

General transmission lemma and Wiener complexity of triangular grids

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Abstract

The Transmission Lemma from [Computer J. 59 (2016) 1174–1179] is extended to the General Transmission Lemma. It gives a formula for the transmission of a vertex u as a function of a collection of edge cuts and an u -routing that uniformly intersects the edge cuts. The applicability of the General Transmission Lemma is demonstrated by computing the Wiener complexity of triangular grid networks.

Keywords: graph distance; transmission of vertex; routing; Wiener index; Wiener complexity; triangular grid network

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1 Introduction

The concept of distance pervades mathematics, many fields of science, and even our daily lives. In particular, distances play a vital role in facility location problems (cf. the introduction [7] to the journal's issue dedicated to models, algorithms and applications for location problems), network design in operations research (cf. [13, 22]), distance based topological indices in mathematical chemistry (cf. [16, 17, 32]), measuring closeness of

groups of individuals in sociology (cf. [8]), identifying role of players in social networks such as the internet (cf. [6]), and so on.

With Wiener's discovery of a close correlation between the boiling points of certain alkanes and the sum of distances in graphs representing their molecular structure [31], it became apparent that graph invariants (alias topological indices in mathematical chemistry) can be used to predict properties of chemical compounds. Consequently numerous new topological indices have been considered over the past decades and their predictive power for various properties tested, cf. the books [14, 15, 27]. Many of these invariants are defined via graph distance. In particular, the celebrated Wiener index of a connected graph is defined as the sum of the distances between all unordered pairs of vertices. In the literature one finds general algorithms for computing the Wiener index (see [23]) as well as special algorithms that are faster on specific families of graphs [9, 18].

The transmission $T(u)$ of a vertex $u \in V(G)$ is a concept closely related to the Wiener index but localized to the selected vertex: $T(u)$ is the sum of distances between u and all the other vertices of G , cf. [1, 21, 26]. In location theory, vertices with the minimum (or maximum) transmission play a special role because they form target sets for locations of facilities. From the Wiener index point of view, the sum of the transmissions of all the vertices of G is twice the Wiener index of G .

Since the transmission is a concept more fundamental than the Wiener index, it deserves a special attention. In our first main result (Lemma 2.1) we generalize the Transmission Lemma [25, Lemma 2.1] and name the new result General Transmission Lemma. This result offers a formula for the transmission of a vertex u in terms of a collection of edge cuts and an u -routing that is compatible with the cuts in a certain way. We then discuss the result and in particular show that the classical cut method can be easily derived from the General Transmission Lemma. Then, in Section 3, we apply the General Transmission Lemma to determine the Wiener complexity of triangular grids. Along the way to obtain this result we also determine the transmission of the vertices of triangular grids.

2 General Transmission Lemma

Let $G = (V(G), E(G))$ be a connected graph. The distance $d_G(u, v)$ (or $d(u, v)$ for short if G is clear from the context) between the vertices u and v of G is the number of edges on a shortest u, v -path. The *diameter* $\text{diam}(G)$ of G is the largest distance between the vertices of G . The *Wiener index* $W(G)$ of G is

$$W(G) = \sum_{\{u,v\} \in \binom{V(G)}{2}} d_G(u, v)$$

and the *transmission* $T_G(u)$ (or $T(u)$ for short) of a vertex $u \in V(G)$ is

$$T_G(u) = \sum_{v \in V(G)} d_G(u, v).$$

Suppose that $\{T(u) : u \in V(G)\} = \{t_1, \dots, t_k\}$ and that G contains s_i vertices w with $T(w) = t_i$, where $i \in \{1, \dots, k\}$. Then, clearly,

$$W(G) = \frac{1}{2} \sum_{i=1}^k s_i t_i. \quad (1)$$

To state the General Transmission Lemma, the following concepts are crucial. Let u be a vertex of a connected graph G , and for each $x \in V(G)$, let P_{ux} be a u, x -path. Then the set of paths $\mathcal{P}_u = \{P_{ux} : x \in V(G)\}$ is a u -routing. A u -routing is *minimal* if each P_{ux} is a u, x -geodesic. Let \mathcal{P} be a u -routing in a (connected) graph G , let $\mathcal{F} = \{F_1, \dots, F_k\}$ be a multi-set of edge cuts of G , and let λ be a positive integer. Then we say that \mathcal{P} is λ -compatible with \mathcal{F} if

- (i) $|P_{ux} \cap F_i| \leq 1$ holds for each $i, 1 \leq i \leq k$, and for each $x \in V(G)$, and
- (ii) every edge $e \in \bigcup_{x \in V(G)} E(P_{ux})$ lies in precisely λ edge cuts from \mathcal{F} .

With these concepts in hand we are ready for:

Lemma 2.1 (General Transmission Lemma) *Let $\mathcal{F} = \{F_1, \dots, F_k\}$ be a multi-set of edge cuts of a connected graph G . Let $u \in V(G)$ and let G_u^i be the component of $G \setminus F_i$ that contains u . If $\{P_{ux} : x \in V(G)\}$ is a minimal u -routing which is λ -compatible with \mathcal{F} , then*

$$T(u) = \frac{1}{\lambda} \sum_{i=1}^k |V(G) \setminus V(G_u^i)|. \quad (2)$$

Proof. By definition of $T(u)$ and by the assumption that $\{P_{ux}\}$ is a minimal u -routing, we have

$$T(u) = \sum_{x \in V(G)} |E(P_{ux})|. \quad (3)$$

Let x be an arbitrary vertex of $V(G)$ and consider the path P_{ux} . Let $e \in E(P_{ux})$. Then for every $F_i \in \mathcal{F}$ such that $e \in F_i$, the vertex x lies in the set $V(G) \setminus V(G_u^i)$. Indeed, let $e = yz$, where $d(u, y) < d(u, z)$. Since F_i is an edge cut, y and z lie in different components of $V(G) \setminus V(G_u^i)$. Since $E(P_{ux}) \cap F_i = \{e\}$, the vertices y and u lie in the same component of $V(G) \setminus V(G_u^i)$ which is different from the component in which z and x lie. Consequently, $x \notin V(G_u^i)$.

Since each edge of G lies in λ cuts from \mathcal{F} , we infer that x lies in precisely $\lambda \cdot d_G(u, x)$ sets from the family of vertex subsets $\{V(G) \setminus V(G_u^i) : x \in V(G), 1 \leq i \leq k\}$. Consequently,

$$\lambda \cdot \sum_{x \in V(G)} |E(P_{ux})| = \sum_{i=1}^k |V(G) \setminus V(G_u^i)|. \quad (4)$$

Combining (4) with (3) yields the result. \square

To see that the condition $|P_{ux} \cap F_i| \leq 1$ must be included in the definition of when \mathcal{P} is λ -compatible with \mathcal{F} in order that Lemma 2.1 holds true, consider the following example.

Let $\mathcal{F} = \{F_1, \dots, F_5\}$ be the collection of edge cuts of the 5-cycle graph, where each of the cuts consists of two incident edges. If u is a vertex of the 5-cycle, then its minimal u -routing is unique and the right-hand side of (2) equals $(4 + 1 + 1 + 1 + 1)/2 = 4$, but on the other hand $T(u) = 6$. The reason for this difference is the fact that each of the two geodesics from the minimal u -routing of length 2 contains two edges from the same cut.

In the rest of this section we list some special cases and applications of the General Transmission Lemma.

First, setting $\lambda = 1$ in the General Transmission Lemma we obtain the Transmission Lemma [25, Lemma 2.1].

Second, let u be a vertex of a connected graph G and let $N_k(u) = \{x : d_G(u, x) = k\}$, where $0 \leq k \leq \text{diam}(G)$. For $k \in \{1, \dots, \text{diam}(G)\}$, let F_i be the set of edges of G that connect a vertex of $N_{i-1}(u)$ with a vertex of $N_i(u)$. Then any minimal u -routing is 1-compatible with $\mathcal{F} = \{F_1, \dots, F_{\text{diam}(G)}\}$ and thus Lemma 2.1 applies. This example in principle says that applying the lemma to the cuts between the distance levels resembles the distances derived via a BFS-tree of G rooted at u .

Third, if G is a connected graph, then an edge cut $F \subseteq E(G)$ is called *convex*, if $G \setminus F$ consists of two convex components. (A subgraph H of G is *convex* if every shortest u, v -path in G between vertices u and v of H lies completely in H .) Let $\lambda E(G)$ denote the collection of edges of G , each edge repeated λ times. Let $\mathcal{F} = \{F_1, \dots, F_k\}$ be a partition of $\lambda E(G)$ into convex cuts. (It is well-known that such a partition exists for $\lambda = 1$ if and only if G is a partial cube [19, Proposition 2.1].) Let $n_i(G)$ be the order of one of these components, so that the other has order $n(G) - n_i(G)$. By the convexity, any minimal u -routing is λ -compatible with \mathcal{F} for any vertex u of G . Hence the General Transmission Lemma applies and consequently,

$$\begin{aligned} W(G) &= \frac{1}{2} \sum_{u \in V(G)} T(u) = \frac{1}{2} \sum_{u \in V(G)} \left(\frac{1}{\lambda} \sum_{i=1}^k (n(G) - |G_u^i|) \right) \\ &= \frac{1}{\lambda} \left(\frac{1}{2} \sum_{i=1}^k [n_i(G) \cdot (n(G) - n_i(G)) + (n(G) - n_i(G)) \cdot n_i(G)] \right) \\ &= \frac{1}{\lambda} \left(\sum_{i=1}^k n_i(G) \cdot (n(G) - n_i(G)) \right). \end{aligned}$$

When $\lambda = 1$, the result is the classical cut method [18], and when λ is arbitrary, the result is due to Chepoi, Deza, and Grishukin [9]. We refer to [20] for a survey on the cut method and to [5, 11, 28, 29] for developments on the method after 2015.

3 Wiener Complexity of Triangular Grid Networks

In this section we apply Lemma 2.1 to determine the Wiener complexity of triangular grid networks. Before doing it we motivate the concept of the Wiener complexity and emphasize the role of grid graphs in digital binary image processing.

In view of (1) it is natural to define the *Wiener complexity* $C_W(G)$ of a graph G as the number of different transmissions of its vertices, that is, $C_W(G) = k$, where k is the value defined in (1). A motivation for the Wiener complexity is that for a given graph G it reveals the variety of the distance function on G . If $C_W(G)$ is large, then the variety of distances in G is large; roughly speaking, G is metrically un-homogeneous. Such graphs were recently investigated in [4]. On the other hand, if $C_W(G)$ is small, then G is, roughly speaking again, rather uniform with respect to its metric properties.

The Wiener complexity was introduced in [2] under the name *Wiener dimension*. Later, in [3], a general framework of complexities was proposed as follows. Suppose that G is a graph and I a topological index I of the form

$$I(G) = \sum_{v \in V(G)} f(v),$$

where $f : V(G) \rightarrow \mathbb{R}_0$. We note that many indices are indeed of such form, say when f is a function of the degree, distance, ... We now count how many different values f takes, that is, we determine the order of the image of f , and say that this order is the *I -complexity* of G and denote it with $C_I(G)$. Hence, in saying that $C_W(G)$ is the Wiener complexity of G , we follow the above terminology and notation.

In digital binary image processing the traditional grids being used are square, but other grids are also used. In particular, the hexagonal and the triangular grids have more symmetry axes than the square grid has which implies that a smaller angle rotations (60°) can transform these non-traditional grids to themselves than the angle (90°) needed for square grids. Due to these better symmetric properties, the triangular grid could over perform the traditional square grid in various applications, cf. the book [12].

The triangular grid graphs can be formally defined as follows, cf. [30, pp. 390–392]. A triangular grid, also called an isometric grid, is a grid formed by tiling the plane regularly with equilateral triangles. The *triangular grid graph* T_n , $n \geq 1$, is the lattice graph obtained by interpreting the order- $(n+1)$ triangular grid as a graph, with the intersection of grid lines being the vertices and the line segments between vertices being the edges. More formally, the vertex set of T_n consists of vertices (i, j, k) such that i, j, k are nonnegative integers summing to n . Two vertices are adjacent if the sum of the absolute differences of their coordinates is 2. In Fig. 1(a) the triangular grid graph T_6 is shown, while in Fig. 1(b) the transmission of each of its vertices is given.

The main result of this section now reads as follows.

Theorem 3.1 *If $n \geq 1$, then*

$$C_W(T_n) = \begin{cases} \frac{1}{6} \left(\binom{n+2}{2} + 3 \lfloor \frac{n}{2} \rfloor + 5 \right); & n \equiv 0 \pmod{3}, \\ \frac{1}{6} \left(\binom{n+2}{2} + 3 \lfloor \frac{n}{2} \rfloor + 3 \right); & \text{otherwise.} \end{cases}$$

In the rest of the section we prove Theorem 3.1. Along the way we also determine the transmission of the vertices of T_n .

Having in mind the standard embedding of T_n into the plane (just as T_6 is in Fig. 1), the edges of T_n can be partitioned into horizontal edges, acute edges, and obtuse edges,

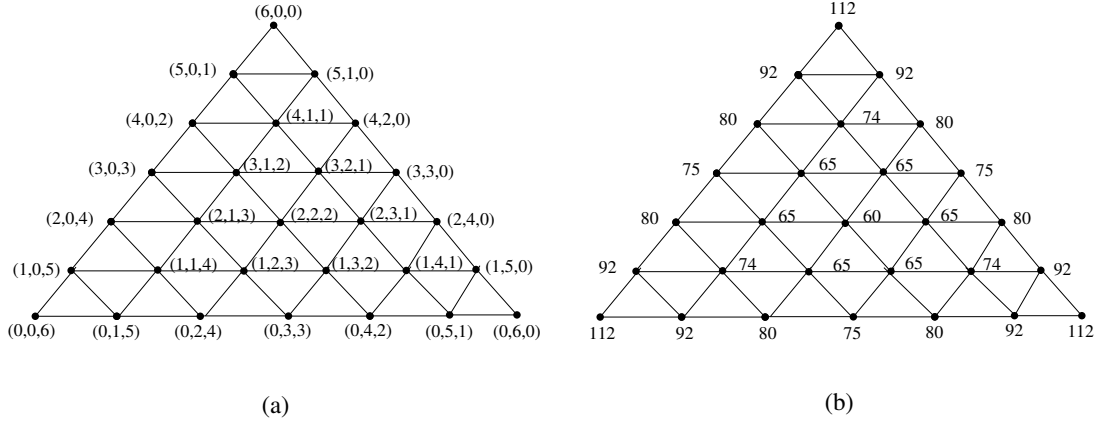


Figure 1: (a) The triangular grid graph T_6 (b) Transmissions of the vertices of T_6

cf. [24]. In Fig. 2(a) this partition is indicated on the case T_2 . The shortest path comprising of horizontal (acute, obtuse) edges with end vertices of degree 2 in T_n is said to be at h -level (a -level, o -level) 0. Inductively the path through the parents of vertices at level i is said to be at h -level (a -level, o -level) $i + 1$, $0 \leq i \leq n - 1$. The vertex $v = (i, j, k)$ in T_n is the point of intersection of the paths representing its h -level i , a -level j , and o -level k . See Fig. 2(b). We call the edge cut of T_n comprising of only acute and obtuse edges (obtuse and horizontal edges, acute and horizontal edges) with end vertices in h -levels $(i - 1)$ and i as H_i (resp. A_i, O_i), $i \in \{1, \dots, n\}$. Sometimes, H_i (resp. A_i, O_i) is also referred to as a horizontal (resp. acute, obtuse) cut, $i \in \{1, \dots, n\}$. See Fig. 2(b) again for the case T_2 .

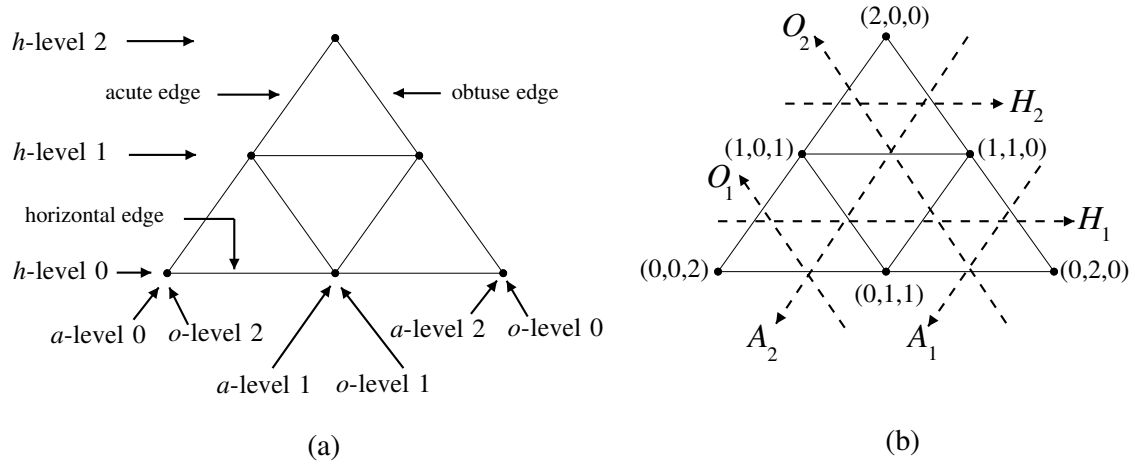


Figure 2: (a) Levels in T_2 (b) $H_i, A_i,$ and O_i cuts, $i \in \{1, 2\}$

In the computation that follows, we will make use of the well-known identity and use

the notation $[\cdot]$ for it:

$$\sum_{i=1}^n \sum_{j=1}^i j = \frac{n(n+1)(n+2)}{6} =: [n].$$

We first determine the transmission of the vertices of T_n .

Theorem 3.2 *If $n \geq 1$ and $v = (i, j, k) \in V(T_n)$, then*

$$T(v) = \frac{1}{6} \left((3n+3)(i^2 + j^2 + k^2) - (i^3 + j^3 + k^3) + 3n^2 + 4n \right).$$

Proof. Since $(i, j, k) \in V(T_n)$ we have $i, j, k \geq 0$ and $i + j + k = n$.

Note that each of the H_i , A_i , and O_i is a convex cut for $i \in \{1, \dots, n\}$. Also, any v -routing is 2-compatible with $\mathcal{E} = \{H_1, \dots, H_n\} \cup \{A_1, \dots, A_n\} \cup \{O_1, \dots, O_n\}$. Hence we may apply the General Transmission Lemma. The contribution to $2T(v)$ due to the horizontal cuts H_1, \dots, H_n is equal to

$$\begin{aligned} & \sum_{r=1}^{n-i} \sum_{s=1}^r s + \left\{ (n-i) \sum_{r=1}^i r + (1 \cdot 2 + 2 \cdot 3 + \dots + i(i+1)) \right\} \\ = & \frac{(n-i)(n-i+1)(n-i+2)}{6} + \left\{ (n-i) \sum_{r=1}^i r + (1 \cdot 2 + 2 \cdot 3 + \dots + i(i+1)) \right\} \\ = & \frac{(n-i)(n-i+1)(n-i+2)}{6} + \frac{(n-i)i(i+1)}{2} + \frac{i(i+1)(2i+1)}{6} + \frac{i(i+1)}{2} \\ = & \frac{(n-i)(n-i+1)(n-i+2)}{6} + \frac{i(i+1)(3n-i+4)}{6}. \end{aligned}$$

Taking into account the horizontal, acute, and obtuse cuts, and having in mind the symmetry of T_n , we get:

$$\begin{aligned} 2T(v) &= \frac{(n-i)(n-i+1)(n-i+2)}{6} + \frac{i(i+1)(3n-i+4)}{6} + \\ & \frac{(n-j)(n-j+1)(n-j+2)}{6} + \frac{j(j+1)(3n-j+4)}{6} + \\ & \frac{(n-k)(n-k+1)(n-k+2)}{6} + \frac{k(k+1)(3n-k+4)}{6} \\ &=: X. \end{aligned}$$

Because $i + j + k = n$ we have

$$\begin{aligned} 0 &= \frac{1}{6} (i + j + k - n)(3n^2 + 3n - 2) \\ &= \frac{1}{3} \left((3n+3)(i^2 + j^2 + k^2) - (i^3 + j^3 + k^3) + 3n^2 + 4n \right) - X, \end{aligned}$$

from which the result follows. \square

To determine the Wiener complexity of T_n , the following lemma is crucial. In the lemma, the sets $\{i, j, k\}$ and $\{p, q, r\}$ are considered as multi-sets, that is, even if two (or three) coordinates are equal, the multi-set has three elements.

Lemma 3.3 Let $u = (i, j, k)$ and $v = (p, q, r)$ be vertices of T_n . Then $T(u) = T(v)$ if and only if $\{i, j, k\} = \{p, q, r\}$.

Proof. If $\{i, j, k\} = \{p, q, r\}$, then $T(u) = T(v)$ follows by the symmetry of T_n .

To prove the converse assume that $\{p, q, r\} \neq \{i, j, k\}$. Since $u \neq v$, the coordinates of u and v differ in at least one entry. Suppose $i \neq p$, $j = q$, and $k = r$. Then $i + j + k \neq p + q + r$, a contradiction to $i + j + k = p + q + r = n$. Hence the coordinates of u and v differ in at least two entries. We now consider two cases.

Case 1: $i \neq p$, $j \neq q$, $k = r$.

We have $i + j = p + q$. Without loss of generality, let $i \geq j$. If $i < p$, then $j > q$ and vice versa. We discuss the case when $i < p$. This implies $n - i > n - p$ and $n - j < n - q$. Hence $[n - i] > [n - p]$ and $[n - j] < [n - q]$, see Fig. 3.

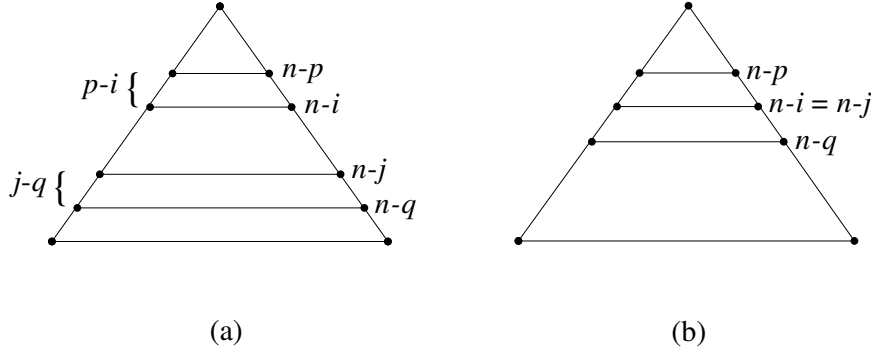


Figure 3: T_n with (a) $i \neq j, i < p$ (b) $i = j, i < p$

We can now compute as follows:

$$\begin{aligned}
T(u) - T(v) &= ([1] + \cdots + [n - p] + [n - p + 1] + \cdots + [n - i]) + \\
&\quad ([1] + \cdots + [n - p] + [n - p + 1] + \cdots + [n - i] + \\
&\quad [n - i + 1] + \cdots + [n - j]) - ([1] + \cdots + [n - p]) + \\
&\quad ([1] + \cdots + [n - p] + [n - p + 1] + \cdots) + \\
&\quad ([n - i] + [n - i + 1] + \cdots + [n - j] + [n - j + 1] + \cdots + [n - q]) \\
&= ([1] + \cdots + [n - p] + [n - p + 1] + \cdots + [n - i]) - \\
&\quad ([1] + \cdots + [n - p] + [n - j + 1] + \cdots + [n - q]) \\
&= ([n - p + 1] + [n - p + 2] + \cdots + [n - p + (p - i)]) - \\
&\quad ([n - j + 1] + [n - j + 2] + \cdots + [n - j + (j - q)]).
\end{aligned}$$

Further, $p - i = j - q$ and $[n - p + t] < [n - j + t]$, $1 \leq t \leq p - i (= j - q)$. Hence $T(u) - T(v) < 0$.

Case 2: $i \neq p$, $j \neq q$, and $k \neq r$.

It is not possible to have $i < p$, $j < q$, and $k < r$ because $i + j + k = p + q + r = n$. Hence the inequality is reversed for at least one pair. Without loss of generality let $i \geq j \geq k$. We

consider the case when $i < p$ and $j < q$. Then $i + j + k = p + q + r$ implies that $k > r$. We discuss the case when $i > j > k$. Other cases are similar. We have $n - i < n - j < n - k$. Further we have $n - i > n - p$, $n - j > n - q$, and $n - k < n - r$. Cf. Fig. 4, where for convenience we have taken $n - j < n - k$.

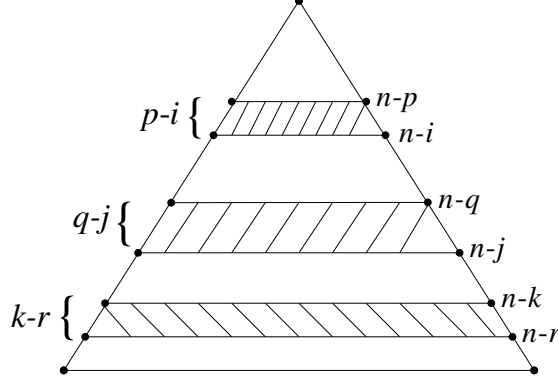


Figure 4: T_n with $i > j > k$ and $i < p, j < q, k > r$

The case when $n - j \geq n - k$ is not ruled out. In either case we have:

$$\begin{aligned}
T(u) - T(v) &= ([1] + [2] + \cdots + [n - p] + [n - p + 1] + \cdots + [n - i]) + \\
& \quad ([1] + \cdots + [n - i] + [n - i + 1] + \cdots + [n - q] + \\
& \quad [n - q + 1] + \cdots + [n - j]) + \\
& \quad ([1] + \cdots + [n - i] + \cdots + [n - j] + [n - j + 1] + \cdots + [n - k]) - \\
& \quad ([1] + \cdots + [n - p]) + ([1] + [2] + \cdots + [n - i] + \\
& \quad [n - i + 1] + \cdots + [n - q]) + \\
& \quad ([1] + \cdots + [n - i] + \cdots + [n - j] + \cdots + [n - k] + \\
& \quad [n - k + 1] + \cdots + [n - r]) \\
&= ([n - p + 1] + \cdots + [n - i] + [n - q + 1] + \cdots + [n - j]) - \\
& \quad ([n - k + 1] + \cdots + [n - r]).
\end{aligned}$$

Since $[n - p + t] < [n - k + t]$ for $1 \leq t \leq p - i$, $[n - p + t] < [n - k + t]$ for $1 \leq t \leq q - j$, and $k - r = (p - i) + (q - j)$, we have $T(u) - T(v) < 0$. \square

We can now complete the proof of Theorem 3.1 and for this sake distinguish three cases.

Case 1: n is even.

A vertex of T_n with exactly two equal coordinates is of the form $(x, x, n - 2x)$ (or a permutation of it). To count the number of such vertices, we take the values of x between 0 and $\frac{n}{2}$. Hence the total number of such tuples is $(\frac{n}{2} + 1)$. Since $T((x, x, n - 2x)) = T((x, n - 2x, x)) = T((n - 2x, x, x))$, we have accounted for a count of $(\frac{n}{2} + 1)$ to $C_W(T_n)$. So far we have considered $3(\frac{n}{2} + 1)$ vertices of T_n which have two equal coordinates. For

the rest of the vertices (x, y, z) we have $x \neq y \neq z$ and $x + y + z = n$. The number of such vertices is

$$|V(T_n)| - 3 \binom{n}{2} + 1 = \frac{(n+1)(n+2)}{2} - \frac{3n}{2} - 3.$$

Again, in view of Lemma 3.3, the six vertices obtained by permuting the coordinates x , y , and z , have the same transmission. Hence,

$$\begin{aligned} C_W(T_n) &= \binom{n}{2} + 1 + \frac{1}{6} \left(\frac{(n+1)(n+2)}{2} - 3 \binom{n}{2} + 1 \right) \\ &= \frac{1}{6} \left(\frac{(n+1)(n+2)}{2} + 3 \binom{n}{2} + 3 \right). \end{aligned}$$

Case 2: $n \equiv 0 \pmod{3}$.

The number of vertices with exactly two equal coordinates is equal to $3(\lfloor \frac{n}{2} \rfloor + 1) - 3$ as $(\frac{n}{3}, \frac{n}{3}, \frac{n}{3})$ is not accounted for in the above computation. Taking into account the vertex $(\frac{n}{3}, \frac{n}{3}, \frac{n}{3})$, the number of vertices considered is $3(\lfloor \frac{n}{2} \rfloor) + 1$. The number of the remaining vertices is

$$\frac{(n+1)(n+2)}{2} - 3 \lfloor \frac{n}{2} \rfloor - 1$$

and consequently

$$\begin{aligned} C_W(T_n) &= \lfloor \frac{n}{2} \rfloor + 1 + \frac{1}{6} \left(\frac{(n+1)(n+2)}{2} - 3 \lfloor \frac{n}{2} \rfloor - 1 \right) \\ &= \frac{1}{6} \left(\frac{(n+1)(n+2)}{2} + 3(\lfloor \frac{n}{2} \rfloor) + 5 \right). \end{aligned}$$

Case 3: n is odd and not a multiple of 3.

In this case we have:

$$\begin{aligned} C_W(T_n) &= \left(\lfloor \frac{n}{2} \rfloor + 1 \right) + \frac{1}{6} \left(\frac{(n+1)(n+2)}{2} - 3 \left(\lfloor \frac{n}{2} \rfloor + 1 \right) \right) \\ &= \frac{1}{6} \left(\frac{(n+1)(n+2)}{2} + 3 \lfloor \frac{n}{2} \rfloor + 3 \right), \end{aligned}$$

which completes the proof of Theorem 3.1.

4 Concluding remarks

In this paper we have proved the General Transmission Lemma and applied it to determine the Wiener complexity of triangular grids. This work can be extended in several ways. Chen et al. [10] have studied the hexagonal mesh as a fixed interconnection network, see Fig. 5. It would be of interest to compute the Wiener complexity of the hexagonal mesh, a task that is not a simple extension of the present work. Another challenge is to

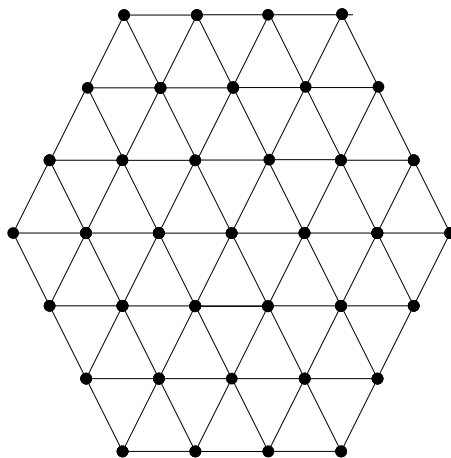


Figure 5: Hexagonal Mesh

compute the Wiener complexity for families of graphs arising from chemical compounds such as different carbon nanotubes, boron nanotubes, etc.

In location theory, sets of vertices with the minimum (or maximum) distance in a graph play a special role because they form target sets for locations of facilities. The framework can be made more general by considering the sum of distances to a specified multiset of vertices referred to as *profiles*. Determining profiles in networks using the transmission of vertices is also a challenging task.

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