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ON Θ-GRAPHS OF PARTIAL CUBES

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Abstract

The Θ-graph $\Theta(G)$ of a partial cube G is the intersection graph of the equivalence classes of the Djoković-Winkler relation. Θ-graphs that are 2-connected, trees, or complete graphs are characterized. In particular, $\Theta(G)$ is complete if and only if G can be obtained from K_1 by a sequence of (newly introduced) dense expansions. Θ-graphs are also compared with familiar concepts of crossing graphs and τ-graphs.

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1. Introduction

Motivated by the conjecture from [8] that the strong chromatic index $s'(G)$ of a bipartite graph G is at most $\Delta^2(G)$, Brešar [1] introduced the Θ-*graph* $\Theta(G)$ of a partial cube G as the intersection graph of the equivalence classes of the relation Θ. More precisely, the vertex set of $\Theta(G)$ consists of the Θ-classes of G, two Θ-classes E and F being adjacent whenever there exist adjacent edges $e \in E$ and $f \in F$. Note that the Θ-graph of a tree T is the line graph $L(T)$ of T.

Median graphs are among most important examples of partial cubes. In [1] a problem was posed whether $\chi(\Theta(G)) = \Delta(G)$ holds for any median graph G. An example is given in [4] demonstrating that this is not true in general. Nevertheless, the concept of the Θ-graph showed to be useful to obtain good upper bounds for the strong chromatic index of special families of partial cubes. For instance, $s'(G) \leq 2\chi(\Theta(G))$ holds for all tree-like partial cubes G. See [2] for definition and results on tree-like partial cubes.

Two concepts similar to the Θ-graph have been previously studied. The first one is the crossing graph of a partial cube [12, 3] and the second the τ-graph of a partial cube [17, 11]. The purpose of this note is to study Θ-graphs and compare them with the crossing graphs and the τ-graphs. The latter two concepts are defined in the last section.

In the rest of this section we define necessary concepts. In Section 2 we characterize partial cubes for which Θ-graphs are 2-connected and follow with characterizations of two extreme cases with respect to connectivity: partial cubes whose Θ-graphs are trees and partial cubes with complete graphs as Θ-graphs. In the last section we compare Θ-graphs with crossing graphs and τ-graphs. We characterize median graphs G for which $\Theta(G)$ equals the τ-graph of G, and median graphs G for which $\Theta(G)$ equals the crossing graph of G. We also give several examples showing that the equality problem is more difficult in the general case and observe that one can build new examples with desired property using the Cartesian product of graphs.

For $u, v \in V(G)$, let $d_G(u, v)$, or $d(u, v)$ for short, denote the length of a shortest path in G from u to v. The *interval* $I(u, v)$ between two vertices u and v in G is the set of all vertices on shortest paths between u and v. A subgraph H of a graph G is *isometric* if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$ and *convex* if $I(u, v) \subseteq V(H)$ for any $u, v \in V(H)$. A connected graph G is a *median graph* if for every triple u, v, w of its vertices $|I(u, v) \cap I(u, w) \cap I(v, w)| = 1$.

The vertex set of the *n-cube* Q_n consists of all n-tuples $b_1 b_2 \dots b_n$ with $b_i \in \{0,1\}$, two vertices being adjacent if the corresponding tuples differ in precisely one place. A graph G is a *partial cube* if G is an isometric subgraph of some Q_n. Partial cubes, being subgraphs of hypercubes, are of course bipartite. It is well known that median graphs are partial cubes [15].

Edges xy and uv of a graph G are in the *Djoković-Winkler relation* Θ if $d(x,u)+d(y,v) \neq d(x,v)+d(y,u)$. For bipartite graphs G, the definition of Θ can be simplified as follows: $e = xy$ and $f = uv$ are in relation Θ if $d(x,u) = d(y,v)$ and $d(x,v) = d(y,u)$, cf [6, 10]. Relation Θ is reflexive and symmetric. In general Θ is not an equivalence relation. Winkler [18] proved that among bipartite graphs Θ is transitive precisely for partial cubes.

For a connected graph $G = (V,E)$ and an edge ab of G set

$$W_{ab} = \{w \in V \mid d(a,w) < d(b,w)\},$$
$$U_{ab} = \{w \in W_{ab} \mid w \text{ has a neighbor in } W_{ba}\} \text{ and}$$
$$F_{ab} = \{e \in E \mid e \text{ is an edge between } W_{ab} \text{ and } W_{ba}\}.$$

We will also use W_{ab} and U_{ab} to denote the graphs induced by W_{ab} and U_{ab}, respectively. Note that if G is bipartite then $V = W_{ab} \cup W_{ba}$. By the Djoković theorem from [7], a connected graph G is a partial cube if and only if G is bipartite and has convex W_{ab}'s. Moreover, F_{ab}'s coincide with the Θ-classes of G. In addition, the set F_{ab} forms a matching between U_{ab} and U_{ba} that induces an isomorphism between the subgraphs induced by U_{ab} and U_{ba}.

2. Θ-Graphs that are 2-Connected, Trees, or Complete Graphs

Let G be a partial cube with at least two edges. Since G is connected so is $\Theta(G)$. In this section we first characterize partial cubes G for which $\Theta(G)$ is 2-connected. Then we consider two extreme cases with respect to connectivity: partial cubes whose Θ-graphs are trees and partial cubes with complete graphs as Θ-graphs.

For our first result we recall the following facts about Θ, see [10] for proofs.

Lemma 2.1. *Let G be a partial cube.*

(i) *Let P be a shortest path in G. Then no two different edges of P are in relation Θ.*

(ii) *Suppose P is a walk connecting the endvertices of an edge e. Then P contains an edge f with $e\Theta f$.*

(iii) *If $e = uv$ is a bridge then no edge from W_{uv} is in relation Θ with an edge from W_{vu}.*

Theorem 2.2. *Let G be a partial cube with at least five edges. Then $\Theta(G)$ is 2-connected if and only if any bridge of G has an endvertex of degree 1.*

Proof. Suppose that ab is a bridge of G such that both a and b are of degree at least 2. Since ab is a bridge, edges e and f from different components of $G \setminus ab$ are not in relation Θ by Lemma 2.1 (iii). Then for any neighbor c of a, $c \neq b$, and any neighbor d of b, $d \neq a$, any F_{ac}, F_{bd}-path in $\Theta(G)$ contains F_{ab}. It follows that F_{ab} is a cut vertex of $\Theta(G)$.

Conversely, suppose that for every Θ-class E with only one edge, at least one of the endvertices of this edge is of degree 1. We need to show that $\Theta(G)$ has no cut vertices.

Let F_{ab} be an arbitrary Θ-class of G and assume first that $|F_{ab}| > 1$. Let $a'b'$ be another edge of F_{ab}. Let F_{uv} and F_{xy} be two Θ-classes such that uv and xy both belong to the subgraph induced by W_{ab}. We may without loss of generality assume that $d(u, x) < d(u, y)$. As W_{ab} is convex, there exists a shortest u, x-path in W_{ab}. Therefore there exists an F_{uv}, F_{xy}-path in $\Theta(G)$ that does not contain F_{ab}. Suppose next F_{uv} lies completely in W_{ab} and F_{xy} completely in W_{ba}. Let P_a be a shortest a, a'-path and P_b a shortest b, b'-path. By the Djoković theorem, P_a lies in W_{ab} and P_b in W_{ba}. Moreover, Lemma 2.1 (i) and (ii) implies that for every $e \in P_a$ there exists $e' \in P_b$ such that $e\Theta e'$. It is now straightforward to find an F_{uv}, F_{xy}-path in $\Theta(G)$ that does not contain F_{ab}. We conclude that F_{ab} is not a cut vertex of $\Theta(G)$ if $|F_{ab}| > 1$.

Suppose now that $|F_{ab}| = 1$, that is, ab is a bridge. Then we may assume that a is of degree 1 while b is of degree at least 2. Then the (open) neighborhood of F_{ab} in $\Theta(G)$ induces a complete graph. Hence also in this case F_{ab} is not a cut vertex, and we conclude that $\Theta(G)$ is 2-connected. ■

Crossing graphs and τ-graphs are universal in the sense that every graph is a crossing graph of some partial cube [12] and that every graph is a τ-graph of some partial cube [11]. This property is not shared with Θ-graphs. For instance, among trees, only paths can be represented as Θ-graphs as the next result asserts.

Proposition 2.3. *Let G be partial cube. Then $\Theta(G)$ is a tree if and only if $G = C_4$ or $G = P_n$, $n \geq 2$. In these cases, $\Theta(G)$ is a path.*

Proof. Suppose first that a partial cube G is not a tree and let $C = C_{2k}$ be a shortest cycle of G. Then C is isometric. If $k \geq 3$, then $\Theta(G)$ contains C_k. Let $k = 2$ and assume that $G \neq C_4$. Then considering an arbitrary edge adjacent to a 4-cycle of G we infer that $\Theta(G)$ contains a triangle. So if G contains a cycle and if $\Theta(G)$ is a tree, G is the 4-cycle. Suppose next that G is a tree. Then, if G contains a vertex of degree at least three, $\Theta(G)$ contains a triangle again. If follows that G must be a path P_n, and in that case $\Theta(P_n) = P_{n-1}$. ■

In the last result of this section we characterize partial cubes whose Θ-graphs are complete graph. For this sake some preparation is needed.

Let H be a connected graph. An *isometric cover* H_0, H_1 consists of two isometric subgraphs H_0 and H_1 of H such that $H = H_0 \cup H_1$ and $H_0 \cap H_1 \neq \emptyset$. For $i = 0, 1$ let H_{0i} be an isomorphic copy of H_i and for any vertex $u \in H_i$, let u_i be the corresponding vertex in H_{0i}. The *expansion of H with respect to* H_0, H_1 is the graph G obtained from the disjoint union $H_{00} \cup H_{01}$, where for each $u \in H_0 \cap H_1$ the vertices u_0 and u_1 are joined by an edge. A *contraction* is the reverse operation to the expansion.

Chepoi [5] proved that a graph is a partial cube if and only if it can be obtained from K_1 by a sequence of expansions. He followed the approach of Mulder [14, 15] who previously proved an analogous result for median graphs.

Let us call the expansion of a partial cube H a *dense expansion* if any Θ-class E contains an edge with an endvertex in $H_0 \cap H_1$ and for any different Θ-classes E and F there exist adjacent edges $e \in E$ and $f \in F$ such that e and f both lie in H_0 or both belong to H_1. Then we have:

Theorem 2.4. *Let G be a partial cube. Then $\Theta(G)$ is a complete graph if and only if G can be obtained from K_1 by a sequence of dense expansions.*

Proof. Suppose G is a partial cube obtained from K_1 by a sequence of dense expansions. Then, by the definition of the dense expansion, the new

Θ-class of G obtained by the expansion is adjacent in $\Theta(G)$ to any other Θ-class of G. Moreover, any two other different Θ-classes remain adjacent in $\Theta(G)$.

Conversely, suppose $\Theta(G) = K_n$. Let E_G be an arbitrary fixed Θ-class of G and let H be the contraction of G with respect to E. Let H_0, H_1 be the corresponding isometric cover of H. Note first that $\Theta(H) = K_{n-1}$. Let F_H be a Θ-class of H. Then F_H contains an edge with an endvertex in $H_0 \cap H_1$, for otherwise the corresponding Θ-class F_G would not be adjacent to E in $\Theta(G)$. Let next F_H and F'_H be two different Θ-classes of H. Then we infer that there must exist adjacent edges $f \in F_H$ and $f' \in F'_H$ such that f and f' both lie in H_0 or both in H_1, for otherwise F_G and F'_G would not be adjacent in $\Theta(G)$. We conclude that G is obtained from H by a dense expansion. Induction completes the proof. ∎

The above theorem is of similar nature as [12, Proposition 4.4], where partial cubes with complete crossing graphs are characterized using the so called all-color expansion. We close the section with the following:

Problem 2.5. *Characterize graphs that can be represented as Θ-graphs. More precisely, for which graphs X there exists a partial cube G such that $X = \Theta(G)$?*

3. Θ-Graphs Versus Crossing Graphs and τ-Graphs

In this section we compare Θ-graphs of partial cubes with their crossing graphs and τ-graphs. We first define the latter two concepts. Let G be a partial cube. Then both, its *crossing graph* $G^{\#}$, and its *τ-graph* G^{τ}, have Θ-classes of G as vertices. In $G^{\#}$ two Θ-classes are adjacent if edges from both classes appear on a common isometric cycle, while in G^{τ} two Θ-classes E and F are adjacent whenever there exist edges $e \in E$ and $f \in F$ which induce a convex path on three vertices.

For a subgraph X of a graph G let ∂X be the set of edges with one endvertex in X and the other in $G \setminus X$. For the proof of the next proposition we need to recall two lemmata.

Lemma 3.1 (Convexity Lemma)**.** *An induced connected subgraph H of a bipartite graph G is convex if and only if no edge of ∂H is in relation Θ to an edge in H.*

Lemma 3.2. *Let G be a median graph. Then Θ-classes E and F are adjacent in G^τ if and only if there exist adjacent edges $e \in E$ and $f \in F$ and there exist no two edges $e' \in E$ and $f' \in F$ that lie in a common C_4.*

Lemmata 3.1 and 3.2 are from [9] and [11], respectively. Using them we arrive at the main theorem of this section.

Proposition 3.3. *Let G be a median graph. Then*

(i) *$G^\tau \subseteq \Theta(G)$, where the equality holds if and only if G is a tree.*

(ii) *$G^\# \subseteq \Theta(G)$, where the equality holds if and only if G is a hypercube.*

Proof. (i) Clearly, if G is a tree then $G^\tau = \Theta(G) = L(T)$. Suppose that G is not a tree. Then G contains a cycle and since G is median, it contains a 4-cycle $uvwz$. Then $F_{uv}F_{vw} \in E(\Theta(G))$ but $F_{uv}F_{vw} \notin E(G^\tau)$ by Lemma 3.2.

(ii) Suppose that $F_{ab}F_{uv} \in E(G^\#)$. Then Θ-classes F_{ab} and F_{uv} appear on a common 4-cycle. This implies that $F_{ab}F_{uv} \in E(\Theta(G))$, hence $G^\# \subseteq \Theta(G)$.

Let $G = Q_n$. Then $G^\# = \Theta(G) = K_n$. Suppose next that G contains n Θ-classes but $G^\# \neq K_n$. Since the crossing graph of a median graph G is K_n if and only if $G = Q_n$, see [13, Theorem 11] (or [12, Proposition 4.1]), we infer that $G \neq Q_n$. In addition, using [10, Lemma 2.41], there exists an edge uv of G, such that $W_{uv} \setminus U_{uv} \neq \emptyset$. So there exists an edge $ab \in W_{uv}$ such that $a \in W_{uv} \setminus U_{uv}$ and $b \in U_{uv}$. Then $F_{ab}F_{uv} \in E(\Theta(G))$. On the other hand, Lemma 3.1 implies that $F_{ab}F_{uv} \notin E(G^\#)$. So $G^\# \neq \Theta(G)$ if G is not a hypercube. ■

We note that Proposition 3.3 (ii) is of similar nature as a characterization of hypercubes among median graphs as the graphs for which the union of two non-empty disjoint half-spaces equals G, see [16, Further Topics 6.35.3].

In the class of all partial cubes the problem of when the Θ-graph is equal to the crossing graph or to the τ-graph seems to be rather difficult. We first compare Θ-graphs with τ-graphs. It is easily seen that also in this case $G^\tau \subseteq \Theta(G)$ holds. But now we have more graphs than trees for which equality holds. Clearly, $G^\tau = \Theta(G)$ holds for any partial cube with no 4-cycles. However, the same is also true for Q_3^-, the vertex deleted 3-cube, as well as the graph obtained from Q_3^- by expanding its isometric 6-cycle (in this case the derived graphs are K_4).

Comparing $G^\#$ with $\Theta(G)$ we find that none of possible inclusions hold. For instance, $C_8^\# = K_4$ and $\Theta(C_8) = C_4$, so $\Theta(C_8) \subset C_8^\#$. Recall that for any tree T on at least three vertices we have $T^{\#} \subset \Theta(T)$. The graphs C_6, Q_3^- and Q_3^- expanded as above are graphs for which the equality holds.

To obtain more graphs G with $G^{\#} = \Theta(G)$ from known graphs with this property we first observe the following lemma. The *Cartesian product* $G \,\Box\, H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(a,x)(b,y) \in E(G \,\Box\, H)$ whenever either $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. Let $G \oplus H$ denote the join of graphs G and H. Then:

Lemma 3.4. *Let G and H be partial cubes. Then* $\Theta(G \,\Box\, H) = \Theta(G) \oplus \Theta(H)$.

Proof. The Θ-classes of $G \,\Box\, H$ are in one-to-one correspondence with the union of the Θ-classes of G and the Θ-classes of H, cf. [10, Lemma 4.3]. In other words, $V(\Theta(G \,\Box\, H)) = V(\Theta(G)) \cup V(\Theta(H))$. By the definition of the Cartesian product, the Θ-class of $G \,\Box\, H$ corresponding to a Θ-class of G and the Θ-class of $G \,\Box\, H$ corresponding to a Θ-classes of H both appear on a common 4-cycle, hence the proposition follows. ■

Proposition 3.5. *Let G and H be partial cubes with $G^{\#} = \Theta(G)$ and $H^{\#} = \Theta(H)$. Then $(G \,\Box\, H)^{\#} = \Theta(G \,\Box\, H)$.*

Proof. Apply Lemma 3.4 and the fact that $(G \,\Box\, H)^{\#} = G^{\#} \oplus H^{\#}$, see [12, Proposition 6.1]. ■

To conclude the paper we note that for trees T_1 and T_2 Lemma 3.4 implies that $L(T_1 \,\Box\, T_2) = L(T_1) \oplus L(T_2)$.

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