge 3$, none of the latter graphs is planar. Any S_p^n contains K_p as a subgraph and is consequently not planar for $p \ge 5$. We have thus arrived at the following result.

Proposition 1.2 *The planar Sierpiński graphs are*
(i) S_p^n, $p \in [3]$, $n \in \mathbb{N}_0$ (ii) $S_p^0 \cong K_1$, $p \in \mathbb{N}$ (iii) $S_4^1 \cong K_4$ and (iv) S_4^2.

For the corresponding statement for Hanoi graphs see [62, Theorem 2].

Crossing number

Because most of the Sierpiński graphs are not planar, it is natural to study their crossing numbers. Recall that the crossing number, $\mathrm{cr}(G)$, of a graph G, is the minimum number of crossings of edges over all possible drawings of G in the plane. In the case the base-4-Sierpiński graphs the following holds.

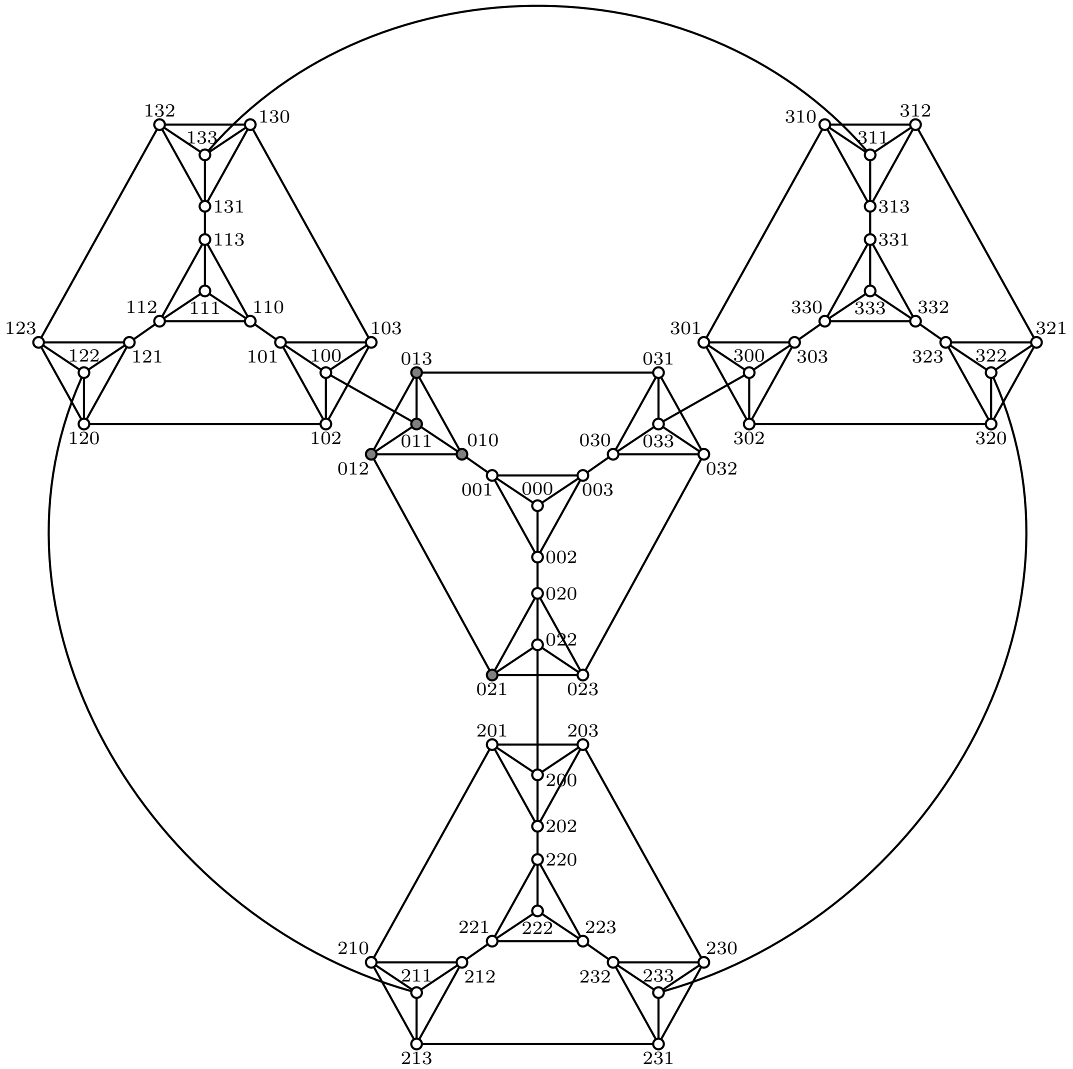

Figure 6: A drawing of S_4^3 with 12 crossings

Theorem 1.3 [79, Proposition 3.2] *If* $n \in \mathbb{N}_3$*, then*

$$\frac{3}{16}4^n \;\le\; \mathrm{cr}(S_4^n) \;\le\; \frac{1}{3}4^n - \frac{12n-8}{3}\,.$$

Note that for $n = 3$ upper and lower bound in Theorem 1.3 coincide, whence $\mathrm{cr}(S_4^3) = 12$ and the drawing of S_4^3 from Figure 6 is optimal. For $p \ge 5$, the following upper bound was proved in [79]:

$$\mathrm{cr}(S_p^n) \leq \frac{p(p^{n-1}-1)}{p-1} \cdot \mathrm{cr}(K_{p+1}) + \mathrm{cr}(K_p)\,.$$

The crossing number of Sierpiński graphs was further studied by Köhler [84]. For $n = 2$ he expressed the crossing number of S_p^2 with the crossing number of the complete graphs and the graphs K_n^- we get from complete graphs K_n by removing one edge as follows.

Theorem 1.4 [84, Satz 3.11] *If* $p \in \mathbb{N}$*, then*

$$\mathrm{cr}(S_p^2) = p \cdot \mathrm{cr}(K_{p+1}^-) + \mathrm{cr}(K_p)\,.$$

To determine the crossing number for all Sierpiński graphs seems to be a very difficult problem; after all, $S_p^1 \cong K_p$. On the other hand, a bit surprisingly, we have the following result for their regularizations ${}^+S_p^n$ and ${}^{++}S_p^n$.

Theorem 1.5 [79, Theorem 4.1] *If* $p \in \mathbb{N}_2$ *and* $n \in \mathbb{N}$*, then*

$$\mathrm{cr}({}^+S_p^n) = \frac{p^n-1}{p-1}\,\mathrm{cr}(K_{p+1})$$

and

$$\mathrm{cr}({}^{++}S_p^n) = \frac{(p+1)p^{n-1}-2}{p-1}\,\mathrm{cr}(K_{p+1})\,.$$

Automorphism group

Since graph automorphisms preserve the degrees, any automorphism of S_p^n maps the set of extreme vertices onto itself. Actually, any permutation of extreme vertices of S_p^n leads to an automorphism of S_p^n. Moreover, these are the only symmetries, which can be proved, for instance, using the fact that any vertex of S_p^n is uniquely determined by its distances from extreme vertices. Hence:

Theorem 1.6 [79, Lemma 2.2] *For any* $p, n \in \mathbb{N}$, $\mathrm{Aut}(S_p^n) \cong \mathrm{Sym}(P)$.

The same result holds for Hanoi graphs; see [95] or [59, Theorem 5.33].

Hamiltonicity

The hamiltonicity of Sierpiński graphs was observed already in the seminal papers on WK-recursive networks and on Sierpiński graphs:

Theorem 1.7 [19, Theorem 2.2], [77, Proposition 3] *If $p \in \mathbb{N}_3$ and $n \in \mathbb{N}$, then the graph S_p^n is hamiltonian.*

A hamiltonian cycle of S_p^n can be constructed as follows. Let $iQ_{j,k}^{(n-1)}$ be a path in iS_p^{n-1} between vertices ij^{n-1} and ik^{n-1}, such that it includes all the vertices from iS_p^{n-1} (such a path exists, for example use induction to prove it). Then we can build a hamiltonian cycle with

$$0Q_{(p-1),1}^{(n-1)} \cup e_{01}^{(n)} \cup 1Q_{0,2}^{(n-1)} \cup e_{12}^{(n)} \cup \cdots \cup e_{(p-2)(p-1)}^{(n)} \cup (p-1)Q_{(p-2),0}^{(n-1)} \cup e_{(p-1)0}^{(n)} .$$

In the case $p = 3$ the Sierpiński graphs contain a unique hamiltonian cycle [76]. Xue, Zuo, and Li [116] deepened the study of hamiltonicity of Sierpiński graphs by proving the following result.

Theorem 1.8 [116, Theorem 3.1] *If $p \in \mathbb{N}_2$ and $n \in \mathbb{N}$, then S_p^n can be decomposed into an edge-disjoint union of $\lfloor \frac{p}{2} \rfloor$ hamiltonian paths the end vertices of which are extreme vertices.*

Moreover, they also determined the number of edge-disjoint hamiltonian cycles of S_p^n.

Theorem 1.9 [116, Theorem 3.2] *If $p \in \mathbb{N}_3$ and $n \in \mathbb{N}$, then S_p^n contains $\lceil \frac{p}{2} \rceil - 1$ edge-disjoint hamiltonian cycles.*

Theorem 1.9 was, apparently unknown to the authors of [116], earlier proved for the case when p is odd [67, Theorem 2].

A graph G is *hamiltonian-connected* if for any two distinct vertices of G there exists a hamiltonian path between them. Fu [38, 39] studied the hamiltonian-connectivity of Sierpiński graphs and proved that if $p \in \mathbb{N}_4$, then S_p^n remains hamiltonian-connected even if $p - 4$ vertices are removed from it.

Number of spanning trees and matchings

The number of spanning trees of a graph G is denoted by $\tau(G)$ and known as the *complexity* of G. The standard approach to obtain the complexity, namely using Kirchhoff's Matrix Tree Theorem (cf. [75]), seems not to be applicable to Sierpiński graphs. Nevertheless, Teufl and Wagner proved the following remarkable result.

Theorem 1.10 [109, p. 892] *If $n \in \mathbb{N}_0$, then the complexity of S_3^n equals*

$$\tau(S_3^n) = 3^{\frac{1}{4}(3^n-1)+\frac{1}{2}n} \cdot 5^{\frac{1}{4}(3^n-1)-\frac{1}{2}n} = \left(\sqrt{\frac{3}{5}}\right)^n \left(\sqrt[4]{15}\right)^{3^n-1} .$$

An entirely combinatorial proof of Theorem 1.10 was given by Zhang et al. in [121], cf. also [59, Theorem 2.24].

Teufl and Wagner [108, Example 6.3] also determined the asymptotic behavior of the number of matchings $m(S_3^n)$, namely $0.6971213284 \cdot 1.77973468825^{3^n}$. Compatible with this result is the calculation by Chen et al. in [20, Proposition 5] of the average entropy $\mu(S_3^n) = \lim_{n\to\infty} 3^{-n} \ln(m(S_3^n)) \approx 0.5764643\ldots$ The exact formula for $m(S_3^n)$, however, is not yet known, cf. [59, Exercise 2.19].

D'Angeli and Donno [24, Section 3] studied weighted spanning trees on S_3^n, while the Tutte polynomial of the same family of graphs was investigated by Donno and Iacono in [26].

Interestingly, base-3 Sierpiński graphs were used as a tool in [120] to enumerate spanning trees of Apollonian networks and in [87] to calculate the Tutte polynomial of these networks. The key observation for these applications is the appealing fact that the inner duals of the Apollonian networks are precisely the base-3 Sierpiński graphs.

2 Metric properties of Sierpiński graphs

As we have already mentioned, there is an intimate relation between Sierpiński graphs and the Tower of Hanoi. More precisely, optimal solutions of the latter bijectively correspond to shortest paths in the state graphs. Hence among all the properties of Sierpiński graphs, metric properties appear to be the most important and have accordingly been most intensely studied. In this chapter we present an overview of these properties.

Let us first briefly recall the concepts involved. The *distance* $\mathrm{d}_G(u, v)$ between vertices u and v of a connected graph G is the usual shortest-path distance, that is, the number of edges on a shortest u, v-path. If the graph G considered is clear from the context we may simply write d instead of d_G. The *(total) distance* $\mathrm{d}_G(u)$ *of a vertex* u in G equals the sum of all distances to u:

$$\mathrm{d}_G(u) = \sum_{v \in V(G)} \mathrm{d}_G(u, v)$$

while the *average distance of* $G \neq (\emptyset, \emptyset)$ is

$$\overline{\mathrm{d}}(G) = \frac{1}{|G|^2} \sum_{u \in V(G)} \mathrm{d}_G(u)\,.$$

The *eccentricity* $\varepsilon_G(u)$ *of a vertex* $u \in V(G)$ is the maximum distance between u and any other vertex,

$$\varepsilon_G(u) = \max\{\mathrm{d}_G(u, v) \mid v \in V(G)\}\,.$$

The *diameter* $\mathrm{diam}(G)$ of G is the maximum eccentricity of its vertices and the *radius* $\mathrm{rad}(G)$ is the minimum eccentricity of its vertices. A vertex with $\varepsilon_G(u) = \mathrm{rad}(G)$ is called a *central vertex* of G and the set of central vertices $C(G) = \{u \in V(G) \mid \varepsilon_G(u) = \mathrm{rad}(G)\}$ is the *center* of G. The *average eccentricity* $\overline{\varepsilon}(G)$ of $G \neq (\emptyset, \emptyset)$ is the arithmetic mean of all eccentricities,

$$\overline{\varepsilon}(G) = \frac{1}{|G|} \sum_{u \in V(G)} \varepsilon_G(u)\,.$$

2.1 Distance between vertices

The first key metric result about Sierpiński graphs is the following formula for the distance between an arbitrary vertex and an extreme vertex.

Lemma 2.1 [77, Lemma 4] *If* $p \in \mathbb{N}$ *and* $n \in \mathbb{N}_0$*, then for any* $j \in P$ *and any vertex* $s = s_n \ldots s_1$ *of* S_p^n*,*

$$\mathrm{d}(s, j^n) = \sum_{d=1}^{n} [s_d \neq j] \cdot 2^{d-1}\,.^{13} \tag{5}$$

Moreover, there is exactly one shortest path between s *and* j^n*. In particular,* $\mathrm{d}(i^n, j^n) = 2^n - 1$ *for any* $\{i, j\} \in \binom{P}{2}$ *and consequently* $\varepsilon(i^n) = 2^n - 1$*.*

Because of the fundamental role played by Lemma 2.1, we will present the proof here.

[13] The *Iverson bracket* (or *Iverson convention*) assigns a value $[\mathfrak{S}] \in \{0, 1\}$ to a statement $\mathfrak{S}$ and is defined as

$$[\mathfrak{S}] = \begin{cases} 1\,, & \text{if } \mathfrak{S} \text{ is true,} \\ 0\,, & \text{if } \mathfrak{S} \text{ is false.} \end{cases}$$

Proof. By induction on n. The statement is trivial for $n = 0$. Let $n \in \mathbb{N}_0$ and $s = s_{n+1}\overline{s}$, $\overline{s} \in P^n$.

If $s_{n+1} = j$, then one can use the shortest path in S_p^n from $\overline{s}$ to j^n and add a j in front of each vertex. Hence

$$\mathrm{d}(s, j^{1+n}) \le \sum_{d=1}^{n} [s_d \neq j] \cdot 2^{d-1} = \sum_{d=1}^{n+1} [s_d \neq j] \cdot 2^{d-1} \ \left(< 2^{n+1}\right) .$$

If $s_{n+1} \neq j$, we can compose a path from s to j^{1+n} by going from $s_{n+1}\overline{s}$ to $s_{n+1}j^{1+n}$ on a (shortest) path of length $\le \sum_{d=1}^{n} [s_d \neq j] \cdot 2^{d-1}$, then moving to js_{n+1}^n on one extra edge and finally from here to j^{1+n} in another $2^n - 1$ steps, altogether

$$\mathrm{d}(s, j^{1+n}) \le \sum_{d=1}^{n} [s_d \neq j] \cdot 2^{d-1} + 1 + 2^n - 1 = \sum_{d=1}^{n+1} [s_d \neq j] \cdot 2^{d-1} \ \left(< 2^{n+1}\right) .$$

To show that these are the unique shortest paths, respectively, we note that no optimal path from s to j^{1+n} can touch a subgraph kS_p^n for $s_{n+1} \neq k \neq j$. Consider any such path. Then it must contain

- the edge $\{ik^n, ki^n\}$ for some $i \neq k$, to enter kS_p^n;
- a path from ki^n to $k\ell^n$, $\ell \neq k$, inside kS_p^n;
- the edge $\{k\ell^n, \ell k^n\}$ to leave kS_p^n, so that $\ell \neq i$, because we are on a path;
- a path from some jh^n, $h \neq j$, to j^{1+n} to finish the path inside jS_p^n.

By induction assumption this would comprise at least $1 + (2^n - 1) + 1 + (2^n - 1) = 2^{n+1}$ edges, such that the path cannot be optimal because we already found a strictly shorter one. □

The following consequences of Lemma 2.1 are due to Parisse.

Corollary 2.2 [94, Proposition 2.5] *If $p \in \mathbb{N}$ and $n \in \mathbb{N}_0$, then for any vertex s of S_p^n,*

$$\sum_{i=0}^{p-1} \mathrm{d}(s, i^n) = (p-1)(2^n - 1) .$$

Corollary 2.3 [94, Corollary 2.2(i)] *If $p \in \mathbb{N}$ and $n \in \mathbb{N}_0$, then for arbitrary vertices js and jt of S_p^{1+n},*

$$\mathrm{d}_{S_p^{1+n}}(js, jt) = \mathrm{d}_{S_p^n}(s, t) .$$

Knowing that the shortest paths to extreme vertices are unique, we can use the recursive structure of Sierpiński graphs to obtain all possible candidates for shortest paths between arbitrary vertices. There are exactly $p-1$ such paths defined as follows.

Let $\mathbb{N}_0 \ni n < N \in \mathbb{N}$, $\underline{s} \in P^{N-n-1}$, $\{i,j\} \in \binom{P}{2}$, and $s,t \in P^n$. Then in S_p^N we have (with $\overline{P} := P \cup \{p\}$):

$$\mathrm{d}(\underline{s}is, \underline{s}jt) = \min\left\{\mathrm{d}_\ell(is, jt) \mid \ell \in \overline{P} \setminus \{i,j\}\right\} ,$$

where $\mathrm{d}_\ell(is,jt) := \mathrm{d}(s,\ell^n) + 1 + 2^n + \mathrm{d}(t,\ell^n)$ for $\ell \in P$ and $\mathrm{d}_p(is,jt) := \mathrm{d}(s,j^n) + 1 + \mathrm{d}(t,i^n)$; the paths realizing these numbers of moves are unique and given by $is \to i\ell^n \to \ell i^n \to \ell j^n \to j\ell^n \to jt$ and $is \to ij^n \to ji^n \to jt$, respectively, where we have omitted the prefix $\underline{s}$ which remains constant throughout the paths. In particular, there are at most two LDMs in any optimal s,t-path in S_p^N.

Remark 2.4 *We have changed the exponent of the Sierpiński graph under investigation to N in order to emphasize the label, namely $n+1$, of the largest disc moved in a shortest path from $\underline{s}is$ to $\underline{s}jt$ in S_p^N and to make ready use of formula* (5) *in S_p^n.*

Definition 2.5 *Let $p \in \mathbb{N}_2$, $n \in \mathbb{N}$ and let $\{i,j\} \in \binom{P}{2}$. Further let $s = \underline{s}i\overline{s}$ and $t = \underline{s}j\overline{t}$ be vertices of S_p^n, where $\overline{s}, \overline{t} \in P^{\delta-1}$ and $\underline{s} \in P^{n-\delta}$ for some $\delta \in [n]$. Then set*

$$\begin{aligned} \forall \ell \in P: \quad \mathrm{d}_\ell(s,t) &= \mathrm{d}(\overline{s}, \ell^{\delta-1}) + 1 + 2^{\delta-1} + \mathrm{d}(\overline{t}, \ell^{\delta-1}) , \\ \mathrm{d}_p(s,t) &= \mathrm{d}(\overline{s}, j^{\delta-1}) + 1 + \mathrm{d}(\overline{t}, i^{\delta-1}) . \end{aligned}$$

The distance $\mathrm{d}_p(s,t)$ is called the direct distance *between s and t, $\mathrm{d}_\ell(s,t)$ for $\ell \in P \setminus \{i,j\}$ are the* indirect distances.

The s,t-path corresponding to the direct distance will be called the *direct s,t-path*. It includes exactly one move of the largest moved disc δ, namely directly from i to j in the STH.

First observe that the vertices s and t in the above definition both belong to the subgraph $\underline{s}S_p^\delta$. For these two vertices the distance $\mathrm{d}_\ell(s,t)$, $\ell \in P \setminus \{i,j\}$, corresponds to the path through the subgraph $\underline{s}\ell S_p^{\delta-1}$, i.e. disc δ moves from i through ℓ to j. It is easy to see that a shortest path between these vertices is one of the paths corresponding to the distances d_ℓ for $\ell \in \overline{P} \setminus \{i,j\}$. Other possibilities would be to go through more than just one subgraph isomorphic to $S_p^{\delta-1}$, but then this path would already be longer than the diameter of the subgraph $\underline{s}S_p^\delta$. Note also that the shortest path between an arbitrary vertex s and an extreme vertex j^n of S_p^n is the direct s, j^n-path.

In Figure 7 we present the graph S_4^4 with emphasized paths that correspond to distances $\mathrm{d}_\ell(0231, 2301)$, $\ell \in \{1, 3, 4\}$. The direct path, that is, the path corresponding to the distance $\mathrm{d}_4(0231, 2301)$, is drawn in red, the path for $\mathrm{d}_1(0231, 2301)$ is green, and the path for $\mathrm{d}_3(0231, 2301)$ is blue. Obviously the shortest path for these two vertices is the direct path and $\mathrm{d}(0231, 2301) = 9$.

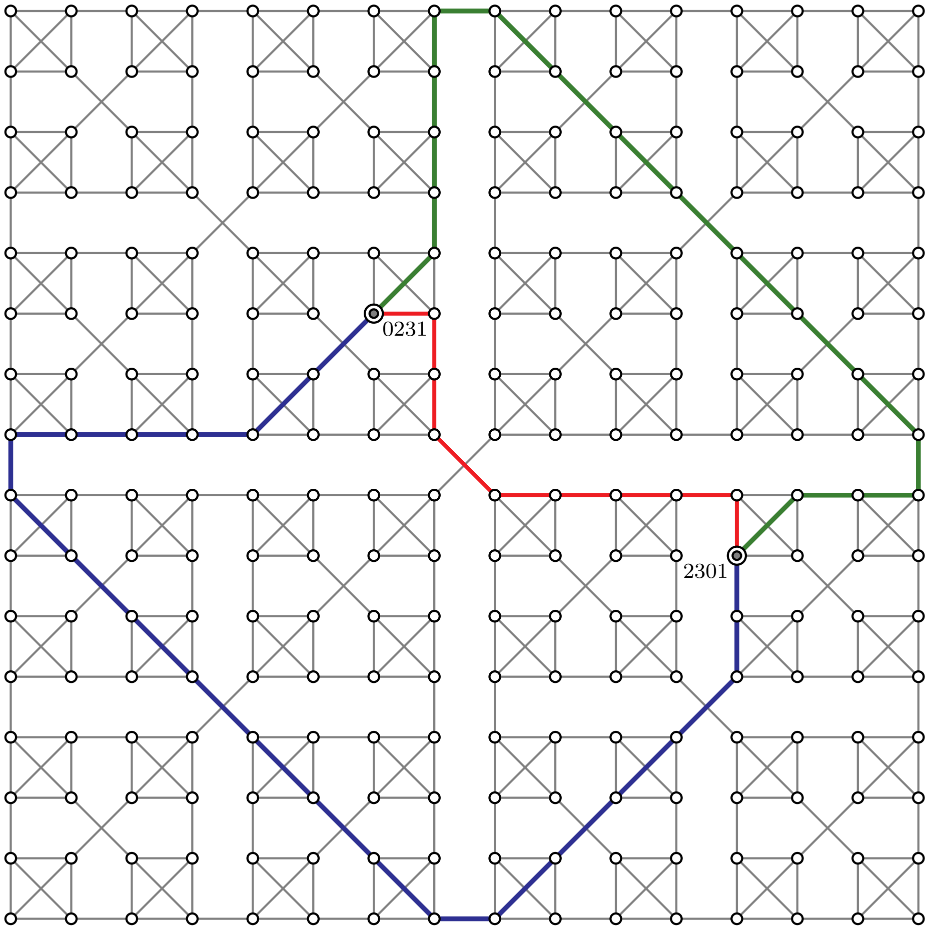

Figure 7: Paths in S_4^4 representing distances $\mathrm{d}_\ell(0231, 2301)$, $\ell \in \{1, 3, 4\}$

The above discussion can be summarized in the following general result.

Theorem 2.6 [77, Theorem 5] *Let $p \in \mathbb{N}_2$ and $n \in \mathbb{N}$. If $s = \underline{s}i\overline{s}$ and $t = \underline{s}j\overline{t}$ are vertices of S_p^n, where $\{i,j\} \in \binom{P}{2}$, $\delta \in [n]$, $\overline{s}$, $\overline{t} \in P^{\delta-1}$, and $\underline{s} \in P^{n-\delta}$, then*

$$\mathrm{d}(s,t) = \min\left\{\mathrm{d}_\ell(s,t) \,|\, \ell \in \overline{P} \setminus \{i,j\}\right\} . \tag{6}$$

The minimum in Theorem 2.6 can be realized by at most two of the distances d_ℓ, $\ell \in \overline{P} \setminus \{i,j\}$, that is, there are at most two shortest paths between any two vertices (cf. the original proof in [77, Theorem 6] or an alternative recent proof [57, Corollary 1.1]). Moreover, if there are two shortest paths between two vertices, one of them is the direct path. From the algorithmic point of view it has been shown in [77, Corollary 7] that the distance between arbitrary vertices of S_p^n can be computed in $O(n)$ time (cf. also [57, Theorem 3.1]).

In order to determine the metric dimension of Sierpiński graphs (cf. Section 2.2), almost-extreme vertices have been introduced in [83] as follows. Let $p \in \mathbb{N}_2$, $n \in \mathbb{N}$. For any $\{i,j\} \in \binom{P}{2}$ the vertex of the form $i^n j$ of the graph S_p^{n+1} is called an *outer almost-extreme vertex* (of S_p^{n+1}) and the vertex ij^n of S_p^{1+n} is an *inner almost-extreme vertex* (of S_p^{1+n}).

The distance between an outer almost-extreme vertex and an arbitrary vertex is given in the next result. For its generality recall the fact that by Corollary 2.3 the distance between arbitrary vertices does not depend on a common prefix.

Proposition 2.7 [83, Proposition 4] *If $p \in \mathbb{N}_2$, $n \in \mathbb{N}$ and $j^n k$ is an outer almost-extreme vertex of S_p^{n+1}, then for $i \in P \setminus \{j\}$,*

$$\mathrm{d}_{S_p^{n+1}}(is, j^n k) = \mathrm{d}_{S_p^n}(s, j^n) + 2^n - [i = k] \,.$$

Almost-extreme vertices are defined for $n \in \mathbb{N}$, nevertheless Proposition 2.7 holds also for $n = 0$. In that case $S_p^1 \cong K_p$, every vertex is extreme, and the distance is $\mathrm{d}_{S_p^1}(i,k) = [i \neq k] = 1 - [i = k]$ as stated in the proposition.

To present a formula for the distance between an inner almost-extreme vertex and an arbitrary vertex, we need the following definition.

Definition 2.8 *Let jk^n be an inner almost-extreme vertex of S_p^{1+n}, $p \in \mathbb{N}_2$, $n \in \mathbb{N}$. Then $s \in P^n$ is called* special w.r.t. jk^n, *if*

$$\exists\, \delta \in [n],\ \underline{s} \in (P \setminus \{j,k\})^{n-\delta} :\ s = \underline{s}k\overline{s} \,.$$

A special vertex s is called odd, *if $\overline{s} = j^{\delta-1}$.*

Note that for fixed $\{j,k\} \in \binom{P}{2}$ there are $\frac{1}{2}\left(p^n - (p-2)^n\right)$ special $s \in P^n$ w.r.t. jk^n, of which 1 is odd, if $p = 2$, n are odd, if $p = 3$, and $\left((p-2)^n - 1\right)/(p-3)$ are odd, if $p \in \mathbb{N}_4$.

Now we can state:

Proposition 2.9 [83, Proposition 7] *If $p \in \mathbb{N}_2$, $n \in \mathbb{N}$ and jk^n is an inner almost-extreme vertex of S_p^{1+n}, then for any $i \in P$ with $i \neq j$,*

$$\mathrm{d}_{S_p^{1+n}}(is, jk^n) = \begin{cases} \mathrm{d}_{S_p^n}(s, j^n) + 1, & \textit{if } i = k\,, \\ \mathrm{d}_{S_p^n}(s, k^n) + 2^n + 1, & \textit{if } i \neq k \textit{ and } s \textit{ is special w.r.t. } jk^n\,, \\ \mathrm{d}_{S_p^n}(s, j^n) + 2^n, & \textit{otherwise}\,. \end{cases}$$

Corollary 2.10 [119, Corollary 3.2] *Let $p \in \mathbb{N}_3$, $n \in \mathbb{N}$ and $\{i, j, k\} \in \binom{P}{3}$. Then the distance between the inner almost-extreme vertex jk^n of S_p^{1+n} and the vertex is where s is odd w.r.t. jk^n is*

$$\mathrm{d}_{S_p^{1+n}}(is, jk^n) = 2^{n+1} - 2^{\delta - 1}\,.$$

To conclude this section we present the total distance of extreme and almost-extreme vertices. The first result of the next theorem can be found in the proof of [94, Corollary 2.6] and in [115, Satz 3.1.10]. The total distance of almost-extreme vertices was determined in [83, Theorems 9 and 11].

Theorem 2.11 *If $p \in \mathbb{N}$ and $n \in \mathbb{N}_0$, then for any $i \in P$ and $\{j, k\} \in \binom{P}{2}$,*

$$\begin{aligned} \mathrm{d}_{S_p^{n+1}}(i^{n+1}) &= \frac{p-1}{p} p^{n+1} \left(2^{n+1} - 1\right)\,, \\ \mathrm{d}_{S_p^{n+1}}(j^n k) &= \frac{p-1}{p}(2p)^{n+1} - \left(1 + \frac{1}{p(p-1)}\right) p^{n+1} + \frac{p}{p-1}\,, \\ \mathrm{d}_{S_p^{n+1}}(jk^n) &= \frac{p^2-2}{p(p+2)}(2p)^{n+1} - \frac{p-2}{2p} p^{n+1} - \frac{p}{2(p+2)}(p-2)^{n+1}\,. \end{aligned}$$

2.2 Distance invariants

We next present central distance invariants of Sierpiński graphs. Using Theorem 2.6 their diameter can be determined easily.

Proposition 2.12 [19, Theorem 2.1], [94, Corollary 2.2(ii)] *If $p \in \mathbb{N}_2$ and $n \in \mathbb{N}_0$, then* $\mathrm{diam}(S_p^n) = 2^n - 1$.

As a consequence of Lemma 2.1 and Proposition 2.12 we have the following interesting result that counts the number of vertices at a given distance ℓ from some fixed extreme vertex.

Corollary 2.13 [94, Corollary 2.4] *If $p \in \mathbb{N}$ and $n \in \mathbb{N}_0$, then for an arbitrary extreme vertex i^n of S_p^n and $\ell \in [2^n]_0$,*

$$|\{s \in P^n \mid \mathrm{d}(s, i^n) = \ell\}| = (p-1)^{q(\ell)} ,$$

where $q(\ell)$ is the number of non-zero binary digits of ℓ. Consequently,

$$\sum_{\ell=0}^{2^n-1} (p-1)^{q(\ell)} = p^n .$$

When it comes to eccentricities, Parisse [94] showed that the eccentricity of an arbitrary vertex is determined by its distances to the extreme vertices:

Proposition 2.14 [94, Lemma 2.3] *If $p \in \mathbb{N}$ and $n \in \mathbb{N}_0$, then for an arbitrary vertex s of S_p^n,*

$$\varepsilon(s) = \max\{\mathrm{d}(s, i^n) \mid i \in P\} .$$

Knowing the eccentricity of vertices, the radius and the center of the Sierpiński graphs can be determined:

Theorem 2.15 [94, Theorem 3.1] *If $p, n \in \mathbb{N}$, then*

$$\mathrm{rad}\left(S_p^n\right) = \lfloor 2^{n-p+1}\left(2^{p-1}-1\right)\rfloor = \begin{cases} 2^n - 1 , & n < p , \\ 2^{n-p+1}\left(2^{p-1}-1\right) , & n \geq p . \end{cases}$$

The center of S_p^n is

$$C\left(S_p^n\right) = \begin{cases} P^n , & n < p , \\ C_p^n , & n \geq p , \end{cases}$$

where

$$C_p^n = \left\{z \in P^n \mid z = z_p \ldots z_2 z_1^{n-p+1} , \{z_p, \ldots, z_1\} = P\right\} .$$

Note that the graph $C(S_p^n)$ induced by the center of S_p^n contains

$$\left|C(S_p^n)\right| = \begin{cases} p^n , & n < p , \\ p! , & n \geq p \end{cases}$$

vertices and

$$\left\|C(S_p^n)\right\| = \begin{cases} \frac{1}{2}p(p^n - 1) , & n < p , \\ \frac{1}{2}p! , & n \geq p \end{cases}$$

edges. In particular, for $n \geq p > 1$, the center of S_p^n induces a 1-regular graph with $\frac{p!}{2}$ independent edges, that is, a subgraph of S_p^n isomorphic to $\frac{p!}{2}K_2$.

Hinz and Parisse [64] determined the average eccentricity of Sierpiński graphs.

Theorem 2.16 [64, Corollary 3.5] *If $p \in \mathbb{N}$ and $n \in \mathbb{N}_0$, then*

$$\overline{\varepsilon}(S_p^n) = \left(1 - \binom{2p}{p-1}^{-1}\right) 2^n - \frac{p-1}{p} - \sum_{k=0}^{p-2} (-1)^{p-k} \frac{p-1-k}{2p-k} \binom{p}{k} \left(\frac{k}{p}\right)^n .$$

The proof of Theorem 2.16 is far from being straightforward due to the possibility of two necessary LDMs in shortest paths.

An even more impressive formula was obtained by Wiesenberger for the average distance in Sierpiński graphs.

Theorem 2.17 [115, Satz 3.1.11] *For $p \in \mathbb{N}$ let*

$$\begin{aligned}
\alpha_p &= p^4 - 12p^3 + 56p^2 - 104p + 68\,, \\
\lambda_{p,\pm} &= \frac{1}{2}p^2 - p + 1 \pm \frac{1}{2}\sqrt{\alpha_p}\,, \\
\gamma_{p,\pm} &= (p^2 + 3p - 2) \mp (p^4 + p^3 - 30p^2 + 58p - 36)\sqrt{\alpha_p}\,.
\end{aligned}$$

Then for all $n \in \mathbb{N}_0$,

$$\begin{aligned}
\overline{\mathrm{d}}(S_p^n) &= \frac{(p-1)(2p^4 + 6p^3 - 17p^2 + 26p - 16)}{p(2p-1)(p^3 + 4p^2 - 4p + 8)} 2^n \\
&\quad - \frac{p-2}{p} + \frac{p^2 + 3p - 6}{(2p-1)(p^2 - 7p + 8)} p^{-n} \\
&\quad - \frac{p(p-1)\gamma_{p,+}}{2(p^2 - 7p + 8)(p^3 + 4p^2 - 4p + 8)} \left(\frac{\lambda_{p,+}}{p^2}\right)^n \\
&\quad - \frac{p(p-1)\gamma_{p,-}}{2(p^2 - 7p + 8)(p^3 + 4p^2 - 4p + 8)} \left(\frac{\lambda_{p,-}}{p^2}\right)^n .
\end{aligned}$$

For particular values of p, the parameter α_p turns out to be a perfect square. For example $\alpha_2 = 4$ and $\alpha_4 = 36$. In such cases the formula from Theorem 2.17 simplifies. For example, in the named cases we have:

$$\begin{aligned}
\overline{\mathrm{d}}(S_2^n) &= \frac{1}{3}\left(2^n - 2^{-n}\right)\,, \\
\overline{\mathrm{d}}(S_4^n) &= \frac{89}{140}2^n - \frac{1}{2} + \frac{1}{4}2^{-n} - \frac{11}{14}4^{-n} + \frac{2}{5}8^{-n}\,.
\end{aligned}$$

Alekseyev and Berger [1] studied expected lengths of random walks on Sierpiński graphs S_3^n from 0^n to 1^n and found the value

$$\frac{(3^n - 1)(5^n - 3^n)}{2 \cdot 3^{n-1}}\,.$$

One of their proofs uses electrical circuit theory. For general S_p^n see the result of Wiesenberger in [59, p. 155].

To determine other invariants of S_p^n it is necessary to decide for each specific pair s, t whether there are two LDMs (necessary) in a shortest s,t-path. This can be done most efficiently with the aid of an automaton. Before we describe this decision algorithm in the next section, let us indicate how use can be made, as mentioned in Section 2.1, of (outer) almost-extreme vertices in order to determine the metric dimension of Sierpiński graphs. This graph invariant has been extensively studied in the past few years; see [5] for different aspects of it and its role in general mathematics. Before moving to the metric dimension of S_p^n, let us define it first in the general case.

A subset R of the vertex set $V(G)$ is a *resolving set* for the graph G, if

$$\forall \{x, y\} \in \binom{V(G)}{2} \, \exists \, r \in R: \ \mathrm{d}_G(r, x) \neq \mathrm{d}_G(r, y) \, .$$

The *metric dimension* of G, $\mu(G)$, is the size of a smallest resolving set. To determine the metric dimension of S_p^n, we first deduce the following consequence of Proposition 2.7.

Corollary 2.18 *If $p \in \mathbb{N}_3$ and $n \in \mathbb{N}_0$, then for any $\{i, j, k\} \in \binom{P}{3}$ and $s \in P^n$,*

$$\mathrm{d}_{S_p^{1+n}}(is, j^n k) = \mathrm{d}_{S_p^{1+n}}(is, j^{n+1}) \, .$$

Let $R \subseteq V(S_p^{n+1})$, $p \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Assume that $R \cap jS_p^n = \emptyset = R \cap kS_p^n$ for some $j \neq k$. It then follows from Corollary 2.18 that for each $r \in R$ we have $\mathrm{d}(r, j^n k) = \mathrm{d}(r, j^{n+1})$. Hence R can not be a resolving set for S_p^{n+1} and so each resolving set must contain at least $p-1$ elements. On the other hand, (any) $p-1$ extreme vertices form a resolving set and, with recourse to the pigeonhole principle, no jS_p^n can contain more than one element of a minimal resolving set. We have thus arrived at:

Theorem 2.19 [83, Corollary 6], [96, Théorème 3.6] *If $p \in \mathbb{N}$ and $n \in \mathbb{N}_0$, then*

$$\mu(S_p^{n+1}) = p - 1 \, .$$

Moreover, if R is a minimum resolving set, then $|R \cap V(jS_p^n)| \leq 1$ holds for every $j \in P$.

2.3 The decision algorithm

The phenomenon that two LDMs may appear or even be necessary in an optimal path beween two vertices of a Sierpiński graph S_p^n can obviously only occur for $p \in \mathbb{N}_3$ and $n \in \mathbb{N}_2$; we will mostly restrict ourselves to these data throughout this section.

The paper [58] demonstrated that the phenomenon is not as rare as might be expected, already in the case of $p = 3$. For instance, extreme vertices are the only vertices that are linked to any other vertex by exactly one shortest path ([58, Corollary 3.7]; cf. also [59, Proposition 2.33]). For each vertex the number of those vertices that are linked to it by two shortest paths (recall that there can not be more than that) can be written down in terms of *Stern's diatomic sequence*; see [58, Theorem 3.8] or [59, Corollary 2.36]. Likewise, there is a formula for the number of vertices whose shortest path from a fixed vertex is unique and contains two LDMs; see [59, Theorem 2.37]. Among all these results we want to present only the following in explicit form, because it was already anticipated in [55, Proposition 6i] (for a more detailed discussion, see [59, Section 2.4.1]):

Proposition 2.20 [58, Proposition 3.9] *Among the 9^n pairs $(s,t) \in (T^n)^2$, $n \in \mathbb{N}_0$, there are*

$$\frac{3}{4}\left(\frac{\sqrt{17}+1}{\sqrt{17}}\Theta_+^n - 2\cdot 3^n + \frac{\sqrt{17}-1}{\sqrt{17}}\Theta_-^n\right)$$

for which two shortest s,t-paths in S_3^n exist; here $\Theta_\pm := (5 \pm \sqrt{17})/2$.

For results about two LDMs in the general case $p \in \mathbb{N}_3$ we may state another consequence of Proposition 2.7:

Corollary 2.21 *Let $p \in \mathbb{N}_2$, $n \in \mathbb{N}$ and $\{i,j\} \in \binom{P}{2}$. Then there are two shortest paths between an arbitrary vertex is of S_p^{1+n} and an outer almost-extreme vertex j^nk if and only if $s = k^n$ and $k \neq i$.*

Proof. Recall from Definition 2.5 and Lemma 2.1 that an indirect distance $\mathrm{d}_\ell(is, j^nk)$, $\ell \in P \setminus \{i,j\}$, can be equal to the direct distance $\mathrm{d}_p(is, j^nk) = \mathrm{d}(is, j^nk)$ iff $\mathrm{d}(s,\ell^n) + 2^n - [k = \ell] = \mathrm{d}(s, j^n) - [i = k]$, which in turn is true iff $i \neq k = \ell$, $\mathrm{d}(s, j^n) = 2^n - 1$, and $\mathrm{d}(s, \ell^n) = 0$. This is only the case if $s = k^n$. □

This result was generalized by Xue et al. in [119]. They characterized all vertices with two shortest paths to an outer almost-extreme vertex j^nk, not just those which are not in the same subgraph jS_p^n.

Theorem 2.22 [119, Theorem 3.3 and Corollary 3.4] *If* $p \in \mathbb{N}_2$, $n \in \mathbb{N}$ *and* $j^n k$ *is an outer almost-extreme vertex of* S_p^{n+1}, *then there are two shortest paths between an arbitrary vertex* s *of* S_p^{n+1} *and* $j^n k$ *if and only if* $s = j^{n-m} i k^m$ *with* $m \in [n]$ *and* $i \in P \setminus \{j, k\}$. *Moreover,* $\mathrm{d}_{S_p^{n+1}}(j^{n-m} i k^m, j^n k) = 2^{m+1} - 1$.

Figure 8 shows the graph S_5^3 with emphasized vertices (red) for which there are two shortest paths to the almost-extreme vertex 002 (gray).

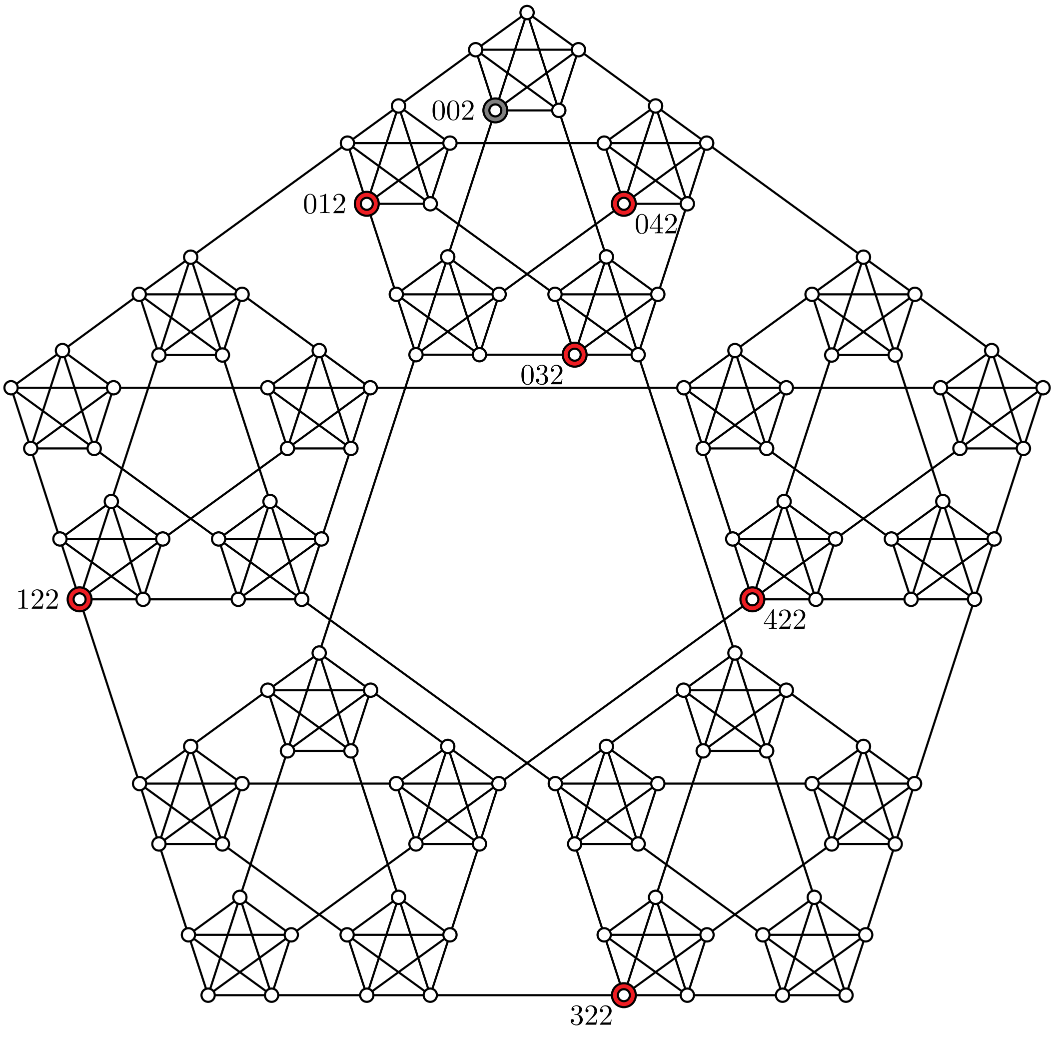

Figure 8: A drawing of Sierpiński graph S_5^3 with vertices emphasized which have two shortest paths to 002

With respect to inner almost-extreme vertices, recall that the direct path between two vertices is and jt of S_p^{1+n} is the path corresponding to the direct distance $\mathrm{d}_p(is, jt)$ (cf. Definition 2.5). Moreover, in Definition 2.8 we said what it means that s is *special* with respect to an inner almost-extreme vertex jk^n in S_p^{1+n}. The choice of this expression becomes apparent in the following proposition.

Proposition 2.23 *If $p \in \mathbb{N}_2$, $n \in \mathbb{N}$ and jk^n is an inner almost-extreme vertex of S_p^{1+n}, then the direct path between an arbitrary vertex is and jk^n, $\{i, j\} \in \binom{P}{2}$, is the only shortest is, jk^n-path if and only if $i = k$ or s is not special with respect to jk^n.*

Similarly, the name for odd s w.r.t. jk^n was chosen because of the next result.

Proposition 2.24 *If $p \in \mathbb{N}_2$, $n \in \mathbb{N}$ and jk^n is an inner almost-extreme vertex of S_p^{1+n}, then there are two shortest paths between the vertex is of S_p^{1+n} and jk^n, $\{i, j\} \in \binom{P}{2}$, if and only if $i \neq k$ and s is odd with respect to jk^n.*

Finally, we have the complementary result

Corollary 2.25 *If $p \in \mathbb{N}_2$, $n \in \mathbb{N}$ and jk^n is an inner almost-extreme vertex of S_p^{1+n}, then an indirect path between the vertex is of S_p^{1+n} and jk^n, $\{i, j\} \in \binom{P}{2}$ is the only shortest is, jk^n-path if and only if $i \neq k$ and s is special, but not odd with respect to jk^n.*

An example of direct and special vertices is illustrated in Figure 9 on the graph S_6^3 for the almost-extreme vertex 144. All vertices circled green are direct for 144 and thus belong to the first line of the formula in Proposition 2.9, for all the others we use the second line. Orange vertices are special for 144, so for these vertices both lines of the equation in Proposition 2.9 hold.

The result of Proposition 2.20 can also be obtained based on an approach by Romik [102], who presented an elegant algorithm to decide about the number of LDMs in shortest paths of S_3^n, based on a finite automaton. (See [59, Figure 2.27] for a nice drawing of Romik's automaton.) This has been replaced recently with an even "charming" algorithm (Bing Xue in MathSciNet®) by Hinz and Holz auf der Heide in [57] which applies to all S_p^n, $p \geq 3$, and whose idea we will now describe. The procedure is based on the automaton depicted in Figure 10.

The input for the automaton are two (arbitrary) vertices is, jt of S_p^{1+n} for $n \in \mathbb{N}$. Without loss of generality we may assume that $i \neq j$, since the distance and the shortest paths between is and jt do not depend on a common prefix. The very first pair fixes all the values i and j in the automaton. For example, in the

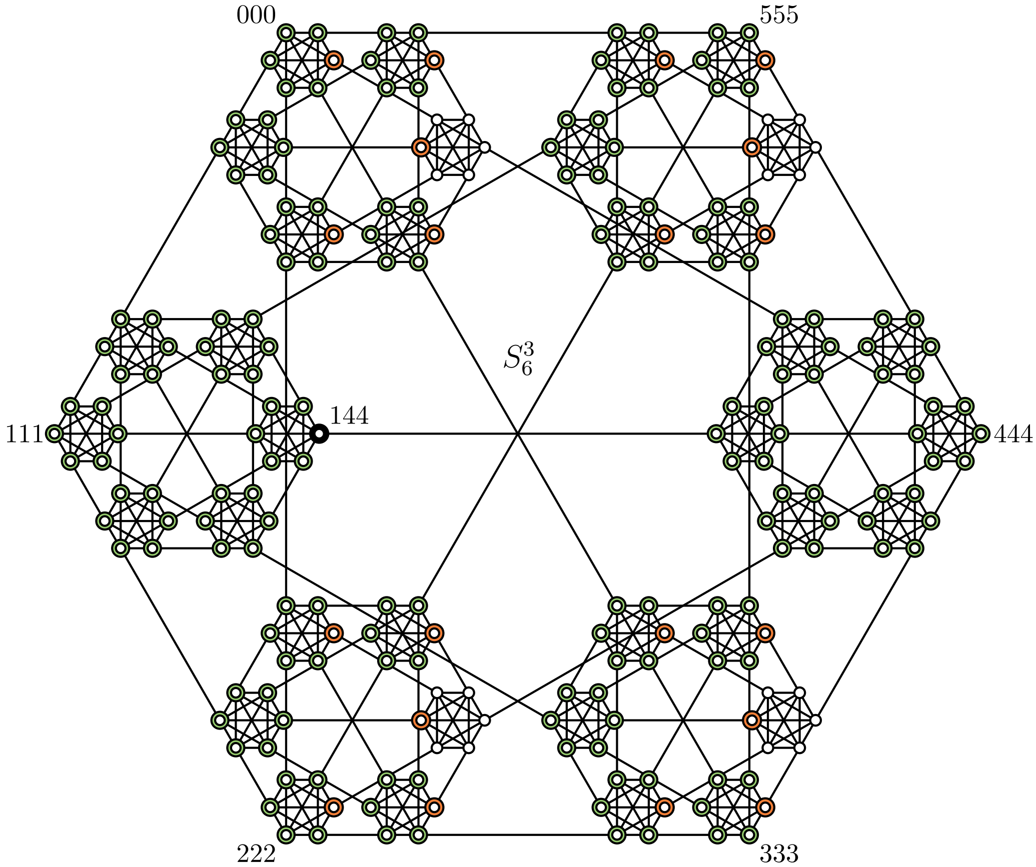

Figure 9: Direct and special vertices with respect to 144 in S_6^3

case of $0s$ and $1t$ we replace in the automaton any i with 0 and j with 1. Vertices is and jt are now entered into the so prepared automaton as pairs (s_d, t_d) one by one starting with $d = n$ and, if necessary, down to 1. Note that all dots in the automaton are arbitrary entries. We begin with the pair $(g, h) = (s_n, t_n)$ in state 0 of the automaton, which is a source state with all links pointing outwards. Here $k \in P \setminus \{i, j\}$ identifies the subgraph kS_p^n the indirect path may pass. Depending on the values of g and h we move either to state 1, one of the states A and B or we terminate the procedure in state D, which, besides E, is one of the two sinks or *absorbing* states, which means that if we come to one of these states, we already know the answer to the decision whether to move the largest disc once or twice (or if there are two shortest paths), although we have possibly entered less than n pairs of components of s and t.

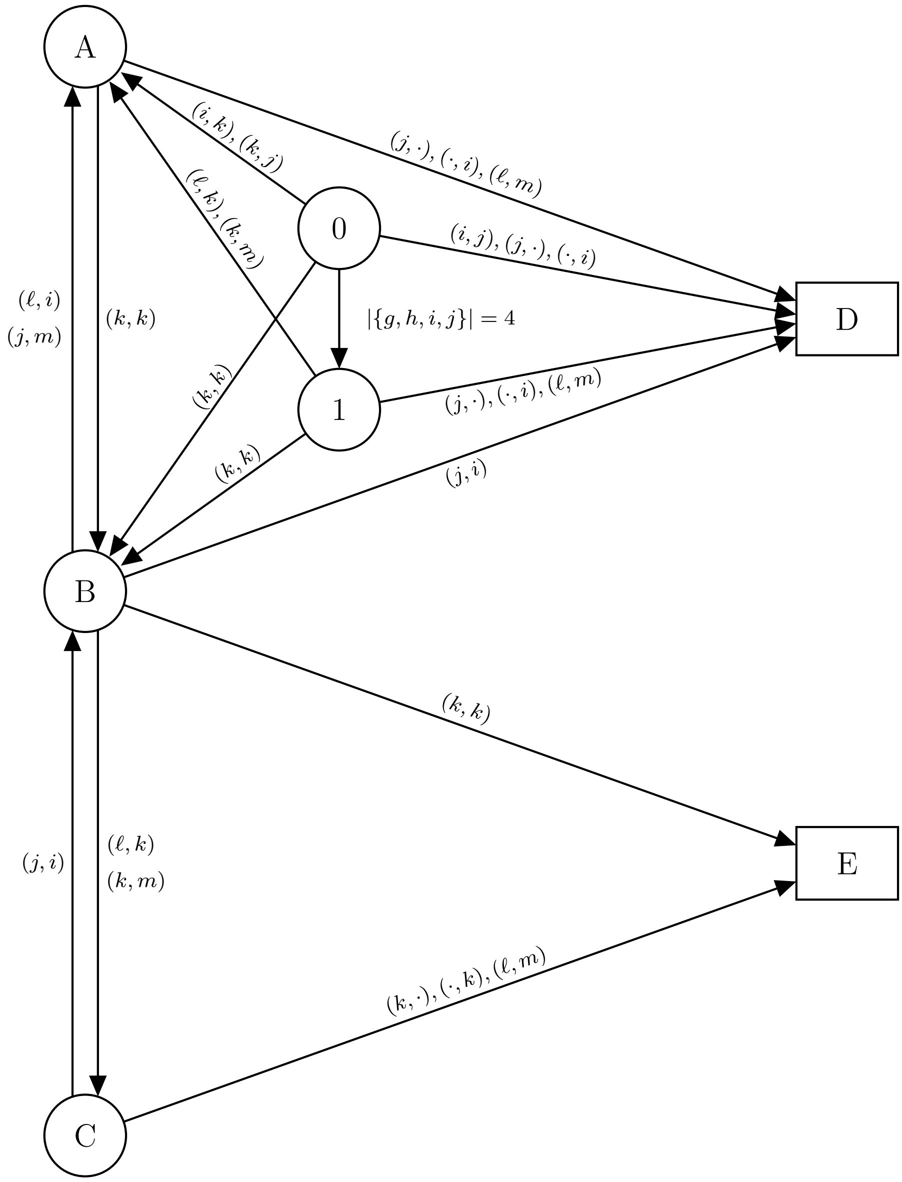

Figure 10: P2 decision automaton for S_p^n

Arriving in state 1 means that k has not been identified yet. This can, of course, happen only if $p \in \mathbb{N}_4$ (for $p = 3$ we have $k = 3 - i - j$) and in fact state 1 is entered only if g, h, i, and j are all different. In this case, we enter more pairs (s_d, t_d), $d \in [n-1]$, until we either exit state 1, where $\ell \in P \setminus \{g, h, j\}$

and $m \in P \setminus \{g,h,i\}$, in which case $k \in \{g,h\}$ *is* fixed, or we run out of input pairs. If the latter happens in one of the non-absorbing states, the decision is made according to Table 1.

1, A, D	the largest disc moves once, i.e. the unique shortest path is the direct path; a value of k is not needed
B	the largest disc moves either once or twice, i.e. there are two shortest paths, the direct path and the path corresponding to d_k
C, E	the largest disc moves twice, i.e. the unique shortest path is the path corresponding to d_k

Table 1: Interpretation of states 1, A, B, C, D, and E

After leaving states 0 or 1, we either finish in D or the value of k is fixed and $\ell \in P \setminus \{j,k\}$, $m \in P \setminus \{i,k\}$.

To get a better idea of the automaton, let us run it in S_4^4 for $is = 0321$ and $jt = 12^3$. We fix $i = 0$ and $j = 1$ in the automaton and insert $g = 3$ and $h = 2$ into state 0. All four values being different, this leads to state 1 and input of the next pair $(s_2, t_2) = (2, 2)$ tells us that $k = h = 2$ and we have to continue in state B. Finally, the last pair $(s_1, t_1) = (1, 2)$ leaves us in B (cases not occuring on the arrows mean no change of state), and we can conclude that both, the direct path and the path through the subgraph $2S_4^3$, are shortest $0321, 12^3$-paths (cf. Proposition 2.24).

However, if $is = 03^3$, then again we insert all pairs, but end up in state 1. Therefore in this case no k is fixed as another candidate for a shortest path because the direct path is the only shortest one (cf. Proposition 2.23).

For a more advanced application of the decision algorithm let us now indicate how to use it to prove all previous results about distances to almost-extreme vertices. Throughout we will assume that $p \in \mathbb{N}_2$ and that $\{i,j\} \in \binom{P}{2}$. We begin with outer almost-extreme vertices $j^n h$, $\{j,h\} \in \binom{P}{2}$, $n \in \mathbb{N}$ (cf. [57, Proposition 2.3]).

If $n = 1$, the input (s_1, h) in state 0 of the automaton leads to and ends in one of the states 1, A or D, except when $s_1 = h \neq i$, in which case we end in B with $k = h$. Otherwise, we either reach D if $s_n \in \{i,j\}$ or fix $k = s_n$ and enter state A. From there we can only get to B in the last step (and therefore never to C or E) and only if $s_d = k$ for all $d \in [n-1]$. This proves Corollary 2.21 again and shows that $\mathrm{d}(is, j^nh) = \mathrm{d}_p(is, j^nk) = \mathrm{d}(s, j^n) + 1 + \mathrm{d}(i^n, j^{n-1}k) = \mathrm{d}(s, j^n) + 2^n - [i = k]$, hence proving Proposition 2.7. Another direct consequence is Theorem 2.22.

We now turn to inner almost-extreme vertices jh^n, $\{j, h\} \in \binom{P}{2}$, $n \in \mathbb{N}$. Let $i \in P \setminus \{j\}$ and $s \in P^n$. Then the automaton is fixed for the pair (i, j) and we enter in state 0 the pair $(g, h) = (s_n, h)$. If $h = i$, then we already terminate in state D, in accordance with the statement in Proposition 2.23, so that we may now assume that $\{h, i, j\} \in \binom{P}{3}$. The input $g = j$ also leads to D and s is not special w.r.t. jh^n. The inputs $g = i$ and $g = h$ result in states A and B, respectively, with k fixed to h. Finally, if $|\{g, h, i, j\}| = 4$ we have to move to state 1 for a decision about k. Here the input (s_d, h), $d \in [n-1]$, leads back to state 1 for $s_d = g$ and to state D for $s_d = j$; $s_d = h$ results in state B and $s_d \notin \{g, h, j\}$ is state A, in both cases with k fixed as h. In state A, the input of (s_d, k), $d \in [n-1]$, results in D for $s_d = j$, in B for $s_d = k$ and in A otherwise. Likewise in B we return to B, if $s_d = j$, we end up in E, if $s_d = k$ and we go to state C otherwise. We cannot escape from the couple of states C and E, i.e. when s is special but not odd, because this would only be possible for an input (j, i). We only end up in state B if there is a $\delta \in [n]$ with $s_\delta = h = k$, $s_d \notin \{j.k\}$ for $d \in [n] \setminus \delta$, and $s_d = j$ for $d \in [\delta - 1]$, i.e. if s is odd. This proves Propositions 2.23, and 2.24 and hence also Proposition 2.9.

We can conclude this chapter by saying that the algorithm based on the automaton of [57] allows to determine the value of the distance between any two vertices explicitly, i.e. without minima or the like. So it is (almost) as good as a closed formula. All metric properties that are derived from the graph distance are determined by the values of the distance function.

3 Perfect codes and related topics

Codes in graphs, introduced by Biggs [10], present a generalization of the classical error-correcting codes. In the graph theory setting, the Hamming codes and the Lee codes are the perfect codes of the Cartesian product of complete graphs and cycles, respectively. As non-trivial codes are rare in coding theory, it is clear that very few graphs contain non-trivial codes. For instance, Kratochvíl [85] proved a remarkable result that there are no (non-trivial) 1-perfect codes in the Cartesian product of several copies of a complete bipartite graph with at least three vertices. It is also known that apart from H_p^n, $n \in [3]_0$, there are no 1-perfect codes in Hanoi graphs for $p \in \mathbb{N}_4$; see [59, p. 195]. It is hence appealing that Sierpiński graphs, and therefore in particular H_3^n, actually *do* possess 1-perfect codes. In the next section we first present this result and then continue with related results on (a, b)-codes, identifying codes, and $L(2, 1)$-labelings. In the subsequent section we move to domination type invariants.

3.1 Perfect and related codes

In this section 1-perfect codes, (a, b)-codes, identifying codes, and locating-dominating codes in Sierpiński graphs are considered. At the end it is then shown that perfect (and almost-perfect) codes can be employed to determine an optimal $L(2, 1)$-labeling of S_p^n.

Let G be a graph and $t \in \mathbb{N}$. A set of vertices $C \subseteq V(G)$ is a *t-code* in G, if for any two distinct vertices u, v of G, $\mathrm{d}_G(u, v) \geq 2t+1$. The set C is called a *t-perfect code*, if for every $v \in V(G)$ there is exactly one $c \in C$ such that $\mathrm{d}(c, v) \leq t$. In particular, if C is a 1-perfect code of G, then $N_G[C] = V(G)$.[14]

The following theorem says that Sierpiński graphs are utmost nice with respect to 1-perfect codes.

Theorem 3.1 [78, Theorem 3.6] *If $p \in \mathbb{N}$ and $n \in \mathbb{N}_0$, then the graph S_p^n has a unique 1-perfect code, if n is even, and there are exactly p 1-perfect codes, if n is odd. Moreover, if n is odd, then each 1-perfect code is determined by the only extreme vertex it contains.*

In Figure 11, 1-perfect codes in S_4^2 and S_3^3 are shown. The three 1-perfect codes of S_3^3 are shown in red, blue and yellow, respectively.

Theorem 3.1 had been proved earlier by Cull and Nelson [23] for the case S_3^n. A shorter argument for the uniqueness part of Theorem 3.1 was given in [44]. In [78] a linear algorithm is presented that, for a given 1-perfect code C of S_p^n and a vertex v of S_p^n, determines whether v is an element of C, and if not, it finds the neighbor of v in C. Li and Nelson [86] also proved that the base-3 Sierpiński graphs contain no non-trivial t-perfect codes for $t \geq 2$. (A code is called *trivial* if it consists of a single vertex. Note that if $t \geq \mathrm{diam}(G)$, then any vertex of G forms a trivial t-perfect code.)

In passing we point to two related papers. In [81] it is proved that S_p^n ($p \geq 3$, $n \geq 2$) contains a set of vertices D such that the (open) neighborhoods of the vertices from D partition $V(S_p^n)$ if and only if p is even. (Note that a graph contains a set whose closed neighborhoods partition its vertex set if and only if

[14] As usual, we denote the *open neighborhood* of a vertex u in G by $N_G(u)$, and $N_G[u] = N_G(u) \cup \{u\}$ is the *closed neighborhood* of u. For $D \subseteq V(G)$ we write $N_G(D)$ and $N_G[D]$ for the union of these sets over all $u \in D$, respectively. Moreover, if $S \subseteq V(G)$ we set $N_S(u) := \{v \in S \mid \{u, v\} \in E(G)\}$, and similarly $N_S[u] := N_S(u) \cup \{u\}$.

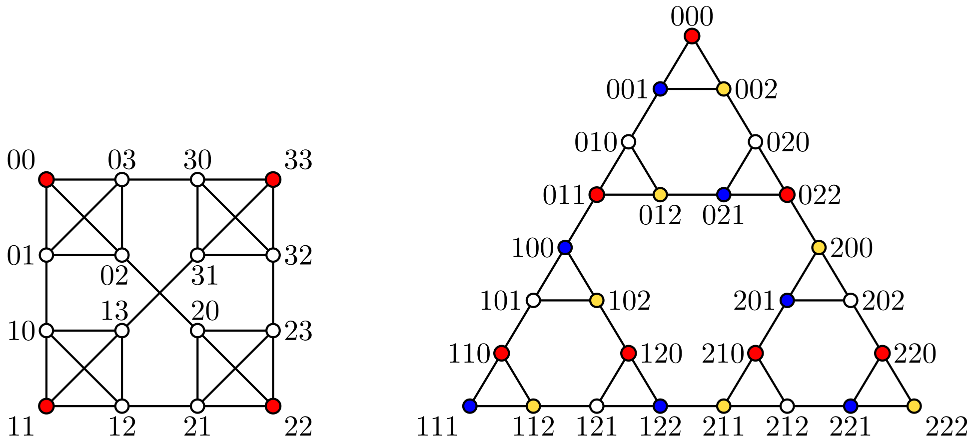

Figure 11: 1-perfect codes of S_4^2 (left) and S_3^3 (right)

the graph has a 1-perfect code.) In the second paper to be mentioned here [111] the graphs S_3^n found an application in the theory of error-correcting codes.

(a, b)-codes

Another type of codes that were studied on the family of Sierpiński graphs are the (a, b)-codes introduced in [27] as follows. If G is a graph and $a, b \in \mathbb{N}_0$, $a + b \geq 1$, then an *(a, b)-code* of G is a set C of vertices with the property that a vertex in C has exactly a neighbors in C and a vertex, which is not in C, has exactly b neighbors in C. Note that $(0, 1)$-codes are precisely the 1-perfect codes. The following result thus widely generalizes Theorem 3.1.

Theorem 3.2 [9, Theorem 1.1] *If $p, n \in \mathbb{N}_2$, then S_p^n contains an (a, b)-code if and only if $a < p$ and one of the following statements holds.*

(i) $a \geq 1$, $b = a$ and p is even;

(ii) $a \geq 2$ is even, $b = a$ and p is odd;

(iii) $a = 0$ and $b = 1$ (the case of 1-perfect codes);

(iv) $a \geq 1$, $b = a + 1$ and n is odd;

(v) $a \geq 1$, $b = a + 2$, $p = 2a + 1$ and $n = 2$.

In [9] it was furthermore observed that all the existing (a, b)-codes in S_p^n are unique up to symmetries.

Identifying codes

To present additional results on codes in Sierpiński graphs some new definitions are needed. Let $G = (V(G), E(G))$ be a graph and let $C \subseteq V(G)$. Then C is a *dominating set*, if every vertex in $V(G) \setminus C$ has at least one adjacent vertex in C, that is, if $N_G[C] = V(G)$. A dominating set C is an *identifying code* if $N[u] \cap C \neq N[v] \cap C$ holds for any two distinct vertices $u, v \in V(G)$ and is a *locating-dominating code* if $N[u] \cap C \neq N[v] \cap C$ holds for any two distinct vertices $u, v \in V(G) \setminus C$. For more information on these codes see the book [21]. Gravier et al. proved:

Theorem 3.3 [45, Theorems 2.1 and 3.1] *If $p \in \mathbb{N}_3$ and $n \in \mathbb{N}_2$, then the minimum cardinality of an identifying code in S_p^n is $p^{n-1}(p-1)$, and the minimum cardinality of a locating-dominating code in S_p^n is $p^{n-1}(p-1)/2$.*

$L(2,1)$**-labelings**

An $L(2,1)$-*labeling* of a graph G is a labeling of its vertices with labels from $[\lambda + 1]_0$ such that vertices at distance 2 get different labels and the labels of adjacent vertices differ by at least 2. The maximum label used in an $L(2,1)$-labeling f is called the *span* of f. The minimum span over all $L(2,1)$-labeling of G is denoted by $\lambda(G)$ and called the λ-*number* of G. For a survey on the $L(2,1)$-labelings (and, more generally, $L(h,k)$-labelings) see [16].

Here is a connection between perfect codes and $L(2,1)$-labelings. By definition, the set of vertices of a given label in an $L(2,1)$-labeling of G forms a 1-code. Hence, if f is an $L(2,1)$-labeling of G with span λ, and C_i, $i \in [\lambda+1]_0$, is the set of vertices u with $f(u) = i$, then the sets $C_0, \ldots, C_\lambda$ form a partition of $V(G)$ and two distinct vertices from C_i are at distance at least 3. Labeling all the vertices from C_i with $2i$ we arrive at the following observation.

Proposition 3.4 [44, Proposition 1.1] *If G is a graph and $\{C_0, \ldots, C_k\}$ is a partition of $V(G)$ such that C_i, $i \in [k+1]_0$, forms a 1-code in G, then $\lambda(G) \leq 2k$.*

Proposition 3.4 was the starting point for the following result.

Theorem 3.5 [44, Theorem 3.2] *If $p \in \mathbb{N}_3$ and $n \in \mathbb{N}_2$, then $\lambda(S_p^n) = 2p$.*

The λ-numbers are, of course, also known for Sierpiński graphs with $p < 3$ or $n < 2$, that is, for paths and complete graphs. They are excluded from the statement to avoid several (uninteresting) cases.

A special type of $L(2,1)$-labelings are equitable $L(2,1)$-labelings. An $L(2,1)$-labeling is *equitable*, if the orders of its labeling classes differ by at most one. The *equitable* $L(2,1)$*-labeling number*, $\lambda_e(G)$, of a graph G is the smallest integer ℓ, such that there is an equitable $L(2,1)$-labeling of G of span ℓ. Fu and Xie [37] determined the equitable $L(2,1)$-labeling number of Sierpiński graphs; it turns out that it equals the λ-number.

Theorem 3.6 [37, Theorem 3.3] *If* $p, n \in \mathbb{N}_2$*, then* $\lambda_e(S_p^n) = 2p$.

3.2 Domination-type invariants

In this section we cover the following domination-type problems studied on Sierpiński graphs: the standard domination number, the total domination number, the generalized power domination, the global strong defensive alliance, and the hub number.

The *domination number* $\gamma(G)$ of a graph G is the order of a smallest dominating set in G. $D \subseteq V(G)$ is a *total dominating set*, if every vertex in $V(G)$ has at least one adjacent vertex in D and the *total domination number* $\gamma_t(G)$ of G is the order of a smallest total dominating set in G. For more information on the domination number and the total domination number see the books [53] and [54], respectively.

By definition, 1-perfect codes are dominating sets. Hence if C is a 1-perfect code of G, then $\gamma(G) \leq |C|$. It was independently proved several times (cf. [53, Theorem 4.2]) that equality holds, as the next result asserts.

Proposition 3.7 *If* C *is a 1-perfect code of a graph* G*, then* $\gamma(G) = |C|$*. In particular, all perfect codes of* G *have the same cardinality.*

Proof. Let $C = \{c_1, \ldots, c_\ell\}$, $\ell = |C|$, be a 1-perfect code of G. Then $N[C] = V(G)$ and this implies $\gamma(G) \leq \ell$.

Let $D = \{d_1, \ldots, d_{\ell'}\}$, $\ell' = |D|$, be a dominating set of G. Then for an arbitrary $i \in [\ell]$ there is a $j \in [\ell']$, so that $d_j \in N[c_i]$. By taking the minimal such j, we get an injective mapping from $[\ell]$ to $[\ell']$. It is indeed injective, since for arbitrary distinct $c, c' \in C$, $N[c] \cap N[c'] = \emptyset$. Now, by using the pigeonhole principle, $\ell \leq \ell'$. □

Combining Theorem 3.1 with Proposition 3.7 we get:

Theorem 3.8 [78, Theorem 3.8] *If* $p \in \mathbb{N}$ *and* $n \in \mathbb{N}_0$*, then*

$$\gamma(S_p^n) = \frac{p^n + p^{[n \text{ even}]}}{p+1}\,.$$

For those Hanoi graphs which do not possess a 1-perfect code the domination number is not known.

With some more effort, Gravier et al. determined the total domination number of Sierpiński graphs.

Theorem 3.9 [45, Theorem 4.1] *If* $p \in \mathbb{N}_3$ *and* $n \in \mathbb{N}_2$*, then* $\gamma_t(S_p^n) = p^{n-1} + [p \text{ odd}]$.

A longest sequence S of distinct vertices of a graph G such that each vertex of S dominates some vertex that is not dominated by its preceding vertices, is called a *Grundy dominating sequence*. The length of S is the *Grundy domination number* of G and denoted by $\gamma_{gr}(G)$. Brešar, Gologranc, and Kos [13] proved the following result.

Theorem 3.10 [13, Theorem 3.1] *If* $p, n \in \mathbb{N}$*, then*

$$\gamma_{gr}(S_p^n) = p^{n-1} + \frac{p(p^{n-1}-1)}{2}\,.$$

The *outer-connected domination number* $\widetilde{\gamma}_c(G)$ of a graph G is the size of a smallest dominating set D of G such that the graph induced by $V(G) \setminus D$ is connected. Chang, Liu, and Wang [17] proved that if $p \in \mathbb{N}_3$ and $n \in \mathbb{N}$, then $\widetilde{\gamma}_c(S_p^n) = \widetilde{\gamma}_c({}^+S_p^n) = p^{n-1}$ and $\widetilde{\gamma}_c({}^{++}S_p^n) = p^{n-1} + p^{n-2}$. Moreover, if $n \in [2]$, then $\widetilde{\gamma}_c(\widehat{S}_3^n) = n$ and if $n \in \mathbb{N}_3$, then $\widetilde{\gamma}_c(\widehat{S}_3^n) = 3^{n-2}$.

The generalized power domination, introduced in [18], was recently studied on Sierpiński graphs [28]. The problem is defined as follows. Let $k \in \mathbb{N}_0$. Then we want to determine a subset S of vertices of a graph G, such that starting with $X = N[S]$, and iteratively adding vertices to X which have a neighbor v in X and at most k neighbors of v are not yet in X, we arrive at $X = V(G)$. The *k-power domination number*, $\gamma_{P,k}(G)$ of G is the size of a smallest subset S of vertices with this property. For $k = 0$ the problem is to determine the domination number, that is, $\gamma_{P,0}(G) = \gamma(G)$, while $\gamma_{P,1}(G)$ is the usual power domination introduced in [51]. Here is the main result of [28].

Theorem 3.11 [28, Theorem 3.1] *If* $p, k \in \mathbb{N}$ *and* $n \in \mathbb{N}_0$*, then*

$$\gamma_{P,k}(S_p^n) = \begin{cases} 1\,, & n \in [2]_0 \text{ or } p \in [k+1]\,, \\ p-k\,, & n = 2 \text{ and } p \in \mathbb{N}_{k+2}\,, \\ (p-k-1)p^{n-2}\,, & \text{otherwise.} \end{cases}$$

Among other tools, the proof of Theorem 3.11 makes use of the fact that Sierpiński graphs are hamiltonian (Theorem 1.7).

The motivation for the concept of power domination was the problem of monitoring electrical power systems using as few measurement devices in the system as possible [7, 51]. Since it is also important in how many steps such a monitoring occurs, the concept of the *propagation radius*, $\mathrm{rad}_{P,k}(G)$ was introduced in [28] as $1 +$ a minimum number of iterations in the process of k-power dominating the graph G, when starting with a k-power dominating set S, taken over all minimum k-power dominating sets of G. The propagation radius of Sierpiński graphs was then determined as follows, where for the cases when p lies between $k+1+\sqrt{k+1}$ and $2k+2$ the exact value is not known but limited to only two possibilities.

Theorem 3.12 [28, Theorem 5.3] *Let $p, k \in \mathbb{N}$ and $n \in \mathbb{N}_3$. Then*

$$\mathrm{rad}_{P,k}(S_p^n) = \begin{cases} 3\,, & p \geq 2k+3\,, \\ 4 \text{ or } 5\,, & 2k+2 \geq p \geq k+1+\sqrt{k+1}\,, \\ 5\,, & k+1+\sqrt{k+1} > p \geq k+2\,, \\ \mathrm{rad}(S_p^n)\,, & p \leq k+1\,. \end{cases}$$

We add here that in [110] the k-power domination number and the corresponding propagation radius of the Hanoi graphs H_p^2 have been determined.

Another domination-type concept, known under the name of defensive alliance, cf. [52], was studied on Sierpiński graphs in [88]. Call a nonempty set of vertices $S \subseteq V(G)$ of a graph G a *defensive alliance*, if $|N_S[v]| \geq |N_{V(G)\setminus S}(v)|$ for every vertex $v \in S$. A subset S of vertices is called a *strong defensive alliance*, if $|N_S[v]| > |N_{V(G)\setminus S}(v)|$ for every vertex $v \in S$. Finally, a strong defensive alliance of G is *global*, if it forms a dominating set in G. Lin, Liu, and Wang [88] determined the global strong defensive alliance number $\gamma_{\hat{d}}$ of S_p^n, that is, the minimum cardinality among all global strong defensive alliances, as follows.

Theorem 3.13 [88, Theorem 3.9] *If $p \in \mathbb{N}_3$ and $n \in \mathbb{N}_2$, then*

$$\gamma_{\hat{d}}(S_p^n) = \frac{p + [p \text{ odd}]}{2} \cdot p^{n-1}\,.$$

The proof of this theorem includes a construction of an optimal global strong defensive alliance. An example in S_4^3 can be seen in Figure 12.

The last domination-type invariant presented here that was studied on Sierpiński graphs is the hub number introduced by Walsh in [112]. A set $Q \subseteq V(G)$ is a *hub set* of a graph G, if for every pair of vertices $u, v \in V(G) \setminus Q$ there exists

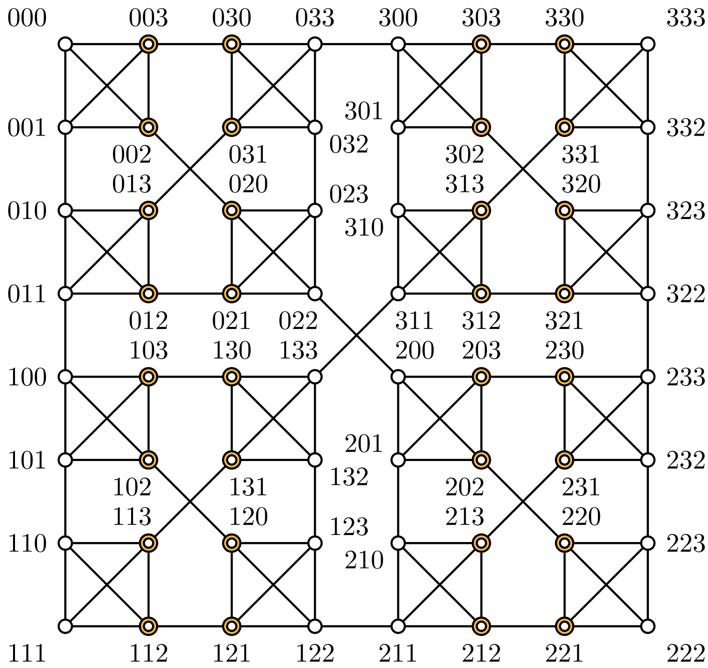

Figure 12: An optimal global strong defensive alliance of S_4^3

a u,v-path such that all intermediate vertices on this path are in Q. The *hub number*, $h(G)$, of a graph G is the size of a smallest hub set of G. A hub set Q of G is *connected*, if Q is a connected set (i.e., the subgraph of G induced by Q is connected). The *connected hub number*, $h_c(G)$, of a graph G is the size of a smallest connected hub set of G. In a similar way one defines the *connected domination number*, $\gamma_c(G)$, of a graph G as the size of a smallest connected dominating set.

In the seminal paper [112] Walsh showed that $\gamma(G) \leq h(G)+1$ holds for every graph G. Moreover, if G is connected, then $h_c(G) \leq \gamma_c(G)$. Grauman et al. [43] combined these properties into the following result.

Theorem 3.14 [43, Theorem 2.1] *If G is a connected graph, then*

$$h(G) \leq h_c(G) \leq \gamma_c(G) \leq h(G) + 1\,.$$

Lin et al. [89] determined the hub number of Sierpiński graphs.

Theorem 3.15 [89, Theorem 9] *If $p, n \in \mathbb{N}$, then*

$$h_c(S_p^n) = h(S_p^n) = 2(p^{n-1} - 1)\,,$$

an optimal hub set of S_p^n being the following:

$$Q_{S_p^n} = \{\underline{s}0\ell^d, \underline{s}\ell 0^d \mid d \in [n-1], \underline{s} \in P^{n-d-1}, \ell \in [p-1]\}\,.$$

An example of an optimal hub set of S_4^3 is shown in Figure 13. Using symmetry we could get p different optimal hub sets by replacing 0 with any $i \in P$ in $Q_{S_p^n}$. Then

$$Q^{(i)}_{S_p^n} = \{\underline{s}i\ell^d, \underline{s}\ell i^d \mid d \in [n-1], \underline{s} \in P^{n-d-1}, \ell \in [p-1]\}$$

is also an optimal hub set for S_p^n.

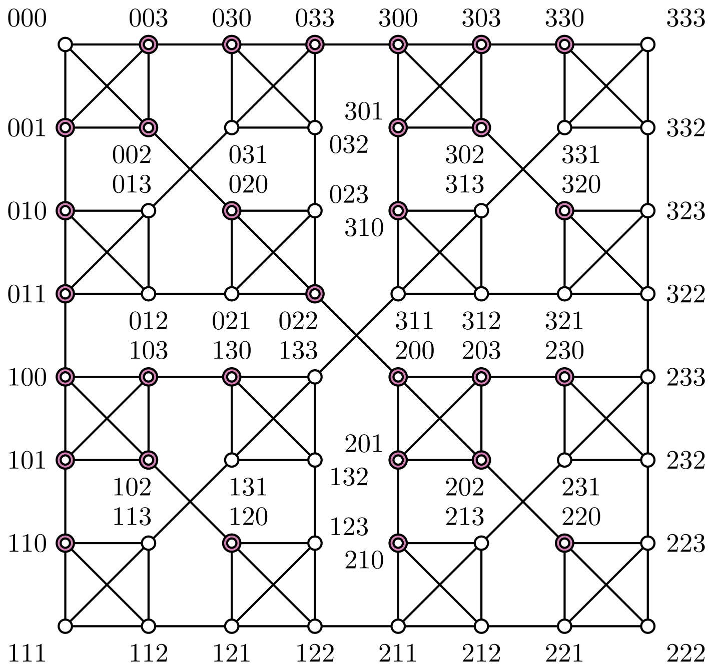

Figure 13: An optimal hub set of S_4^3

Finally, for results on the so-called 2- and 3-rainbow domination number of Sierpiński graphs see [92].

4 Coloring Sierpiński graphs

Many different colorings of Sierpiński graphs have been investigated so far; in this chapter we survey the corresponding results. In the next section we consider the classical colorings (vertex, edge, and total), while in the subsequent section other types of colorings studied on Sierpiński graphs are presented.

4.1 Vertex, edge, and total colorings

A *k-coloring* of a graph $G = (V(G), E(G))$ is a mapping $c : V(G) \to X$, where X is a k-set of *colors*, such that adjacent vertices receive different colors. If there is a k-coloring of G, then G is *k-colorable*. The *chromatic number*, $\chi(G)$, of G is the minimum integer k, such that G is k-colorable.

Let $n \in \mathbb{N}$. Since K_p is a subgraph of S_p^n, we infer that $\chi(S_p^n) \geq p$. On the other hand, Parisse [94, p. 147] observed that the mapping $c : P^n \to P$ defined by

$$c(s_n \dots s_1) = s_1$$

is a p-coloring of $\chi(S_p^n)$. Hence we have the following fact.

Proposition 4.1 [94] *If $p, n \in \mathbb{N}$, then $\chi(S_p^n) = p$.*

A *k-edge-coloring* of a graph $G = (V(G), E(G))$ is a mapping c' from $E(G)$ to a k-set of colors, such that adjacent edges receive different colors. If there is a k-edge-coloring of G, we say that G is *k-edge-colorable*. The *edge-chromatic number* or *chromatic index*, $\chi'(G)$, of G is the minimum integer k, such that G is k-edge-colorable. For the chromatic index of Sierpiński graphs we have:

Theorem 4.2 [72, Theorem 4.1] *If $p, n \in \mathbb{N}_2$, then $\chi'(S_p^n) = p$.*

Theorem 4.2 was proved earlier in [76] for S_3^n, where in addition it was shown that these graphs are uniquely 3-edge-colorable. For another proof of Theorem 4.2 see [63, Theorem 3]. Figure 14 (left) presents a 5-edge-coloring of S_5^2.

A *k-total-coloring* of a graph G is a mapping from the combined set of vertices end edges of G to a k-set (of colors), such that adjacent vertices or edges and incident vertices and edges receive different colors. If there is a k-total-coloring of G, we say that G is *k-total-colorable*. Then the *total chromatic number*, $\chi''(G)$, of G is the minimum integer k, such that G is k-total-colorable. For the total chromatic number of Sierpiński graphs we have the following result:

Theorem 4.3 [63, Theorem 4] *If $p, n \in \mathbb{N}_2$, then $\chi''(S_p^n) = p + 1$.*

For the case when p is odd, Theorem 4.3 was proved earlier by Jakovac and Klavžar [72]. They also showed that $\chi''(S_4^n) = 5$ and conjectured that the total chromatic number of S_p^n for even $p > 4$ equals $p + 2$. Hinz and Parisse [63] disproved the conjecture and rounded off the problem as stated in Theorem 4.3. See Figure 14 (right) for a 5-total coloring of S_4^2.

We finally mention that in [117] the chromatic number of the square of S_p^n was determined to be $p + 1$, where the *square* of a graph G is the graph obtained from G by adding edges between all pairs of vertices at distance 2.

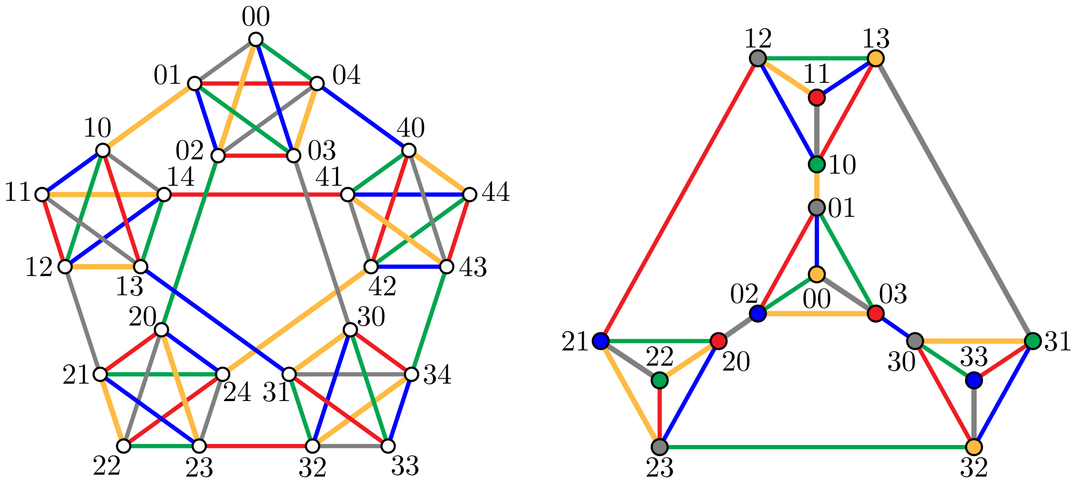

Figure 14: A 5-edge-coloring of S_5^2 (left) and a 5-total-coloring of S_4^2 (right)

4.2 Other colorings

We next present additional colorings of Sierpiński graphs: b-colorings, $\{P_r\}$-free colorings, path t-colorings, linear t-colorings, the edge ranking number, and the packing chromatic number.

A vertex coloring of a graph G is a *b-coloring* if each color class contains a vertex u, such that in each other color class there is a vertex adjacent to u. The *b-chromatic number*, $\varphi(G)$, of G is the maximum integer k, such that there exists a b-coloring of G with k colors. This concept was introduced by Irving and Manlove [69] and received a lot of attention, especially in the last few years, cf. [2, 6] and references therein.

For small p and n, the b-chromatic number of S_p^n is well known: $\varphi(S_p^1) = \varphi(K_p) = p$, $\varphi(S_1^n) = \varphi(K_1) = 1$, $\varphi(S_2^2) = \varphi(P_4) = 2$, and for $n \in \mathbb{N}_3$, $\varphi(S_2^n) = \varphi(P_{2^n}) = 3$. For the other values the b-chromatic number was determined by Jakovac in his PhD thesis as follows.

Proposition 4.4 [70, Trditev 5.1] *If $p \in \mathbb{N}_3$ and $n \in \mathbb{N}_2$, then $\varphi(S_p^n) = p + 1$.*

Suppose $\mathcal{F}$ is a nonempty family of connected bipartite graphs, where each member F of $\mathcal{F}$ has at least 3 vertices. Then a k-coloring of a graph G is *$\mathcal{F}$-free* if G contains no 2-colored subgraphs isomorphic to any graph F of $\mathcal{F}$. The $\mathcal{F}$-free chromatic number, $\chi_{\mathcal{F}}(G)$, of G is the minimum integer k, such that there exists an $\mathcal{F}$-free coloring of G with k colors. In the case $\mathcal{F} = \{P_4\}$ this concept coincides with the the so-called star coloring introduced in [33]; see also [73]. Fu [36] studied the $\{P_r\}$-free colorings of Sierpiński graphs and proved:

Theorem 4.5 [36, Lemma 4.1, Theorem 4.2] *If $p, n \in \mathbb{N}_2$, then*

$$\chi_{\{P_3\}}(S_p^n) = p + 1 = \chi_{\{P_4\}}(S_p^n)\,.$$

As a consequence of this result, Fu deduced in [36, Corollary 4.4] that for every $p \in \mathbb{N}_2$, $n \in \mathbb{N}$, and $5 \le r \le p^n$,

$$p \le \chi_{\{P_r\}}(S_p^n) \le p + 1\,.$$

Path t-colorings and linear t-colorings of Sierpiński graphs were studied by Xue, Zuo, and Li [116] and by Xue et al. [118], respectively.

Let G be a graph and let c be a mapping from $V(G)$ to $[t]$. Then c is a *path t-coloring* if for each $i \in [t]$, the subgraph induced by the vertices of color i is a union of vertex-disjoint paths. The *vertex linear arboricity*, $\mathrm{vla}(G)$, of G is the minimum t such that there exists a path t-coloring of G. Now we have:

Theorem 4.6 [116, Theorem 4.1] *If $p \in \mathbb{N}_2$ and $n \in \mathbb{N}$, then*

$$\mathrm{vla}(S_p^n) = \frac{p + [p \text{ odd}]}{2}\,.$$

Xue, Zuo, and Li then followed with stronger results by establishing additional properties that the corresponding path t-colorings possess. In [116, Theorem 4.2] they did this for p odd and in [116, Theorem 4.3] for p even.

A proper vertex coloring of a graph is called *linear* if the subgraph induced by the vertices colored by any two colors is a union of vertex-disjoint paths. The linear chromatic number of a graph G, denoted by $\mathrm{lc}(G)$, is the minimum number of colors among linear colorings of G. For the linear chromatic number of Sierpiński graphs we have the following result:

Theorem 4.7 [118, Theorem 3.4] *If $p, n \in \mathbb{N}$, then $\mathrm{lc}(S_p^n) = p$.*

The next coloring invariant we present is the edge ranking number defined as follows. Let $c' : E(G) \to [t]$ be a t-edge-coloring of a graph G. Then c' is an *edge t-ranking* if for any two edges of the same color, every path between them contains an intermediate edge with a larger color value. The *edge ranking number*, $\chi'_r(G)$, is the smallest integer t, such that there exists an edge t-ranking of G. Lin, Juan, and Wang [90] proved the following relation between the edge ranking number of Sierpiński graphs and the edge ranking number of complete graphs.

Theorem 4.8 [90, Theorem 7] *If* $p, n \in \mathbb{N}_2$, *then* $\chi'_r(S_p^n) = n \cdot \chi'_r(K_p)$.

Combining Theorem 4.8 with the result for the edge ranking number of complete graphs from [11] yields:

Corollary 4.9 [90, Corollary 8] *If* $p, n \in \mathbb{N}_2$, *then*

$$\chi'_r(S_p^n) = \frac{n}{3}(p^2 + g(p)),$$

where g *is the* Bodlaender function, *defined by* $g(1) = -1$ *and*

$$g(m) = \begin{cases} g\left(\frac{m}{2}\right), & m \text{ even}, \\ g\left(\frac{m+1}{2}\right) + \frac{m-1}{2}, & m \text{ odd}. \end{cases}$$

We conclude this chapter by describing the following very recent coloring study of Sierpiński graphs. If G is a graph, then the *packing chromatic number* $\chi_\rho(G)$ of G is the smallest integer k such that there exists a k-vertex coloring of G in which any two vertices receiving color i are at distance at least $i+1$. This concept was introduced in [41] and given the present name in [14]. It turned out that the packing chromatic number is an intrinsically difficult graph invariant. This can be argued best by the celebrated related result of Fiala and Golovach which asserts that determining χ_ρ is NP-complete even when restricted to trees [34]. On the positive side, the packing chromatic number has been determined for several classes of graphs. For the present survey, the following result from [15] is relevant.

Theorem 4.10 *If* $n \in [4]$, *then*

$$\chi_\rho(S_3^n) = \begin{cases} 3, & n = 1, \\ 5, & n = 2, \\ 7, & n = 3, 4. \end{cases}$$

Moreover, if $n \in \mathbb{N}_5$, *then* $\chi_\rho(S_3^n)$ *is either* 8 *or* 9.

5 Embeddings of Sierpiński graphs

In this chapter we consider embeddings of Sierpiński graphs into Hanoi graphs and Cartesian product graphs. By an *embedding* of a graph G into a graph H we mean an injective mapping $f : V(G) \to V(H)$, such that if uv is an edge in G, then $f(u)f(v)$ is an edge of H. (In short, an embedding is an injective homomorphism.)

The resulting subgraph of H, denoted by $f(G)$, is not necessarily induced. In the case that $f(G)$ is an induced subgraph of H we speak of an *induced embedding*. In addition, an embedding $f : V(G) \to V(H)$ is *isometric* if $\mathrm{d}_H(f(u), f(v)) = \mathrm{d}_G(u, v)$ holds for every pair of vertices $u, v \in V(G)$. Note that every isometric embedding is induced but not the other way round.

5.1 Embeddings into Hanoi graphs

Recall from Section 0.1 that Sierpiński graph S_p^n and Hanoi graph H_p^n are defined on the same vertex set, that S_3^n is isomorphic to H_3^n, and that S_p^n is no longer isomorphic to H_p^n for $p \in \mathbb{N}_4$ and $n \in \mathbb{N}_2$. Since the Hanoi graph H_p^n has significantly more edges than the Sierpiński graph S_p^n it seems reasonable to expect that S_p^n embeds into H_p^n. However, this in not always the case as the next result asserts.

Theorem 5.1 [60, Theorem 3.1] *If $p, n \in \mathbb{N}$, then S_p^n can be embedded into H_p^n if and only if p is odd or $n = 1$.*

Let us prove the non-existence part of Theorem 5.1. For $n \in \mathbb{N}_2$, we have $\|S_2^n\| = 2^n - 1 > 2^{n-1} = \|H_2^n\|$, so that S_2^n cannot be embedded into H_2^n. (In fact, H_2^n is a spanning subgraph of S_2^n.)

Let next $p \in \mathbb{N}_4$ be even and $n = 2$. Assume that there is an embedding $\alpha : S_p^2 \to H_p^2$. By Proposition 0.3, the p-cliques of S_p^2 are mapped onto the p-cliques of H_p^2. The remaining edges of S_p^2, these are exactly the edges $e_{ij}^{(2)}$, $i, j \in P$, $i \neq j$, have to be mapped by α to edges in H_p^2 corresponding to moves of disc 2. There are $\binom{p}{2}$ edges $e_{ij}^{(n)}$ of S_p^2 and they are pairwise non-adjacent. On the other hand, edges in H_p^2 corresponding to moves of disc 2 induce p cliques of order $p-1$. Among the edges of these cliques, we can select at most $p\left\lfloor\frac{p-1}{2}\right\rfloor$ independent ones. Since p is even, $p\left\lfloor\frac{p-1}{2}\right\rfloor < p\,\frac{p-1}{2} = \binom{p}{2}$. We conclude that S_p^2 cannot be embedded into H_p^2.

We will now reduce the more general case for even p and $n \in \mathbb{N}_3$ to the case just dealt with. Let α' be an embedding of S_p^n into H_p^n. The key idea is to consider the image $\alpha'(0^{n-2}S_p^2)$. Since non-extreme vertices of S_p^n are of degree p, they cannot be mapped by α' to perfect vertices. Hence, the p extreme vertices of S_p^n are mapped to p perfect vertices of H_p^n so that $\alpha'(0^n) = j^n$ for some j. Using Proposition 0.3 again, $\alpha'(0^{n-1}S_p^1) = j^{n-1}H_p^1$. Moreover, the subgraph $0^{n-2}S_p^2$ of S_p^n contains $p-1$ p-cliques that are at distance 1 from the clique $0^{n-1}S_p^1$. All the other cliques of S_p^n are at distance larger than 1 from $0^{n-1}S_p^1$. Similarly, the subgraph $j^{n-2}H_p^2$ of H_p^n contains p p-cliques that are pairwise at distance 1.

Every other p-clique of H_p^n is at distance at least 2 from $j^{n-1}H_p^1$. (Indeed, suppose another clique which is not in $j^{n-2}H_p^2$, say $j^{n-3}iH_p^2$, $i \neq j$, would be connected to a vertex $j^{n-1}k$ of $j^{n-1}H_p^1$. Then the vertex of $j^{n-3}iH_p^2$ would have the form $j^{n-3}ijk$, leading to a contradiction with the definition of the Hanoi graphs.) Therefore, $\alpha'(0^{n-2}S_p^2) = j^{n-2}H_p^2$. Hence α' would embed $0^{n-2}S_p^2 \cong S_p^2$ into $j^{n-2}H_p^2 \cong H_p^2$, a possibility which we already excluded.

An embedding of S_p^n into H_p^n for even p was constructed in [60]. We do not give the details here, instead we refer to Figure 15 for an embedding of S_5^2 into H_5^2.

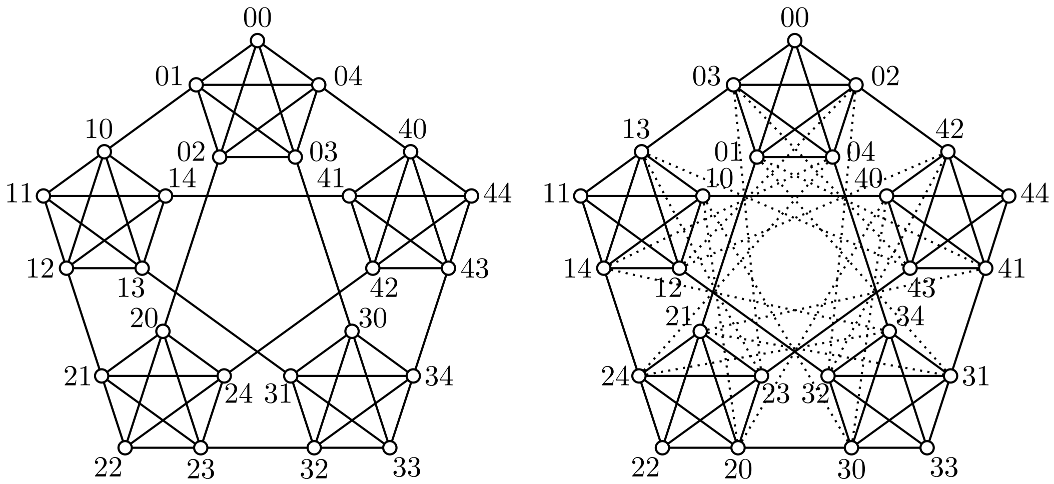

Figure 15: An embedding of S_5^2 into H_5^2

5.2 Embeddings into Cartesian products

In this section we discuss embeddings, induced embeddings, and isometric embeddings of Sierpiński graphs into Cartesian products of graphs.

The *Cartesian product* $G \,\square\, H$ of graphs G and H is the graph on the vertex set $V(G \,\square\, H) = V(G) \times V(H)$, vertices (g, h) and (g', h') being adjacent if either $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $h = h'$. Cartesian products of complete graphs are called *Hamming graphs*. In particular, the n-tuple Cartesian product of the complete graph K_p is denoted by $K_p^{\square,n}$ for brevity.

As observed in [68, Section 2.2], Hanoi graphs H_p^n are spanning subgraphs of $K_p^{\square,n}$. Combining this fact with Theorem 5.1 we get an embedding of odd-base Sierpiński graphs into Hamming graphs.

Corollary 5.2 *If $p \in \mathbb{N}$ is odd and $n \in \mathbb{N}_0$, then S_p^n is a spanning subgraph of $K_p^{\square,n}$.*

The embeddings from Corollary 5.2 being not induced, we next turn to induced embeddings into Hamming graphs. When embedding a graph G into a Cartesian product graph, factor graphs and their vertices that are not used in the image of G play no (essential) role. This is formalized in the following definition.

Definition 5.3 *Let G be a graph and $H = \mathop{\square}\limits_{i=1}^{k} H_i$ be a Cartesian product graph. An embedding $f : V(G) \to V(H)$ is* irredundant *if (i) $|V(H_i)| \geq 2$ for every $i \in [k]$, and (ii) every vertex $h \in \bigcup\limits_{i=1}^{k} V(H_i)$ occurs as a component in the image of some vertex $g \in V(G)$, i.e. $h = f(g)_i$ for some $i \in [k]$. If f is irredundant, then $f(G)$ is called an* irredundant subgraph *of $H = \mathop{\square}\limits_{i=1}^{k} H_i$.*

The embeddings from Theorem 5.1 (and hence also from Corollary 5.2) are clearly irredundant. Now, the Hamming dimension of a graph was introduced in [82] as follows.

Definition 5.4 *Let G be a graph. The* Hamming dimension, $\mathrm{Hdim}(G)$, *of G is the largest number of factors of a Hamming graph into which G embeds as an irredundant induced subgraph. If G is not an induced subgraph of any Hamming graph we set* $\mathrm{Hdim}(G) = \infty$.

To determine the Hamming dimension of a graph, the theory of induced embeddings into Hamming graphs developed in [80] is useful. Let G be a graph and let $\mathcal{F} = \{F_1, \ldots, F_\rho\}$ be a partition of $E(G)$. Such a partition naturally yields the corresponding labeling (of the edge set) $\mathcal{L} : E(G) \to [\rho]$ by setting $\mathcal{L}(e) = i$ for $e \in F_i$. We say that a labeling is *non-trivial* if $\rho \in \mathbb{N}_2$. Further, we introduce two conditions for a labeling:

Condition A. *An edge labeling of a graph G fulfills Condition A, if for any triangle of G, its edges have the same label.*

Condition B. *An edge labeling of a graph G fulfills Condition B, if for any vertices u and v of G at distance at least 2, there exist different labels i and j which both appear on any induced u,v-path. (An induced path in our case is an induced subgraph X of G isomorphic to a path graph.)*

With these two condition we can formulate:

Theorem 5.5 [80, Theorem 3.3] *If G is a connected graph, then* $\mathrm{Hdim}(G) < \infty$ *if and only if there exists a labeling of the edges of G that fulfills Conditions A and B.*

The proof of Theorem 5.5 demonstrates that if a k-labeling of the edges of G that fulfills Conditions A and B is given, then G admits an induced, irredundant embedding into a Hamming graph with k factors, that is, $\mathrm{Hdim}(G) \leq k$. Using this approach the following results on the Hamming dimension of Sierpiński graphs were proved in [82]. First, the following exact values are known.

Theorem 5.6 *(i)* $\mathrm{Hdim}(S_3^2) = 3$, $\mathrm{Hdim}(S_3^3) = 6$,
(ii) $\mathrm{Hdim}(S_p^2) = 2$ *for* $p \in \mathbb{N}_4$,
(iii) $\mathrm{Hdim}(S_p^3) = 4$ *for* $p \in \mathbb{N}_4$.

For the base-3 Sierpiński graphs the dimension can be bounded as follows:

Theorem 5.7 *If $n \in \mathbb{N}_4$, then*

$$\frac{7}{4} \cdot 3^{n-3} + 3 \cdot 2^{n-4} + \frac{3}{2}n - \frac{9}{4} \leq \mathrm{Hdim}(S_3^n) \leq 5 \cdot 3^{n-3} + 1\,.$$

Finally, for $p \in \mathbb{N}_4$ we have the following upper bound.

Theorem 5.8 *If $p \in \mathbb{N}_4$ and $n \in \mathbb{N}_2$, then*

$$\mathrm{Hdim}(S_p^n) \leq \frac{2}{p-1}p^{n-2} + \frac{2p-4}{p-1}\,.$$

Combining the above theory with the Sierpiński triangle labeling, an induced embedding of Sierpiński graphs into the Cartesian product of Sierpiński triangle graphs can also be deduced.

Theorem 5.9 [82, Theorem 3.2] *If $p \in \mathbb{N}_3$ and $n \in \mathbb{N}$, then there exists an induced embedding*

$$S_p^n \to \mathop{\Box}_{i=0}^{n-1} \widehat{S}_p^i\,.$$

After embeddings and induced embeddings we finally consider isometric embeddings into Cartesian products. The classical theory due to Graham and Winkler [42] asserts that there is a unique isometric, irredundant embedding of a graph into a Cartesian product graph with the largest possible number of factors.

It is called the *canonical metric representation.* We first briefly present this embedding and then describe it for Sierpiński graphs. For more details on canonical metric representation see [49, Chapters 11 and 13] and [68, Chapter 14].

Let $e = uv$ and $f = xy$ be edges of a graph G. The edges e and f are in relation Θ if $\mathrm{d}(u,x) + \mathrm{d}(v,y) \neq \mathrm{d}(u,y) + \mathrm{d}(v,x)$. Relation Θ is reflexive and symmetric; let Θ^* be its transitive closure. Then Θ^* partitions $E(G)$ into equivalence classes $F_1 \ldots, F_k$. This partition is called *trivial* if $k = 1$. For any $i \in [k]$ let G/F_i be the graph whose vertex set consists of the connected components of the graph $G - F_i$, two components C and C' being adjacent if there exists an edge $uv \in F_i$ such that $u \in C$ and $v \in C'$. Let in addition $\alpha_i : V(G) \to V(G/F_i)$ be the mapping that maps a vertex v to the connected component of $G - F_i$ that contains v. Then the *canonical metric representation* of G is the mapping

$$\alpha : V(G) \to V\left(\mathop{\square}_{i=1}^{k} G/F_i\right)$$

defined by

$$\alpha(v) = (\alpha_1(v), \ldots, \alpha_k(v))\,.$$

The key property of α is that it is an isometric embedding [42]. Note that if the Θ^*-partition of $E(G)$ is trivial, then α is the identity mapping on $V(G)$.

The canonical metric representation is trivial for $S_p^1 = K_p$, $p \in \mathbb{N}$, and $S_1^n = K_1$, $n \in \mathbb{N}$. For $S_2^n = P_{2^n}$, every edge of S_2^n forms its own Θ^*-class and hence α isometrically embeds S_2^n into the hypercube Q_{2^n-1}. For $p \in \mathbb{N}_4$ the canonical metric representation is also trivial.

Proposition 5.10 [82, Proposition 6.2] *If $p \in \mathbb{N}_4$ and $n \in \mathbb{N}$, then the canonical isometric representation of S_p^n is trivial.*

Thus our only hope for a non-trivial canonical metric representation remains the case S_3^n. And it is indeed the case! For some initial base-3-Sierpiński graphs it is easy to determine Θ^*-classes, see Figure 16 for S_3^2 and S_3^3.

To describe the Θ^*-classes of S_3^n for all values of n explicitly, recall that $i^{n-m}e_{j\ell}^{(m)}$ is the (only) edge between the subgraphs $i^{n-m}jS_3^{m-1}$ and $i^{n-m}\ell S_3^{m-1}$ of S_3^n. Now let $T = \{i, j, \ell\}$ and set

$$\begin{aligned} F_i^n &= \big\{\{i^n, i^{n-1}j\}, \{i^n, i^{n-1}\ell\}\big\} \cup \big\{i^{n-m}e_{j\ell}^{(m)} \mid m \in [n]\big\}\,, \\ \widetilde{F}^n &= E(S_3^n) \setminus (F_0^n \cup F_1^n \cup F_2^n)\,. \end{aligned}$$

Then we have:

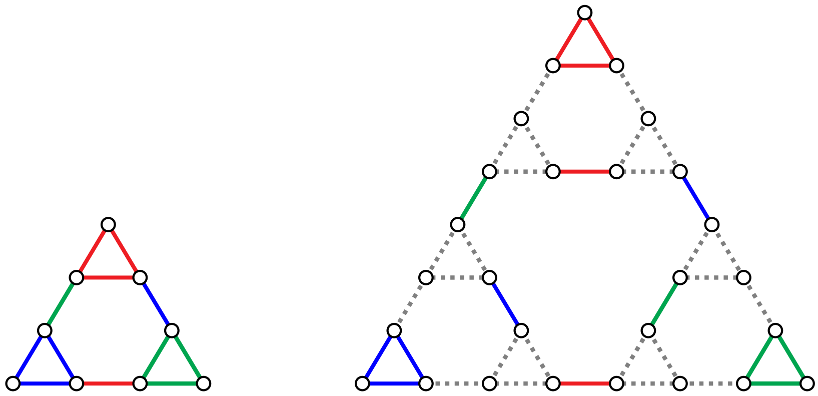

Figure 16: Θ^*-classes of S_3^2 (left) and S_3^3 (right)

Theorem 5.11 [82, Theorem 6.5] *If $n \in \mathbb{N}_2$, then the Θ^*-classes of S_3^n are F_0^n, F_1^n, F_2^n, and $\widetilde{F}^n$.*

In Figure 16, Θ^*-classes F_0^2 and F_0^3 are drawn in red, F_1^2 and F_1^3 in blue, and F_2^2 and F_2^3 in green. Note that $\widetilde{F}^2 = \emptyset$ and $\widetilde{F}^3$ is drawn with dotted gray lines. The quotient graph $S_3^4/\widetilde{F}^4$ is shown in Figure 17.

We conclude the chapter by noting that embeddings of graphs, in which one does not require that edges are mapped into edges, are also intensely investigated. For such embeddings an important measure is the *dilation* of an embedding defined as the maximum distance between pairs of vertices of the host graph that are images of end-vertices of edges of a guest graph. In this direction Rajan et al. [98] determined the dilation of embeddings of specific circulant graphs into Sierpiński graphs S_3^n.

6 Perspectives

Hasunuma [50] recently introduced an interesting construction that generalizes Sierpiński graphs. It allowed him to extend (often with simpler proofs) several results about Sierpiński graphs and to obtain some new results about them. His idea is the following.

Let $G = (V(G), E(G))$ be a graph that may contain loops; as we will see shortly, loops are utmost important in the construction. The following two operations are the key ingredients of Hasunuma's approach.

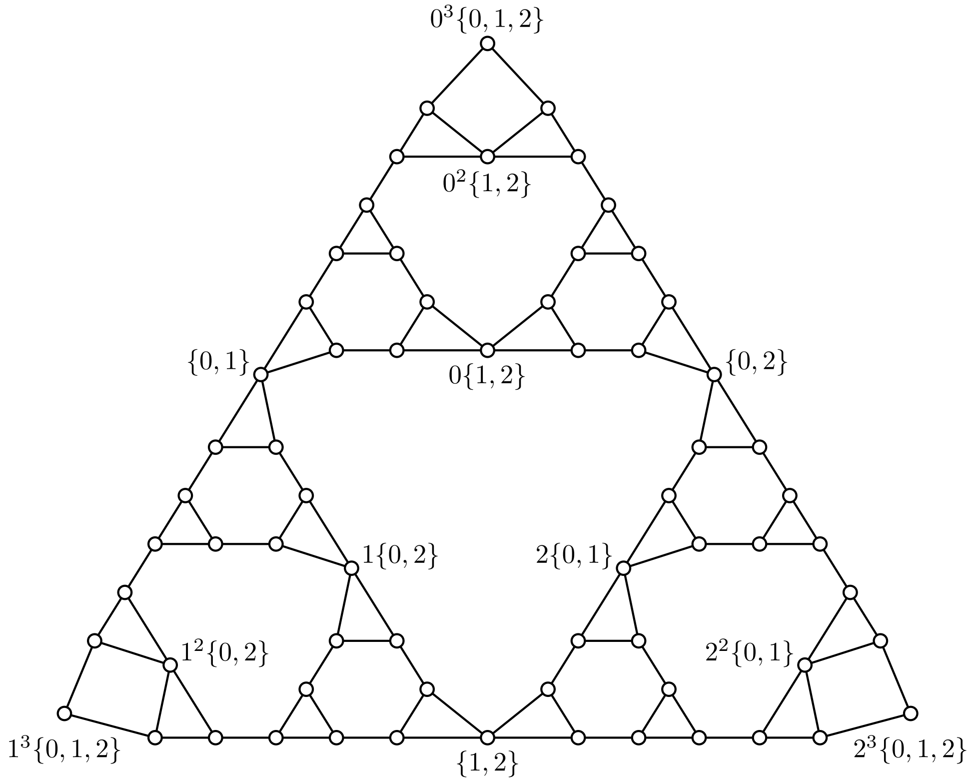

Figure 17: The quotient graph $S_3^4/\widetilde{F}^4$

- The *barycentric subdivision* $B(G)$ of G is the graph obtained from G by subdividing every edge of G by one vertex, respectively, except loops.
- The *line graph* $L(G)$ of G is the graph whose vertex set is $E(G)$ and in which two different vertices are adjacent if they are adjacent as edges in G. A vertex of $L(G)$ that corresponds to a loop of G also has a loop in $L(G)$.

Now, the *subdivided-line graph* $\Gamma(G)$ of G is the line graph of the barycentric subdivision of G, that is, $\Gamma(G) = L(B(G))$, and the *n-iterated subdivided-line graph* $\Gamma^n(G)$ of G is the graph obtained from G by iteratively applying the subdivided-line graph operation n times. The connection between subdivided-line graphs and Sierpiński graphs should be clear from the following basic result.

Proposition 6.1 [50, Proposition 3.3] *Let G be a graph and $n \in \mathbb{N}$. Then each vertex of $\Gamma^n(G)$ can be assigned a label of the form $v_0v_1 \dots v_n$, where $v_0 \in V(G)$*

and $v_1, \dots, v_n$ are neighbors of v_0, such that vertices $v_0v_1 \dots v_n$ and $w_0w_1 \dots w_n$ are adjacent in $\Gamma^n(G)$ if and only if there exists an $h \in [n+1]_0$ with $v_0 \dots v_{h-1} = w_0 \dots w_{h-1}$, $v_h \neq w_h$, and $v_j = w_h$, $w_j = v_h$ for $h < j \leq n$. Moreover, a vertex of the form $v_0v_0 \dots v_0$ has a loop.

Let $(S_p^n)^\circ$ be the graph obtained from S_p^n by adding a loop to each of its extreme vertices, let K_n° be the graph obtained from K_n by adding a loop to each of it vertices, and let $K_1^{p,\circ}$ be the graph obtained from K_1 by adding p loops to it. From Proposition 6.1 we get:

Corollary 6.2 [50, Lemma 3.4] *If $p, n \in \mathbb{N}$, then*

$$(S_p^n)^\circ \cong \Gamma^n(K_1^{p,\circ}) \cong \Gamma^{n-1}(K_p^\circ)\,.$$

Moreover, $\Gamma^{n-1}(K_{p+1}) \cong {}^{++}S_p^n$, where ${}^{++}S_p^n$ is the regularization of S_p^n from Definition 0.10.

By Corollary 6.2, any result on $\Gamma^n(G)$ has a specialization to $(S_p^n)^\circ$ which in all practical cases also gives a specialization to S_p^n. For instance, the next result generalizes Proposition 1.1.

Proposition 6.3 [50, Proposition 3.7] *If G is a graph and $n \in \mathbb{N}$, then*

$$\kappa(\Gamma^n(G)) = \kappa'(\Gamma^n(G)) = \kappa'(G)\,.$$

Concerning hamiltonicity, Hasunuma proved in [50, Corollary 4.6] that if G is a graph, $\ell \leq \delta(G)/2$, and there exists ℓ edge-disjoint hamiltonian cycles (resp., paths) in G, then there exist ℓ edge-disjoint hamiltonian cycles (resp., paths) in $\Gamma^n(G)$, $n \in \mathbb{N}$. In this way he has generalized Theorems 1.8 and 1.9. With respect to hamiltonicity, Hasunuma in part extended results from [38, 39].

Hasunuma also generalized Theorem 3.15 concerning the hub number of Sierpiński graphs. Considering independent spanning trees in subdivided-line graphs he proved that there are $p-1$ independent spanning trees rooted at any vertex of S_p^n [50, Theorem 6.3]. Moreover, he studied the so-called book-embeddings of subdivided-line graphs (and consequently of Sierpiński graphs).

Another interesting generalization of Sierpiński graphs is due to Gravier, Kovše, and Parreau [46]. Their idea is that instead of having a complete graph as a building block, any graph can serve for this purpose. More precisely, if G is a graph with $V(G) = P$, then the *generalized Sierpiński graph $S_p^n(G)$* is the graph with the vertex set P^n, and $u = u_1 \dots u_n$ is adjacent to $v = v_1 \dots v_n$ if and only if there exists an $i \in [n]$ such that $u_j = v_j$ for $j < i$, $u_i \neq v_i$ and $u_iv_i \in E(G)$, and $u_j = v_i$ and $v_j = u_i$ for $j < i$. The authors of [46] announced some appealing results about generalized Sierpiński graphs concerning their automorphism groups, their distinguishing number, and perfect codes; here the distinguishing number of a graph is the smallest number of colors needed to color its vertices such that no non-trivial automorphism preserves the coloring. These results definitely deserve to be elaborated in detail.

Several papers that examined the structure of generalized Sierpiński graphs already appeared. Geetha and Somasundaram [40] determined the total chromatic number for certain families of these graphs. In [101] their chromatic number, vertex cover number, clique number, and domination number, are investigated, while in [99] the so-called Roman domination number of generalized Sierpiński graphs is considered. On the other hand, Rodríguez-Velázquez and Tomás-Andreu [100] obtained closed formulas for the Randić index of several families of generalized Sierpiński graphs, while in [30] this work was extended to the so-called generalized Randić index. Here the *Randić index* is one of the central invariants in chemical graph theory and is, for a graph G, defined as $\sum_{uv \in E(G)} 1/\sqrt{\deg(u)\deg(v)}$. Finally, just before submitting the final version of this survey an extensive paper on the distances in generalized Sierpiński graphs has been posted [31].

Given the interpretation of Sierpiński graphs as the state graphs of the Switching Tower of Hanoi game, there is a natural extension to an infinite graph assuming an infinite supply of discs. This graph, called the *Sisyphean Sierpiński graph*, was introduced in [59, pp. 142f] and still needs to be explored further. A similar remark applies to infinite Sierpiński triangle graphs.

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