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Θ-graceful labelings of partial cubes[☆]

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Abstract

Partial cubes are graphs that allow isometric embeddings into hypercubes. Θ-graceful labelings of partial cubes are introduced as a natural extension of graceful labelings of trees. It is shown that several classes of partial cubes are Θ-graceful, for instance even cycles, Fibonacci cubes, and (newly introduced) lexicographic subcubes. The Cartesian product of Θ-graceful partial cubes is again such and we wonder whether in fact any partial cube is Θ-graceful. A connection between Θ-graceful labelings and representations of integers in certain number systems is established. Some directions for further investigation are also proposed.

Keywords: Graceful labelings; Trees; Ringel–Kotzig conjecture; Partial cubes; Fibonacci cubes; Hypercubes

1. Introduction

Different graph labelings have been introduced so far, the origin of most of them is the seminal paper [22] by Rosa. For an overview of extensive developments in the area see the dynamic survey of graph labelings [11] as well as references therein.

Let T be a tree on n vertices. A bijection $f : V(T) \to \{0, 1, \dots, n-1\}$ is called a *graceful labeling of* T if the resulting edge labels, obtained by $|f(x) - f(y)|$ on every edge xy, are all distinct. Note that in a graceful labeling of T labels form a sequence $1, 2, \dots, n-1$. More generally, a graph G with m edges is called *graceful* if there is an injection $f : V(G) \to \{0, 1, \dots, m\}$ such that the labels of edges are pairwise distinct. The Ringel–Kotzig conjecture asserting that all trees are graceful remains one of the central problems (if not the central problem) in the area.

As already observed by Rosa [22], the cycles C_{4k+2} are not graceful. Cartesian products of two paths, a path by a cycle, and two cycles, have been investigated in many papers for their gracefulness, see [11, Section 2.3] for a survey on these results and, for instance, the papers [10,17]. In addition, the so-called book graphs—the Cartesian product of a star by an edge—have also been studied [6,20], as well as have been cyclic snakes [1].

As soon as the above mentioned graphs are bipartite, which is in most of the cases, they admit isometric embeddings into hypercubes. This fact, together with several difficulties in the above investigations, lead us to a doubt whether the concept of a graceful labeling is natural on such graphs. Instead, in this paper we propose a new kind of labeling, called

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Θ-graceful labeling, that is defined on isometric subgraphs of hypercubes, in particular on trees. In fact, Θ-graceful labelings and graceful labelings coincide on trees. Our results support the feeling that Θ-graceful labelings form a more natural concept on partial cubes than graceful labelings.

In the next section we recall the notion of partial cubes and introduce Θ-graceful labelings. In these terms we extend the Ringel–Kotzig conjecture and observe that hypercubes are Θ-graceful. In Section 3 we show that even cycles and Fibonacci cubes are Θ-graceful and that the Cartesian product of Θ-graceful factors is again Θ-graceful. We also introduce the concept of down-closed partial cubes and show that so-called lexicographic subcubes are Θ-graceful as well. Then we introduce consistent Θ-graceful labelings and show that they give rise to representations of integers in related number systems. We conclude with a question whether all partial cubes are Θ-graceful and propose a study of strongly Θ-graceful labelings and of partial cubes with unique Θ-graceful labelings.

2. Partial cubes and Θ-graceful labelings

The *hypercube* or *n-cube* Q_n, $n \geqslant 1$, is the graph with the vertex set $\{0, 1\}^n$, two vertices being adjacent if the corresponding tuples differ in precisely one place. The set of edges of Q_n whose endvertices differ in the ith place ($1 \leqslant i \leqslant n$), is called the ith *parallel class* of Q_n.

Let us label the vertices of Q_n by numbers $0, 1, \ldots, 2^n - 1$, that are obtained by considering the label of each vertex as a binary number and transforming it into the decimal number. For instance, the vertex $00\ldots0$ gets the label 0, while $11\ldots1$ receives the label $2^n - 1$. In addition, assign labels to the edges of Q_n in the usual way, that is, as the absolute difference between the labels of its endvertices. Then any edge of the ith parallel class receives the label 2^{n-i}. We will extend this idea to all isometric subgraphs of hypercubes. For this sake some preparation is needed.

A subgraph H of a graph G is called *isometric* if for every two vertices u, v of H there exists a shortest u, v-path that lies in H. Isometric subgraphs of hypercubes are called *partial cubes*, [16] and have been extensively studied in recent years [2–5,7,8,19]. Probably the most important characterization of partial cubes is due to Winkler [23], and involves the following edge-parallelism property. We say that two edges $e = xy$ and $f = uv$ of G are in the Djoković–Winkler [9,23] relation Θ if

$$d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u)\,,$$

where $d_G(w, w')$ (or $d(w, w')$ for short) denote the usual shortest path distance in G between w and w'. All graphs considered in this paper are bipartite. In this case $xy\Theta uv$ implies that the notation can be chosen such that $d(x, u) = d(y, v) = d(y, u) - 1 = d(x, v) - 1$. Relation Θ is reflexive and symmetric. Winkler [23] proved that a connected bipartite graph is a partial cube if and only if $\Theta = \Theta^*$, where Θ^* denotes the transitive closure of Θ. Hence Θ is an equivalence relation on a partial cube G and so partitions $E(G)$ into the so-called *Θ-classes*.

Let G be a partial cube on n vertices. We say that a bijection $f : V(G) \to \{0, 1, \ldots, n-1\}$, is a *$\Theta$-graceful labeling of G* if all edges in each Θ-class of G receive the same label, and distinct Θ-classes get distinct labels (where the labeling of the edges is defined in the usual way). If such a labeling exists, then G is called a *Θ-graceful partial cube*. For instance, Fig. 1 shows two Θ-graceful partial cubes.

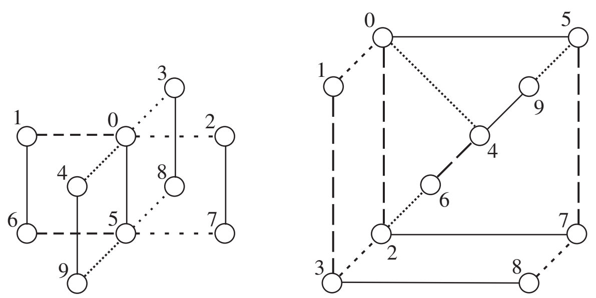

Fig. 1. Partial cubes with Θ-graceful labelings.

Note that the above defined labeling of Q_n is Θ-graceful. Since trees are partial cubes and every Θ-class of a tree consists of a single edge, Θ-graceful labelings of trees are just their graceful labelings. Hence we have the following connection between Θ-graceful labelings of partial cubes and graceful labelings of trees.

Proposition 2.1. *Suppose that there exists a class $\mathscr{C}$ of partial cubes such that* (i) *$\mathscr{C}$ contains all trees and* (ii) *all graphs of $\mathscr{C}$ admit a Θ-graceful labeling. Then all trees are graceful.*

3. Classes of Θ-graceful partial cubes

In the previous section we have observed that hypercubes admit Θ-graceful labelings. Another class of Θ-graceful partial cubes are graceful trees. In this section we present some particular classes of partial cubes that are Θ-graceful, but first we present a result on partial cubes that might be of independent interest.

Let G be a partial cube. It follows from definitions that vertices of G can be represented by binary k-tuples such that the distance between two vertices equals the number of coordinates in which the corresponding k-tuples differ (the latter number is known as the *Hamming distance*).

For a k-tuple $x = x_1 \ldots x_k$ and $1 \leqslant i \leqslant k$, let x^{*i} be the k-tuple obtained from $x_1 \ldots x_k$ by setting x_i to 0. Observe that $x^{*i} = x$ whenever $x_i = 0$. We say that a partial cube G is *down-closed*, if for every vertex x of G and for every i, x^{*i} is also a vertex of G. In addition, if G is a partial cube and $m < |V(G)|$ we denote by G^{-m} the graph obtained from G by deleting the m largest vertices of G with respect to the lexicographic order of their k-tuples.

Theorem 3.1. *Let G be a down-closed partial cube and $0 < m < |V(G)|$. Then G^{-m} is a partial cube.*

Proof. Let that x and y be two vertices of G^{-m}. We need to show that there is a shortest x, y-path of G that lies entirely in G^{-m}. Let z be the k-tuple defined with

$$z_i = \begin{cases} 1 & \text{if } x_i = y_i = 1, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, \ldots, k$. Since G is down-closed, z is in G^{-m}, moreover, every k-tuple between x and z (that is, a k-tuple w with $w_i = 1$ if $z_i = 1$, and $w_i = 0$ if $x_i = 0$) is also in G^{-m}. Hence there is a shortest path in G^{-m} between x and z. Likewise, there is a shortest path in G^{-m} between z and y. Clearly the join of these two paths is a shortest path from x to y. $\square$

3.1. Lexicographic subcubes

For any hypercube Q_n and $m < 2^n$ the graph Q_n^{-m} is defined as above: its vertices are binary n-tuples from $00\ldots0$ up to the $(m+1)$st largest n-tuple with respect to the lexicographic order. We will call all such graphs *lexicographic subcubes*. By Theorem 3.1 and the fact that hypercubes are down-closed, every lexicographic subcube is a partial cube.

Let G be a lexicographic subcube. Without loss of generality we may assume that $m < 2^{n-1}$, since $Q_n^{-(2^{n-1})}$ is isomorphic to Q_{n-1}. Let us label the vertices of G by numbers $0, 1, \ldots, 2^n - 1 - m$, that are obtained by considering the label of each vertex as a binary number and transforming it into the decimal number. The labeling is a bijection, and the obtained edge-labels depend on particular Θ-class, which receive labels $1, 2, 4, \ldots, 2^{n-1}$. We thus infer:

Proposition 3.2. *Lexicographic subcubes are Θ-graceful.*

3.2. Fibonacci cubes

We continue with Fibonacci cubes, an interesting class of partial cubes that was introduced as a model for interconnection networks in [14,15] and studied in several papers afterwards, see [18,21] and references therein.

A *Fibonacci string* of length n is a binary string $b_1 b_2 \ldots b_n$ with $b_i b_{i+1} = 0$ for $1 \leqslant i < n$, that is, a binary string without two consecutive ones. The *Fibonacci cube* Γ_n $(n \geqslant 1)$ is the subgraph of Q_n induced by all Fibonacci strings of length n, see Fig. 2 where Γ_4 is shown.

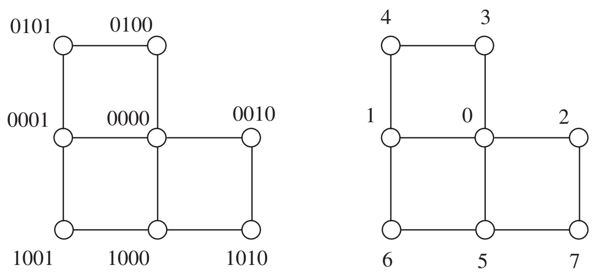

Fig. 2. Γ_4 and its Θ-graceful labeling.

We now verify that Fibonacci cubes are partial cubes. Let x and y be arbitrary vertices of Γ_n. We need to show that $d_{\Gamma_n}(x, y) = d_{Q_n}(x, y)$. Clearly, $d_{\Gamma_n}(x, y) \geqslant d_{Q_n}(x, y)$. For the other inequality first change one by one those coordinates in x that are equal to 1 but the corresponding coordinates in y are 0. After that, change one by one those coordinates in x that are equal to 0 where the corresponding coordinates in y are 1. It is straightforward to verify that in this way we obtain an x, y-path in Γ_n of length $d_{Q_n}(x, y)$.

Recall that the Fibonacci numbers F_n are defined with $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$, $n > 2$.

Proposition 3.3. *Γ_n is Θ-graceful for any $n \geqslant 1$. Moreover, its Θ-classes can be labeled with $F_2, F_3, \ldots, F_{n+1}$.*

Proof. Assign the following label to a vertex $x = x_1 x_2 \ldots x_n$ of Γ_n (cf. Fig. 2).

$$\ell(x) = \sum_{i=1}^{n} x_i F_{n-i+2}.$$

The well-known Zeckendorf theorem (cf. [12, p. 295]) implies that the labels of $V(\Gamma_n)$ are $0, 1, \ldots, F_{n+2} - 1$, cf. Fig. 2. Since $|V(\Gamma_n)| = F_{n+2}$, it follows that the labeling ℓ is bijective.

Since Γ_n is a partial cube, a Θ-class of Γ_n consists of precisely those edges that differ in a fixed coordinate. Therefore, the labeling ℓ is Θ-graceful and moreover, the Θ-classes are labeled with $F_2, F_3, \ldots, F_{n+1}$. □

Since Fibonacci cubes are obviously down-closed, Theorem 3.1 and Proposition 3.3 imply

Corollary 3.4. *For every $n \geqslant 1$ and $m < F_{n+2}$, Γ_n^{-m} is a Θ-graceful partial cube.*

3.3. Even cycles

Proposition 3.5. *Even cycles are Θ-graceful.*

Proof. Denote vertices of C_n by $v_0, v_1, \ldots, v_{n-1}$, and define adjacencies in the natural way.

Consider first the cycles C_{4k+2}, $k \in \mathbb{N}$, and set

$$f(v_{2i}) = i \quad \text{for } i = 0, \ldots, 2k,$$

$$f(v_{2i-1}) = 3k + 1 - i \quad \text{for } i = 1, \ldots, k$$

and

$$f(v_{2i+1}) = 5k + 1 - i \quad \text{for } i = k, \ldots, 2k.$$

(A more intuitive way to describe this labeling is as follows. Starting at v_0, label every second vertex by an increasing sequence of labels from 0 to $2k$. Then continue in v_{2k+1} and label every second vertex (going in the same direction as before) by decreasing sequence of labels from $4k + 1$ to $2k + 1$, cf. the labeling of C_{10} in Fig. 3.) It is straightforward to check that this labeling is Θ-graceful with Θ-classes labeled by all natural numbers between $k + 1$ and $3k + 1$.

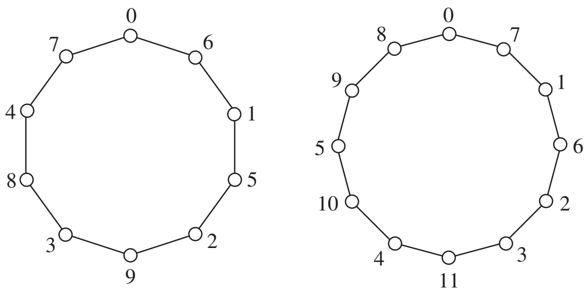

Fig. 3. Θ-graceful labelings of C_{10} and C_{12}.

In the second case we consider the cycles C_{4k}, $k \in \mathbb{N}$, and define the corresponding labelings as follows:

$$f(v_{2i}) = i \quad \text{for } i = 0, \ldots, k-1,$$

$$f(v_{2i-1}) = 3k - 1 - i \quad \text{for } i = 1, \ldots, k-1,$$

$$f(v_{2i-1}) = i \quad \text{for } i = k, \ldots, 2k-1,$$

$$f(v_{2(k+i)}) = 4k - 1 - i \quad \text{for } i = 0, \ldots, k-1$$

and

$$f(v_{4k-1}) = 3k - 1.$$

(See the labeling of C_{12} in Fig. 3.) Again a straightforward check shows that this labeling is Θ-graceful, with Θ-classes labeled by 1 and all numbers between $k+1$ and $3k-1$. $\square$

3.4. Cartesian products of partial cubes

The last result of this section asserts that the Cartesian product of Θ-graceful partial cubes is again Θ-graceful. (It is well-known that the Cartesian product of partial cubes is a partial cube.) Since Q_n can be represented as the Cartesian product of n factors isomorphic to K_2, this result in particular implies that hypercubes are Θ-graceful.

Theorem 3.6. *Let G and H be Θ-graceful partial cubes. Then $G \,\square\, H$ is Θ-graceful.*

Proof. Let G and H be partial cubes on m and k vertices, respectively. We may without loss of generality assume $m \leqslant k$. Let $f : V(G) \to \{0, 1, \ldots, m-1\}$, and $g : V(H) \to \{0, 1, \ldots, k-1\}$ be Θ-graceful labelings of G and H. For an arbitrary vertex $w = (u, v)$ of $G \,\square\, H$ (where $u \in V(G)$, $v \in V(H)$) set $h(w) = h((u, v)) = kf(u) + g(v)$. We claim that h is a Θ-graceful labeling of $G \,\square\, H$. Let $h((u, v)) = h((u', v'))$. Then for some $i, i' j, j'$ we have $ki + j = ki' + j'$ and hence $k(i - i') = j - j'$. This is only possible if $i = i'$ and $j = j'$, and so $(u, v) = (u', v')$. Hence, h is injective. Moreover, $h((x, y)) = 0$, where $f(x) = g(y) = 0$, and $h((a, b)) = km - 1$, where $f(a) = m - 1$, $g(b) = k - 1$. It follows that $h : V(G \,\square\, H) \to \{0, 1, \ldots, km - 1\}$ is a bijection.

Note that the label of an edge of a G-layer of $G \,\square\, H$ is equal to k times the label of its projection onto G. On the other hand, edges in H-layers of $G \,\square\, H$ receive the same labels as in H. Clearly, edges of distinct Θ-classes get distinct labels, while edges of the same Θ-class get the same label. $\square$

For $n \geqslant i \geqslant 0$, the *extended Fibonacci cubes* Γ_n^i were introduced in [24] as the following partial cubes. $\Gamma_i^i = Q_i$ and $\Gamma_{i+1}^i = Q_{i+1}$, while for $n \geqslant i + 2$, the vertex set V_n^i of Γ_n^i is defined recursively with $V_{n+2}^i = 0V_{n+1}^i \cup 10V_n^i$. In [13] it is observed that $\Gamma_n^i = \Gamma_{n-i} \square Q_i$ holds for $n \geqslant i \geqslant 0$. Hence, combining this fact with Theorem 3.6 and Proposition 3.3 we also get

Corollary 3.7. *For every $n \geqslant i \geqslant 0$, the extended Fibonacci cube Γ_n^i is a Θ-graceful partial cube.*

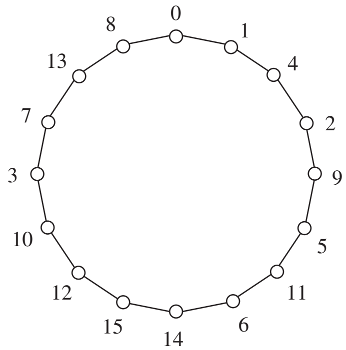

Fig. 4. Non-consistent Θ-graceful labeling of C_{16}.

4. Consistent Θ-graceful labelings

In the previous section we have observed a connection between Θ-graceful labelings and representations of integers in the Fibonacci number system. In this section we extend this correspondence to more Θ-graceful labelings, in fact, it seems that there is such a correspondence in most of the cases.

Let f be a Θ-graceful labeling of a partial cube G. We say that f is a *consistent Θ-graceful labeling* if for every edges xy and uv of G with $xy\Theta uv$ the following condition is fulfilled

$$d(x,u) < d(x,v) \quad \Rightarrow \quad f(x)+f(v)=f(y)+f(u).$$

Graphs in all the preceding figures are labeled with consistent Θ-graceful labelings; in particular, the labelings of Fibonacci cubes are such. Non-consistent Θ-graceful labelings seem to be rather rare, Fig. 4 shows such a labeling of C_{16}. On the other hand the proof of Proposition 3.5 gives a consistent labeling of C_{16}. Note also that every graceful labeling of a tree is a consistent Θ-graceful labeling, where the consistency condition is trivially fulfilled.

Let G be a partial cube with k Θ-classes and let g be an isometric embedding of G into Q_k. For a vertex x of G we will write $g(x)=g_1(x)g_2(x)\ldots g_k(x)$. Since Q_k is vertex transitive we may choose arbitrarily a vertex $x \in V(G)$ such that $g(x)=00\ldots0 =: 0^k$.

Theorem 4.1. *Let f be a consistent Θ-graceful labeling of a partial cube G with k Θ-classes. Let g be an isometric embedding of G into Q_k with $g(f^{-1}(0))=0^k$. Then there exist integers $b_1,\ldots,b_k$, $|b_i|<|V(G)|$, such that for any $x\in V(G)$,*

$$f(x)=\sum_{i=1}^{k} g_i(x)b_i.$$

Proof. Let $E_1, E_2, \ldots, E_k$ be the Θ-classes of G and let $x_iy_i \in E_i$, $1\leqslant i\leqslant k$. Then we may without loss of generality assume that $g_i(x_i)\neq g_i(y_i)$ and that $g_i(x_i)=1$ and $g_i(y_i)=0$. Then we define

$$b_i = f(x_i) - f(y_i).$$

Let uv be an arbitrary edge of E_i, and let u be the vertex $g_i(u)=1$. Then $d(x_i,u)<d(x_i,v)$ and hence by the consistency condition, $f(x_i)+f(v)=f(y_i)+f(u)$, that is, $f(x_i)-f(y_i)=f(u)-f(v)$. It follows that b_i is well-defined.

Let x be an arbitrary vertex of G. We must show that $f(x)=\sum_{i=1}^{k} g_i(x)b_i$. Let w be the vertex with $f(w)=0$. If $x=w$, then since $g(f^{-1}(0))=0^k$ we have $f(x)=\sum_{i=1}^{k} 0\cdot b_i=0$. Otherwise let $w=w_0, w_1, \ldots, w_d=x$ be a shortest w,x-path in G. Since $f(x)=f(w_d)$ and $f(w_0)=0$ we have

$$f(x)=\sum_{i=1}^{d}(f(w_i)-f(w_{i-1})).$$

By the definition of the b_i's, $f(w_i)-f(w_{i-1})=b_j$, where $g_j(w_i)=1$ and $g_j(w_{i-1})=0$. Note that $g_j(x)=1$ if and only if there is an index $1<i\leqslant d$, such that $g_j(w_i)=1$ and $g_j(w_{i-1})=0$. Hence, $f(x)=\sum_{j=1}^{k} g_j(x)b_j$. $\square$

From a different perspective, Theorem 4.1 says that integers from 0 up to $n-1$ can be represented in a special number system with integers $b_1, \dots, b_n$ as base numbers, where $-n < b_i < n$, such that $|b_1|, \dots, |b_n|$ are pairwise distinct positive integers. This gives the following related problem.

The number system problem: Given a set S of n binary k-tuples, can we find a number system with integers $b_1, \dots, b_k$ as base numbers, where $-n < b_i < n$ and $|b_1|, \dots, |b_n|$ are pairwise distinct positive integers, such that

$$\left\{ \sum_{i=1}^{k} x_i b_i ;\ x = (x_1, \dots, x_k) \in S \right\} = \{0, 1, \dots, n-1\}.$$

The problem need not have a positive solution. For instance, for the set of triples {000, 011, 110, 101} no such number system exists. Indeed, there are only three distinct absolute values available for the bases 1, 2 and 3, and clearly one of the base numbers must be $+3$. Hence the other two must be -1 and -2 otherwise we get over range. But then the remaining label is $-1 + (-2) = -3$.

On the other hand if G is a partial cube with a consistent Θ-graceful labeling, Theorem 4.1 ensures such a solution. More precisely, we infer

Corollary 4.2. *Let G be a partial cube. Then G has a consistent Θ-graceful labeling if and only there exists an isometric embedding g of G into Q_k such that for the set of k-tuples $\{g(u);\ u \in V(G)\}$ the number system problem has a solution. Moreover, in the solution the Θ-classes are labeled with the absolute values of base numbers.*

5. Concluding remarks

The results of the previous sections lead to the following

Question 5.1. *Is every partial cube Θ-graceful?*

Note that an affirmative answer to the question in particular solves the Ringel–Kotzig conjecture, so it seems easier to search for a possible example of a partial cube that is not Θ-graceful.

It may also be interesting (in particular, if the question has an affirmative answer) how many different such labelings exist for particular graphs. Let $f: V(G) \to \{0, 1, \dots, n-1\}$ be a Θ-graceful labeling of a partial cube G on n vertices. Let $\bar{f}: V(G) \to \{0, 1, \dots, n-1\}$ be a labeling defined with $\bar{f}(u) = n - 1 - f(u)$. It is clear that $\bar{f}$ is a Θ-graceful labeling of G. Let, in addition, $\psi: V(G) \to V(G)$ be an automorphism of G. Then also $f \circ \psi$ is a Θ-graceful labeling of G. We say that f, $\bar{f}$, and $f \circ \psi$ are *equivalent* Θ-graceful labelings of G. It seems an interesting problem to describe the partial cubes with only one non-equivalent Θ-graceful labeling. C_4 is a trivial example of such a graph. Another partial cube with only one non-equivalent Θ-graceful labeling is the bipartite wheel BW_5. It is shown on Fig. 5, together with its Θ-graceful labeling.

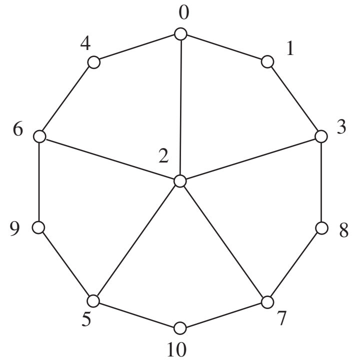

Fig. 5. Θ-graceful labeling of BW_5.

That BW_5 has only the Θ-graceful labeling from Fig. 5 was checked by a computer program. The program found 20 solutions, however they can all be obtained from the above by the action of the dihedral group D_5 (which is the automorphism group of BW_5) and the complementation operation.

Let f be a Θ-graceful labeling of a partial cube G. If the labels of Θ-classes form a sequence $1, 2, \ldots, k$, where k is the number of Θ-classes of G, we say that f is a *strongly Θ-graceful labeling* and that G is a *strongly Θ-graceful partial cube*. For instance, the labeling of the left-hand side partial cube from Fig. 1 is strongly Θ-graceful.

Clearly, graceful trees, Θ-graceful trees and strongly Θ-graceful trees are the same concepts. However, not all partial cubes are strongly Θ-graceful. For instance, we have checked by a computer search that none of the cycles C_6, C_8, and C_{14} is strongly Θ-graceful. On the other hand, C_4 is strongly Θ-graceful as well as are C_{10}, C_{12}, C_{16}, and C_{18}. For instance, the following respective labelings of these cycles are strongly Θ-graceful:

0, 1, 3, 7, 4, 9, 8, 6, 2, 5,

0, 1, 3, 7, 2, 5, 11, 10, 8, 4, 9, 6,

0, 1, 4, 2, 9, 5, 11, 6, 14, 15, 12, 10, 3, 7, 13, 8,

0, 1, 4, 2, 11, 3, 9, 5, 10, 17, 16, 13, 15, 6, 14, 8, 12, 7.

Hence another interesting problem is to characterize strongly Θ-graceful even cycles.

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