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Stern polynomials ☆

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Abstract

Stern polynomials $B_k(t)$, $k \geqslant 0$, $t \in \mathbb{R}$, are introduced in the following way: $B_0(t) = 0$, $B_1(t) = 1$, $B_{2n}(t) = tB_n(t)$, and $B_{2n+1}(t) = B_{n+1}(t) + B_n(t)$. It is shown that $B_n(t)$ has a simple explicit representation in terms of the hyperbinary representations of $n-1$ and that $B'_{2n-1}(0)$ equals the number of 1's in the standard Gray code for $n-1$. It is also proved that the degree of $B_n(t)$ equals the difference between the length and the weight of the non-adjacent form of n.

MSC: 11B83

Keywords: Stern (diatomic) sequence; Stern polynomials; Hyperbinary representation; Standard Gray code; Non-adjacent form

1. Introduction

Stern sequence [17] or, as it is often called, *Stern diatomic series* $b(n)$ is defined recursively by

$$b(0) = 0, \qquad b(1) = 1,$$

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$$b(2n) = b(n), \quad n \geqslant 1,$$
$$b(2n+1) = b(n) + b(n+1), \quad n \geqslant 1.$$

The sequence thus starts as

$$0,\ 1,\ 1,\ 2,\ 1,\ 3,\ 2,\ 3,\ 1,\ 4,\ 3,\ 5,\ 2,\ 5,\ 3,\ 4,\ 1,\ 5,\ 4,\ 7,\ 3,\ \ldots$$

and can, for instance, be obtained as a one-dimensional extract of the so-called Stern–Brocot array. This sequence is `A002487` in Sloane's online database of integer sequences [16].

Stern sequence has been studied in several different fields of mathematics, as a sample of references we propose [9,11,12,15] and a comprehensive survey [18]. The sequence also appears in a very general theory of k-regular sequences due to Allouche and Shallit [1,2].

A nice application of the Stern sequence is given in [4], where Calkin and Wilf prove that the sequence defined by the quotients $b(n)/b(n+1)$, $n \geqslant 1$, encounters every positive rational exactly once. The Calkin and Wilf encoding of positive rationals hence begins as:

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, \frac{4}{1}, \frac{1}{5}, \frac{5}{4}, \frac{4}{7}, \frac{7}{3}, \ldots.$$

Motivated by the definition of the Stern sequence and the above application we now introduce *Stern polynomials* $B_k(t)$, $k \geqslant 0$, $t \in \mathbb{R}$, in the following way.

$$B_0(t) = 0, \qquad B_1(t) = 1,$$
$$B_{2n}(t) = tB_n(t), \quad n \geqslant 1,$$
$$B_{2n+1}(t) = B_{n+1}(t) + B_n(t), \quad n \geqslant 1.$$

The first few of them are: $B_0(t) = 0$, $B_1(t) = 1$, $B_2(t) = t$, $B_3(t) = t + 1$, $B_4(t) = t^2$, $B_5(t) = 2t + 1$, $B_6(t) = t^2 + t$, $B_7(t) = t^2 + t + 1$, and $B_8(t) = t^3$, see also Table 1. Note that

$$B_n(1) = b(n), \quad n \geqslant 0. \tag{1}$$

It is also interesting to observe that the sequence of natural numbers can be encoded as $B_n(2) = n$.

Several well-known sequences of polynomials are defined in a way similar to the one in which we define the Stern polynomials. For instance, the Fibonacci polynomials $F_n(t)$ are defined with $F_0(t) = 0$, $F_1(t) = 1$, and $F_n(t) = tF_{n-1}(t) + F_{n-2}(t)$, see, for example, [19,20]; for recent results on these polynomials cf. also [7,22]. Another such class is formed by the Lucas polynomials $L_n(t)$ defined with $L_0(t) = 2$, $L_1(t) = t$, and $L_n(t) = tL_{n-1}(t) + L_{n-2}(t)$, see [19,21]. Analogously to (1), the Fibonacci numbers and the Lucas numbers are obtained as $F_n(1)$ and $L_n(1)$, respectively. It is interesting to add that Lucas and Fibonacci polynomials have several applications, even in mathematical physics [5].

The main purpose of this paper is to introduce Stern polynomials and to demonstrate that they have many appealing properties. We begin by showing that the polynomial $B_n(t)$ has a simple explicit representation in terms of the hyperbinary representations of n. More precisely,

$$B_n(t) = \sum_{\ell \geqslant 0} \left| \begin{matrix} n-1 \\ \ell \end{matrix} \right| t^{\ell},$$

Table 1
Polynomials $B_n(t)$ obtained from hyperbinary representations of $n-1$

n	0	1	2	3	4	5	6	7	8	9
$\mathcal{H}(n-1)$		$(0)_{[2]}$	$(1)_{[2]}$	$(10)_{[2]}$ $(2)_{[2]}$	$(11)_{[2]}$	$(100)_{[2]}$ $(12)_{[2]}$ $(20)_{[2]}$	$(101)_{[2]}$ $(21)_{[2]}$	$(110)_{[2]}$ $(102)_{[2]}$ $(22)_{[2]}$	$(111)_{[2]}$	$(1000)_{[2]}$ $(112)_{[2]}$ $(120)_{[2]}$ $(200)_{[2]}$
$B_n(t)$	0	1	t	$t+1$	t^2	$2t+1$	t^2+t	t^2+t+1	t^3	t^2+2t+1

n	10	11	12	13	14	15	16
$\mathcal{H}(n-1)$	$(1001)_{[2]}$ $(121)_{[2]}$ $(201)_{[2]}$	$(1010)_{[2]}$ $(1002)_{[2]}$ $(122)_{[2]}$ $(210)_{[2]}$ $(202)_{[2]}$	$(1011)_{[2]}$ $(211)_{[2]}$	$(1100)_{[2]}$ $(1012)_{[2]}$ $(1020)_{[2]}$ $(212)_{[2]}$ $(220)_{[2]}$	$(1101)_{[2]}$ $(1021)_{[2]}$ $(221)_{[2]}$	$(1110)_{[2]}$ $(1102)_{[2]}$ $(1022)_{[2]}$ $(222)_{[2]}$	$(1111)_{[2]}$
$B_n(t)$	$2t^2+t$	t^2+3t+1	t^3+t^2	$2t^2+2t+1$	t^3+t^2+t	t^3+t^2+t+1	t^4

where we use the symbol $\left|{m \atop k}\right|$ to denote the number of hyperbinary representations of m containing exactly k digits 1. Then we prove that $B'_{2n-1}(0)$, $n \geqslant 1$, equals the number of 1's in the standard Gray code for $n-1$. We conclude the paper by proving that the degree of $B_n(t)$, $\deg(B_n(t))$, equals the difference between the length and the weight of the non-adjacent form of n.

2. Explicit representation of Stern polynomials

A *hyperbinary representation* of a non-negative integer n is a representation of n as a sum of powers of 2, each power being used at most twice. We will employ the notation $(a_1a_2\dots a_m)_{[2]}$ to describe the hyperbinary representation $\sum_{i=1}^{m} a_i 2^{m-i}$, $a_i \in \{0,1,2\}$. Let $\mathcal{H}(n)$ denote the set of all hyperbinary representations $(a_1a_2\dots a_m)_{[2]}$ of n, where any two representations of the same integer differing only in zeros on the left-hand side are identified. For instance, $(1)_{[2]}$ is the same representation of 1 as $(01)_{[2]}$. It is well-known, see [4,15], that $b(n)$ counts the number of hyperbinary representations of $n-1$.

Theorem 1. *For any $n \in \mathbb{N}$, $b(n) = |\mathcal{H}(n-1)|$.*

The idea of the proof for Theorem 1 is that the recursive formulas are established by noting that when $n-1 = (a_1a_2\dots a_m)_{[2]}$ is odd, then a_m must be 1, and if $n-1$ is even, a_m may be 0 or 2, but not 1. This theorem can be extended to the Stern polynomials in the following way.

Theorem 2. *For any $n \in \mathbb{N}$,*

$$B_n(t) = \sum_{(a_1a_2\dots a_m)_{[2]} \in \mathcal{H}(n-1)} t^{|\{i \mid a_i = 1\}|}.$$

Proof. The assertion is easily verified to be true for small n. Suppose the result is true up to $n-1$ for some n.

Let n be even, say $n = 2k$, and consider an arbitrary hyperbinary representation $n-1 = (a_1a_2\dots a_m)_{[2]}$. Since $n-1$ is odd, $a_m = 1$ by the observation before the theorem. As $k-1 = (a_1a_2\dots a_{m-1})_{[2]}$, the polynomial that counts the number of 1's in the representation $(a_1a_2\dots a_m)_{[2]}$ is obtained from the polynomial for $(a_1a_2\dots a_{m-1})_{[2]}$ by multiplication by t. As $B_{2k}(t) = tB_k(t)$, the result holds for $n = 2k$ by the induction hypothesis.

Suppose next that n is odd, say $2k+1$. Then $n-1 = (a_1a_2\dots a_m)_{[2]}$ is even and a_m must be 0 or 2. Hence no multiplication by t based on counting the number of 1's in $(a_1a_2\dots a_{m-1})_{[2]}$ appears. Now, if $a_m = 0$ then $(a_1a_2\dots a_{m-1})_{[2]}$ is a hyperbinary representation of k, and if $a_m = 2$, then $(a_1a_2\dots a_{m-1})_{[2]}$ is a hyperbinary representation of $k-1$. Applying the recursive formula $B_{2k+1}(t) = B_{k+1}(t) + B_k(t)$ one gets the result for $n = 2k+1$ by the induction hypothesis. $\square$

Theorem 2 is illustrated in Table 1 for $n \leqslant 16$.

Recall that by the symbol $\left|{m \atop k}\right|$ we denote the number of hyperbinary representations of m containing exactly k digits 1. Then Theorem 2 can be rewritten in the following way.

Corollary 3.

$$B_n(t) = \sum_{\ell \geqslant 0} \left| {n-1 \atop \ell} \right| t^{\ell}.$$

We close this section by the following property of the Stern polynomials.

Proposition 4. *For any $m \geqslant 0$ and any $k \geqslant 1$,*

$$B_{2^{k-1}(2m+1)}(t) = \frac{1}{t}\big(B_{2^k m}(t) + B_{2^k(m+1)}(t)\big).$$

Proof. Compute as follows:

$$\begin{aligned}
\frac{1}{t}\big(B_{2^k m}(t) + B_{2^k(m+1)}(t)\big) &= \frac{1}{t}\big(t^k B_m(t) + t^k B_{m+1}(t)\big)\\
&= t^{k-1}\big(B_m(t) + B_{m+1}(t)\big)\\
&= t^{k-1} B_{2m+1}(t)\\
&= B_{2^{k-1}(2m+1)}(t). \qquad \square
\end{aligned}$$

3. Stern polynomials and standard Gray code

The *standard Gray code* of n is defined as the binary representation of $g(n)$, where $g : \mathbb{N} \longrightarrow \mathbb{N}$ is defined by

$$g(0) = 0, \qquad g\big(2^p + j\big) = 2^p + g\big(2^p - 1 - j\big) \quad \text{for } 0 \leqslant j < 2^p, \tag{2}$$

Table 2
Standard Gray code $g(n)_{(2)}$ of n and the number of 1's in it

n	$g(n)$	$g(n)_{(2)}$	
0	0	$0\,0\,\boxed{0\,0\,0}$	$0 = x(1)$
1	1	$0\,0\,\boxed{0\,0\,1}$	$1 = x(2)$
2	3	$0\,0\,\boxed{0\,1\,1}$	$2 = x(3)$
3	2	$0\,0\,\boxed{0\,1\,0}$	$1 = x(4)$
4	6	$0\,0\,\boxed{1\,1\,0}$	$2 = x(5)$
5	7	$0\,0\,\boxed{1\,1\,1}$	$3 = x(6)$
6	5	$0\,0\,\boxed{1\,0\,1}$	$2 = x(7)$
7	4	$0\,0\,\boxed{1\,0\,0}$	$1 = x(8)$
8	12	$0\,\boxed{1}\,1\,0\,0$	$2 = x(9)$
9	13	$0\,\boxed{1}\,1\,0\,1$	$3 = x(10)$
10	15	$0\,\boxed{1}\,1\,1\,1$	$4 = x(11)$
11	14	$0\,\boxed{1}\,1\,1\,0$	$3 = x(12)$
12	10	$0\,\boxed{1}\,0\,1\,0$	$2 = x(13)$
13	11	$0\,\boxed{1}\,0\,1\,1$	$3 = x(14)$
14	9	$0\,\boxed{1}\,0\,0\,1$	$2 = x(15)$
15	8	$0\,\boxed{1}\,0\,0\,0$	$1 = x(16)$
16	24	$1\,1\,0\,0\,0$	$2 = x(17)$
17	25	$1\,1\,0\,0\,1$	$3 = x(18)$

see [6]. This looks like a complicated definition but the construction of the standard Gray code is simple. The first two words of the code are 0 and 1. Suppose that for some $k \geqslant 1$, the first 2^k words are already known, where every word is written using k digits. Then the next 2^k words of the code are obtained by attaching 1 on the left of each of the first 2^k words in the reverse order. See Table 2 where this construction is indicated for $k = 4$.

The principal interest of the Gray code(s) is that the expansions of two consecutive integers differ at only one place. For some algorithmic aspects on the standard Gray code we refer to [10]. In this section we show that certain coefficients of the Stern polynomials are closely related to this Gray code. More precisely, let $x(n)$ be the coefficient at t^1 of the polynomial $B_{2n-1}(t)$, that is,

$$x(n) = B'_{2n-1}(0). \tag{3}$$

The first few values of this sequence are shown in Table 2.

To establish the connection between the sequence $x(n)$ and the standard Gray code, we need the following auxiliary sequence. For $n \geqslant 0$ let $y(n)$ be the coefficient at t^1 of the polynomial $B_{2n}(t)$, that is,

$$y(n) = B'_{2n}(0). \tag{4}$$

Lemma 5. *For any $n \geqslant 0$, $y(n) = 0$, if n is even, and $y(n) = 1$, if n is odd.*

Proof. It is easily seen that the lemma holds for small n. Then, using $B'_{2n}(t) = tB'_n(t) + B_n(t)$, we get $y(2k) = B'_{4k}(0) = B_{2k}(0) = 0 \cdot B_k(0) = 0$ and $y(2k+1) = B'_{4k+2}(0) = B_{2k+1}(0) = B_k(0) + B_{k+1}(0) = 1$. (Here we use that $\{B_k(0), B_{k+1}(0)\} = \{0, 1\}$ which is easily proved by induction.) $\square$

Lemma 6. *For any* $\ell \geqslant 0$,

$$\begin{aligned} x(4\ell) &= x(2\ell), \\ x(4\ell+1) &= x(2\ell+1), \\ x(4\ell+2) &= x(2\ell+1)+1, \\ x(4\ell+3) &= x(2\ell+2)+1. \end{aligned}$$

Proof. Applying Lemma 5 and having in mind that $B'_{2n+1}(t) = B'_n(t) + B'_{n+1}(t)$ we compute as follows:

$$\begin{aligned} x(4\ell) &= B'_{8\ell-1}(0) = B'_{4\ell-1}(0) + B'_{4\ell}(0) = x(2\ell) + y(2\ell) = x(2\ell), \\ x(4\ell+1) &= B'_{8\ell+1}(0) = B'_{4\ell}(0) + B'_{4\ell+1}(0) = y(2\ell) + x(2\ell+1) = x(2\ell+1), \\ x(4\ell+2) &= B'_{8\ell+3}(0) = B'_{4\ell+1}(0) + B'_{4\ell+2}(0) = x(2\ell+1) + y(2\ell+1) \\ &= x(2\ell+1)+1, \qquad \text{and} \\ x(4\ell+3) &= B'_{8\ell+5}(0) = B'_{4\ell+2}(0) + B'_{4\ell+3}(0) = y(2\ell+1) + x(2\ell+2) \\ &= x(2\ell+2)+1. \qquad \square \end{aligned}$$

We now prove that the sequence $x(n)$ satisfies the following recursive formula:

Lemma 7. $x(1) = 0$, $x(2^k + i) = x(2^k - i + 1) + 1$, *for* $k \geqslant 0$ *and* $0 < i \leqslant 2^k$.

Proof. For $k = 0, 1, 2$ we easily check that $x(2) = x(1) + 1$, $x(3) = x(2) + 1$, $x(4) = x(1) + 1$, $x(5) = x(4) + 1$, $x(6) = x(3) + 1$, $x(7) = x(2) + 1$, $x(8) = x(1) + 1$. Suppose $k \geqslant 2$. Then, using Lemma 6, we inductively get (for appropriate values of ℓ):

$$\begin{aligned} x\left(2^k+4\ell\right) &= x\left(2^{k-1}+2\ell\right) = x\left(2^{k-1}-2\ell+1\right)+1 = x\left(2^k-4\ell+1\right)+1, \\ x\left(2^k+4\ell+1\right) &= x\left(2^{k-1}+2\ell+1\right) = x\left(2^{k-1}-2\ell\right)+1 = x\left(2^k-4\ell\right)+1, \\ x\left(2^k+4\ell+2\right) &= x\left(2^{k-1}+2\ell+1\right)+1 = x\left(2^{k-1}-2\ell\right)+2 \\ &= x\left(2^k-4\ell-1\right)+1, \\ x\left(2^k+4\ell+3\right) &= x\left(2^{k-1}+2\ell+2\right)+1 = x\left(2^{k-1}-2\ell-1\right)+2 \\ &= x\left(2^k-4\ell-2\right)+1. \qquad \square \end{aligned}$$

Now everything is ready for the main result of this section.

Theorem 8. *For any* $n \geqslant 1$, $x(n)$ *equals the number of* 1*'s in the standard Gray code for* $n - 1$.

Proof. Compare Lemma 7 with (2) and keep in mind that 2^p adds an additional digit 1 to the binary representation of the second summand of (2). (Beware of the 1-shift!) $\square$

Corollary 9. *The number of* 1*'s in the standard Gray code for n is the same as the number of hyperbinary representations of* $2n$ *containing exactly one digit* 1.

Proof. By Theorem 8, the number of 1's in the standard Gray code for n equals $x(n+1)$, which by definition equals to the coefficient at t^1 of the polynomial $B_{2n+1}(t)$, i.e. to $\left|{2n \atop 1}\right|$, by Corollary 3. By definition the symbol is equal to the number of hyperbinary representations of $2n$ containing exactly one digit 1. □

The sequence $x(n)$ is `A005811` from [16]. It appears under the name of Kuczma's sequence in [2], where it is proved that $x(n)$ is 2-regular.

4. Stern polynomials and the NAF

A *signed bit representation* of a positive integer is a base 2 representation of the integer in which digits -1, 0, and 1 are allowed. A signed bit representation $n = \sum_{0 \leqslant i \leqslant m} s_i 2^i = s_m \ldots s_0$ is called *non-adjacent form*, NAF for short, if $s_m \neq 0$ and if $s_i \neq 0$ implies $s_{i-1} = 0$ for $i \geqslant 1$. It is well-known that every positive integer has a unique NAF, see [3,14]. The second column of Table 3 gives the NAFs for positive integers up to 17, where $\bar{1}$ stands for -1.

NAF proved to be very useful in computer science, especially for fast computations and in coding theory, see [3,8,13]. This is related to the following well-known remarkable property: among all signed bit representations of an integer n, the NAF minimizes the weight, that is, the number of non-zero digits of a representation. This follows from the facts, that the operation of replacing any block of identical non-zero digits by $100\cdots00\bar{1}$ or $\bar{1}00\cdots001$ (as in $1111 = 1000\bar{1}$ and $\bar{1}\bar{1} = \bar{1}01$) applied to any signed bit representation does not increase its weight, and that the NAF can be obtained from any signed bit representation in finitely many applications of the replacement operation [3].

Table 3
The NAFs of n and $z(n)$—the number of 0's in it

n		$z(n)$
1	1	0
2	10	1
3	$10\bar{1}$	1
4	100	2
5	101	1
6	$10\bar{1}0$	2
7	$100\bar{1}$	2
8	1000	3
9	1001	2
10	1010	2
11	$10\bar{1}0\bar{1}$	2
12	$10\bar{1}00$	3
13	$10\bar{1}01$	2
14	$100\bar{1}0$	3
15	$1000\bar{1}$	3
16	10000	4
17	10001	3

The weight of the NAF of n is denoted by $w(n)$. Let in addition the length $\ell(n)$ of the NAF of n be the number of digits in the NAF, that is, if $s_m \dots s_0$ is the NAF of n, then $\ell(n) = m + 1$. Here is the main result of this section.

Theorem 10. *For any $n \geqslant 1$,*

$$w(n) = \ell(n) - \deg\bigl(B_n(t)\bigr).$$

In the rest of the paper we prove Theorem 10. To shorten the notation set $z(n) = \deg(B_n(t))$.

For the proof it suffices to show that $z(n)$ equals the number of zero digits in the NAF of n. From the definition of $z(n)$ it follows that $z(0)$ is not defined and that $z(1) = 0$, $z(2n) = z(n) + 1$, $z(2n + 1) = \max\{z(n + 1), z(n)\}$, for $n \geqslant 1$. The first few values of this sequence are presented in Table 3.

Lemma 11. *For any $n > 1$, $z(n) + 1 \geqslant \max\{z(n - 1), z(n + 1)\}$. Also, $z(1) + 1 \geqslant z(2)$.*

Proof. The claim obviously holds for small n. For the induction step we have

$$\begin{aligned} z(2n + 1) + 1 &= \max\bigl\{z(n + 1), z(n)\bigr\} + 1 \\ &= \max\bigl\{z(n + 1) + 1, z(n) + 1\bigr\} \\ &= \max\bigl\{z(2n), z(2n + 2)\bigr\}. \end{aligned}$$

For the even case we proceed as follows. From $z(n + 1) + 1 \geqslant \max\{z(n), z(n + 2)\}$ it follows that

$$\begin{aligned} z(n + 1) + 1 &\geqslant \max\bigl\{z(n), z(n + 1), z(n + 2)\bigr\} \\ &= \max\bigl\{\max\bigl\{z(n), z(n + 1)\bigr\}, \max\bigl\{z(n + 1), z(n + 2)\bigr\}\bigr\}, \end{aligned}$$

and thus $z(2n + 2) \geqslant \max\{z(2n + 1), z(2n + 3)\}$. It follows that

$$z(2n + 2) + 1 > \max\bigl\{z(2n + 1), z(2n + 3)\bigr\}. \qquad \square$$

Proposition 12. *The sequence $z(n)$, $n \geqslant 1$, is defined recursively as follows: $z(1) = 0$, $z(2n) = z(n) + 1$, $z(4n - 1) = z(n) + 1$, $z(4n + 1) = z(n) + 1$, for $n \geqslant 1$.*

Proof. Using Lemma 11, we get

$$\begin{aligned} z(4n - 1) &= \max\bigl\{z(2n - 1), z(2n)\bigr\} \\ &= \max\bigl\{\max\bigl\{z(n - 1), z(n)\bigr\}, z(n) + 1\bigr\} \\ &= z(n) + 1, \end{aligned}$$

as well as

$$\begin{aligned} z(4n+1) &= \max\{z(2n), z(2n+1)\} \\ &= \max\{z(n)+1, \max\{z(n), z(n+1)\},\} \\ &= z(n)+1. \qquad \square \end{aligned}$$

Proposition 12 and the definition of the sequence $z(n)$ immediately imply the following recursive form.

Corollary 13. *The sequence $z(n)$, $n \geqslant 1$, is defined recursively as follows: $z(1) = 0$, $z(2n) = z(n) + 1$, $z(4n+1) = z(2n)$, $z(4n+3) = z(2n+2)$, for $n \geqslant 1$.*

From Corollary 13 it follows that $z(m)$, $m \geqslant 1$, counts the number of zero digits in the NAF of m. In other words, if $m = \sum s_i 2^i$ is the NAF of m, then the digit s_0 is 0 if and only if m is an even number, in which case the remaining digits coincide with the digits of $m/2$. (This corresponds to $z(2n) = z(n) + 1$.) In addition, the digit s_0 of $m = 4n + 1$ is 1, and the remaining digits coincide with the digits of $2n$. (This corresponds to $z(4n+1) = z(2n)$.) Finally, the digit s_0 of $m = 4n+3$ is -1, and the remaining digits coincide with the digits of $2n+2$. (This corresponds to $z(4n+3) = z(2n+2)$.) The proof of Theorem 10 is complete.

The sequence $z(n)$, $n \geqslant 1$, is the sequence `A057526` from [16].

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