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Partial cubes and their τ-graphs

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Abstract

The τ-graph G^τ of a partial cube G has the equivalence classes of the Djoković–Winkler relation as vertices, two classes E and F being adjacent if some edges $e \in E$ and $f \in F$ induce a convex P_3. It is shown that for every graph G there exists a median graph M such that $G = M^\tau$, that G^τ is connected if and only if G is a Cartesian prime graph, and that for a median graph G its τ-graph is K_n-free if and only if G contains no convex $K_{1,n}$.

1. Introduction

The celebrated Djoković–Winkler relation Θ [8,17] is defined on the edge set of a graph G in the following way. Edges xy and uv of G are in relation Θ if

$$d(x,u) + d(y,v) \neq d(x,v) + d(y,u).$$

In general Θ is not an equivalence relation. Winkler [17] proved that among bipartite graphs Θ is transitive precisely for graphs isometrically embeddable into hypercubes. These graphs are known as *partial cubes*; see [6,7,9,11,14].

Let e and f be two edges of a graph G; then e and f are in relation τ (see [10,13] or [12, pp. 121]) if $e = f$ or if they form a convex path on three vertices. (That is, $e = uv$, $f = vw$, with $uw \notin E(G)$ and v the only common neighbor of u and w.) For a partial cube G its *τ-graph*

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G^τ is defined as follows. $V(G^\tau)$ consists of the Θ-equivalence classes of G, where Θ-classes E and F are adjacent whenever $E \neq F$ and there exist edges $e \in E$ and $f \in F$ with $e\tau f$.

This concept found a very appealing application in mathematical chemistry. A *hexagonal graph* G is a 2-connected subgraph of the hexagonal lattice so that every bounded face is a hexagon. A hexagonal graph is *catacondensed* if all its vertices lie on its perimeter. The *resonance graph* of G has 1-factors of G as vertices, two 1-factors being adjacent whenever their symmetric difference is the edge set of a hexagon of G. Then Vesel [16] characterized the resonance graphs of catacondensed hexagonal graphs as those median graphs for which G^τ is a tree T with largest degree at most 3 such that the vertices of T of degree 3 correspond to the peripheral Θ-classes of G.

Further motivation for this note comes from the fact that two similar concepts have been studied previously — crossing graphs [2,14] and Θ-graphs [5]. In both cases the derived graphs have Θ-classes as vertices while the adjacencies are defined according to the respective interplays of the classes.

We proceed as follows. In the rest of this section we define concepts needed throughout the note. Then, in Section 2, we show that every graph is a τ-graph of some median graph. In the subsequent section we prove that G^τ is connected if and only if G is a prime graph with respect to the Cartesian product and that the τ-graph of a median graph is K_n-free if and only if it contains no convex $K_{1,n}$. We conclude with two open problems.

For $u, v \in V(G)$, let $d_G(u, v)$, or $d(u, v)$ for short, denote the length of a shortest path in G from u to v. The *interval* $I(u, v)$ between two vertices u and v in G is the set of all vertices on shortest paths between u and v. A subgraph H of a graph G is *isometric* if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$ and *convex* if $I(u, v) \subseteq V(H)$ for any $u, v \in V(H)$. For a connected graph $G = (V, E)$ and an edge ab of G set

$$W_{ab} = \{w \in V \mid d(a, w) < d(b, w)\} \text{ and}$$
$$U_{ab} = \{w \in W_{ab} \mid w \text{ has a neighbor in } W_{ba}\}.$$

Note that if G is bipartite then $V = W_{ab} \cup W_{ba}$.

A graph G is a *median graph* if G is a connected graph such that, for every triple u, v, w of its vertices, $|I(u, v) \cap I(u, w) \cap I(v, w)| = 1$. A subgraph of a median graph G induced by W_{ab} is called *peripheral* if $U_{ab} = W_{ab}$. Among important subclasses of median graphs we mention trees, hypercubes, complete grid graphs, and graphs of acyclic cubical complexes [2]. It is well known that median graphs are partial cubes [15]; see also [12, Proposition 2.22].

2. Representing graphs as τ-graphs

For the representation of graphs as τ-graphs we first recall the following result from [11]. For a subgraph X of a graph G let ∂X be the set of edges with one endvertex in X and the other in $G \setminus X$. Then:

Lemma 2.1 (*Convexity Lemma*). *An induced connected subgraph H of a bipartite graph G is convex if and only if no edge of ∂H is in relation Θ to an edge in H.*

Before stating the representation theorem we prove:

Lemma 2.2. *Let G be a median graph. Then Θ-classes E and F are adjacent in G^τ if and only if there exist adjacent edges $e \in E$ and $f \in F$ and there exist no two edges $e' \in E$ and $f' \in F$ that lie in a common C_4.*

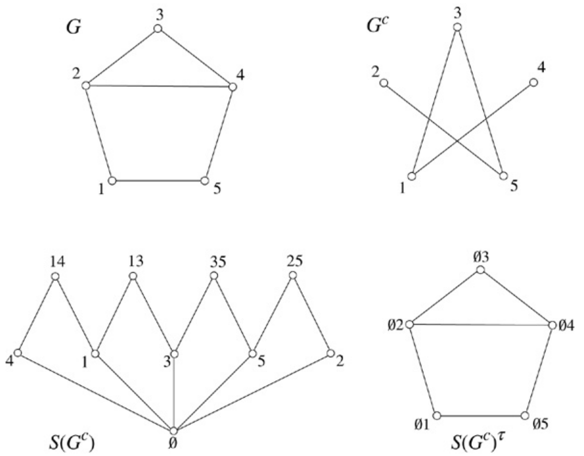

Fig. 1. The house as the τ-graph of the simplex graph of P_5.

Proof. Suppose that E and F are adjacent in G^τ. Let a, b, c denote vertices that induce a convex P_3, where $ab \in E$ and $bc \in F$. Then $c \in W_{ba} \setminus U_{ba}$ for otherwise ab and bc would lie in a 4-cycle. In other words, $bc \in \partial U_{ba}$. Since G is median, U_{ba} is convex; cf. [12, pp. 67]. Then by Lemma 2.1, bc is not in relation Θ with any edge of U_{ba}. Therefore, any edge of F lies either in ∂U_{ba} or in $W_{ba} \setminus U_{ba}$. We conclude that no two edges, one from E and the other from F, lie in a common C_4.

The converse is clear. $\square$

Consider the vertex deleted 3-cube to see that Lemma 2.2 cannot be extended to all partial cubes.

For a graph G, the *simplex graph* $S(G)$ of G is the graph whose vertices are the complete subgraphs of G (including the empty graph), two vertices being adjacent if, as complete subgraphs of G, they differ in exactly one vertex. The simplex graph has been introduced in [4] and later studied elsewhere; see for instance [3]. In particular, simplex graphs are median graphs. Let G^c denote the complement of G, then we have:

Theorem 2.3. *Every graph is a τ-graph of a median graph. More precisely, $G = S(G^c)^\tau$ holds for any graph G.*

Proof. Let $V(G) = \{1, 2, \ldots, n\}$. From the proof of [14, Theorem 3.1] we infer that the simplex graph of G^c has n Θ-classes. More precisely, $S(G^c)$ has Θ-classes E_i with representatives $e_i = \{\emptyset, \{i\}\}$, $1 \le i \le n$. Note that e_i is adjacent to e_j for $i \neq j$.

By Lemma 2.2, E_i and E_j are adjacent in $S(G^c)^\tau$ if and only if e_i and e_j do not belong to a common 4-cycle of $S(G^c)$. This holds if and only if $ij \notin E(G^c)$ which is in turn so if and only $ij \in E(G)$. $\square$

The construction of Theorem 2.3 is illustrated in Fig. 1.

Let H^c be the subgraph of $S(G^c)$ induced by the complete subgraphs of G^c of order at most two. Then H^c is an isometric subgraph of $S(G^c)$ and hence a partial cube. Moreover, it follows from the above proof that $G = (H^c)^\tau$. This gives another construction, simpler than that of Theorem 2.3; however, H^c is in general not median—cf. [14].

3. Connectedness and triangle-free τ-graphs

The *Cartesian product* $G \,\square\, H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(a, x)(b, y) \in E(G \,\square\, H)$ whenever either $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. Every connected graph has a unique prime factor decomposition with respect to the Cartesian product; see [12, Theorem 4.9]. G is called *prime* if its unique prime factor decomposition has only one factor, that is, G itself. Let R^* denote the transitive closure of a given relation R; then Feder [10] (see also [12, Theorem 4.8]) proved:

Theorem 3.1. *Let G be a connected graph. Then $(\Theta \cup \tau)^*$ is a product relation. In particular, G is prime if and only if $(\Theta \cup \tau)^*$ has one equivalence class.*

The proof of the following lemma is straightforward.

Lemma 3.2. *Let G and H be partial cubes. Then $(G \,\square\, H)^\tau = G^\tau \cup H^\tau$.*

Theorem 3.3. *Let G be a partial cube with at least two edges and let $G = G_1 \,\square\, G_2 \,\square\, \cdots \,\square\, G_k$ be its unique prime factor decomposition. Then the connected components of G^τ are G_i^τ, $1 \le i \le k$. In particular, G^τ is connected if and only if G is prime.*

Proof. Suppose first that G is a prime graph. Let E and F be arbitrary different Θ-classes of G. We need to show that there exists an E, F-path in G^τ. As G is prime, Theorem 3.1 implies that the relation $\sigma = (\Theta \cup \tau)^*$ has a single equivalence class. Therefore, for an edge e of E and an edge f of F there exists a sequence of edges of G such that

$$e = e_1 \sigma e_2 \sigma \ldots \sigma e_{k-1} \sigma e_k = f,$$

where for each $1 \le i \le k-1$ either $e_i \Theta e_{i+1}$ or $e_i \tau e_{i+1}$. Then at least one σ must be τ, for otherwise $E = F$. This implies that there exists an E, F-path in G^τ and consequently G^τ is connected.

Suppose now that G is not prime and let $G = G_1 \,\square\, G_2 \,\square\, \cdots \,\square\, G_k$, $k \ge 2$, be the unique prime factor decomposition of G. By the above, G_i^τ is connected for $1 \le i \le k$. An inductive use of Lemma 3.2 completes the argument. □

In [1] it is proved that median graphs with no convex $K_{2,3} - e$ and no convex $K_{1,3}$ can be characterized as the Cartesian products of paths. Moreover, if only convex $K_{2,3} - e$ are forbidden, Cartesian products of trees are characterized. Hence we can ask for the remaining case: which are the median graphs with forbidden convex $K_{1,3}$? An answer is included as a particular case of the following result.

Theorem 3.4. *Let $n \ge 2$. Then a median graph G contains no convex $K_{1,n}$ if and only if G^τ is K_n-free.*

Proof. The result is clear for $n = 2$: G contains no convex path on three vertices if and only if no two edges of G are in relation τ. Assume in the remainder that $n \ge 3$.

Since a convex $K_{1,n}$ of G induces K_n in G^τ it follows that a K_n-free G^τ cannot contain a convex $K_{1,n}$.

Conversely, suppose that G^τ contains $K = K_n$. We claim that for any triangle K_3 from K there exists a convex $K_{1,3}$ in G. Let E_1, E_2, E_3 be the Θ-classes of G that induce such a triangle in G^τ. Let ab be an edge of E_1 such that there exists a vertex c with $ac \in E_2$. Then we may assume that $c \in W_{ab}$. Moreover, $c \in W_{ab} \setminus U_{ab}$, for otherwise ab and ac would lie in a 4-cycle.

Similarly we note that all the edges of E_2 lie in W_{ab}. Let $a'b'$ be an edge of E_1 that is in relation τ with an edge e' of E_3, where $a' \in U_{ab}$. Then e' lies in W_{ab}, for otherwise E_2 and E_3 would not be adjacent in G^τ. Hence $e' = a'c'$ for some vertex c' of W_{ab}. As above, E_3 is contained in W_{ab}. As E_2 and E_3 are adjacent in G^τ there is a vertex u that is an endvertex of an edge from E_2 and an edge from E_3. Note that $u \in W_{ab}$. Let x be the median of a, a', u. Then, as G is median and hence U_{ab} is convex, we have $x \in U_{ab}$. It follows that x is incident with an edge e_1 of E_1. By reversing the roles of a, a', u we similarly get that x is also incident with an edge e_2 from E_2 and an edge e_3 from E_3. But then e_1, e_2, and e_3 induce a convex $K_{1,3}$ and the claim is proved.

Hence for any triangle of K from G^τ there exists a convex $K_{1,3}$ in G. If $n = 3$ we are done. For $n > 3$ we proceed by induction. Let $E_1, \ldots, E_n$ be the Θ-classes of G that induce K in G^τ. Then, by the induction hypothesis, for $E_1, \ldots, E_{n-1}$ there exists a star $A = K_{1,n-1}$ in G, for $E_1, \ldots, E_{n-2}, E_n$ there exists a star $B = K_{1,n-1}$ in G, and for $E_2, \ldots, E_n$ there is a star $C = K_{1,n-1}$ in G. Let a, b, c be the central vertices of the stars A, B, and C, respectively. Then by the same arguments as in the previous paragraph, the median x of a, b, c is adjacent to edges from each of the classes $E_1, \ldots, E_n$. Lemma 2.2 implies that these edges induce a convex $K_{1,n}$ which completes the proof. $\square$

To see that Theorem 3.4 does not extend to partial cubes note that $C_6^\tau = K_3$ but C_6 is $K_{1,3}$-free.

4. Two problems

Among median graphs only stars $K_{1,n}$ have complete τ-graphs. The smallest non-median partial cubes with complete τ-graphs are C_6 and the vertex deleted 3-cube. Note that the vertex deleted 3-cube is obtained by an expansion from C_4, so the property of having a complete τ-graph is not preserved by the expansion. In addition, it is not difficult to see that the τ-graph of the vertex deleted Q_n is K_n.

Problem 4.1. Characterize partial cubes with complete τ-graphs.

Our starting motivation was a characterization of the resonance graphs of catacondensed hexagonal graphs using τ-graphs. We close this note with:

Problem 4.2. Can the resonance graphs of hexagonal graphs be characterized using τ-graphs?

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